Calc. Var.
DOI 10.1007/s00526-012-0519-y

Calculus of Variations

Variational formulation and structural stability of monotone equations

Augusto Visintin

Received: 1 February 2010 / Accepted: 10 March 2012

Abstract After Fitzpatrick's seminal work [MR 1009594], it is known that in a real Banach space V any maximal monotone operator $\alpha : V \to \mathcal{P}(V')$ may be given a variational representation. This is here illustrated on some examples. On this basis, De Giorgi's notion of Γ-convergence is then applied to the analysis of monotone inclusions, like $D_t u + \alpha(u) \ni h$. The compactness and the structural stability are studied, with respect to variations of the operator α and of the datum h. The possible onset of long memory in the limit is also discussed.

Mathematics Subject Classification 35K60 · 47H05 · 49J40 · 58E

1 Introduction

A number of equations that include maximal monotone nonlinearities may be set in variational form. The Γ-convergence with respect to suitable topologies is here used to study the compactness and the structural stability of a class of quasilinear monotone flows.

1.1 The Fenchel System

Let V be a real Banach space, and $\varphi : V \to \mathbf{R} \cup \{+\infty\}$ be a proper function(al). With standard notation, we shall denote the subdifferential and the convex conjugate function of

Communicated by L. Ambrosio.

A. Visintin (✉)
Dipartimento di Matematica, Università degli Studi di Trento, via Sommarive 14,
Povo, 38050 Trento, Italy
e-mail: Visintin@science.unitn.it

Published online: 08 May 2012

φ respectively by $\partial\varphi : V \to \mathcal{P}(V')$ and $\varphi^* : V' \to \mathbf{R} \cup \{+\infty\}$, see e.g. [22,23,28,36]. A classical result due to Fenchel [23] mutually relates φ, φ^* and $\partial\varphi$:

$$\varphi(v) + \varphi^*(v^*) \ \geq \langle v^*, v\rangle \quad \forall (v, v^*) \in V \times V', \tag{1.1}$$

$$\varphi(v) + \varphi^*(v^*) \ = \langle v^*, v\rangle \quad \Leftrightarrow \quad v^* \in \partial\varphi(v). \tag{1.2}$$

Defining the function

$$J(v, v^*) := \varphi(v) + \varphi^*(v^*) - \langle v^*, v\rangle \quad \forall (v, v^*) \in V \times V', \tag{1.3}$$

(1.2) also reads

$$J(v, v^*) = \inf J = 0. \tag{1.4}$$

We shall label this as a problem of *null-minimization.*

1.2 The Fitzpatrick theory

It is known that the operator $\partial\varphi$ is cyclically monotone (and maximal monotone if the function φ is proper, convex and lower semicontinuous), see e.g. [22,36]. In [24] Fitzpatrick extended the system (1.1) and (1.2) to non-cyclically monotone operators. For any proper operator $\alpha : V \to \mathcal{P}(V')$, he defined the convex and lower semicontinuous function

$$\begin{aligned} f_\alpha(v, v^*) : \ &= \ \langle v^*, v\rangle + \sup\left\{\langle v^* - v_0^*, v_0 - v\rangle : v_0^* \in \alpha(v_0)\right\} \\ &= \sup\left\{\langle v^*, v_0\rangle - \langle v_0^*, v_0 - v\rangle : v_0^* \in \alpha(v_0)\right\} \quad \forall (v, v^*) \in V \times V', \end{aligned} \tag{1.5}$$

and proved that, whenever α is maximal monotone,

$$f_\alpha(v, v^*) \geq \langle v^*, v\rangle \quad \forall (v, v^*) \in V \times V', \tag{1.6}$$

$$f_\alpha(v, v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \tag{1.7}$$

This result went essentially unnoticed for several years, until it was rediscovered by Martinez-Legaz and Théra [32] and (independently) by Burachik and Svaiter [15]. This started an intense research, that bridged monotone operators and convex functions; see e.g. [16,25,29–31,34], and Ghossoub's monograph [26]—just to mention few contributions from a growing literature. See also the related notion of *bipotential* [13,14].

Extending the theorem of Fitzpatrick, whenever a convex and lower semicontinuous function $f : V \times V' \to \mathbf{R} \cup \{+\infty\}$ fulfills the system (1.6) and (1.7), nowadays one says that f (variationally) *represents* the operator α. So e.g., for any convex and lower semicontinuous function $\varphi : V \to \mathbf{R} \cup \{+\infty\}$, $f(v, v^*) := \varphi(v) + \varphi^*(v^*)$ represents the operator $\partial\varphi$, because of (1.1) and (1.2). (We shall refer to f as the *Fenchel function* of $\partial\varphi$.) Representable operators are monotone, but need not be either cyclically or maximal monotone [16].

1.3 Γ-Convergence

The formulation in terms of null-minimization based on the Fitzpatrick theory offers the possibility to apply variational techniques to problems that so far did not look prone to that. These include e.g. first-order monotone flows, that miss a formulation as a minimum problem. We refer to typically dissipative evolutions, rather than Euler-Lagrange equations associated with action functionals. This issue has been widely investigated by Ghoussoub and coworkers, see e.g. [25–27] and references therein. In this paper we illustrate the role that may be played in this theory by De Giorgi's notion of Γ-convergence, see e.g. [2,7,8,18,19]. For instance, given a sequence $\{f_n\}$ of representative functions, known results of Γ-compactness yield the existence of a Γ-limit with respect to a suitable weak topology, up to subsequences.

This may provide the *structural stability* of the corresponding null-minimization problem. By this we mean that the mapping that transforms any set of data (including the operator) into the solution is (sequentially) closed with respect to prescribed topologies. The structure of the problem is thus preserved by perturbations of data and operators. This extends more customary results on the closure of the dependence of the solution on data, by including variations of the operator.

If (i) operators, data and solutions range in compact spaces, (ii) the problem is structurally stable, and (iii) the solution is unique, then it is easily seen that the solution depends continuously on operators and data. The structural stability may be regarded as a surrogate of the continuous dependence on the operator, whenever the solution is not known to be unique.

The structural stability seems a natural requirement for the applicative soundness of a model. The approximation of operators may also be of interest for numerical analysis, where the finite-dimensional approximation of infinite-dimensional differential operators is ubiquitous.

1.4 Monotone flows

Let us assume that we are given a Gelfand triplet of (real) Banach spaces

$$V \subset H = H' \subset V' \quad \text{with continuous and dense injections,} \tag{1.8}$$

and fix any $h \in L^{p'}(0, T; V')$ ($2 \le p < +\infty$, $p' = p/(p-1)$) and any $u^0 \in H$. Whenever an operator $\alpha : V \to \mathcal{P}(V')$ is represented by a function f_α, the Cauchy problem

$$\begin{cases} D_t u + \alpha(u) \ni h \quad \text{in } V', \text{a.e. in }]0, T[\ (D_t := \partial/\partial t), \\ u(0) = u^0 \end{cases} \tag{1.9}$$

is equivalent to the null-minimization problem of the convex and lower semicontinuous functional

$$J : \left\{ v \in L^p(0, T; V) \cap W^{1,p'}(0, T; V') : v(0) = u^0 \right\} \to \mathbf{R} \cup \{+\infty\},$$

$$J(v) := \int_0^T [f_\alpha(v, h - D_t v) - \langle h, v \rangle] \, dt + \tfrac{1}{2}\|v(T)\|_H^2 - \tfrac{1}{2}\|u^0\|_H^2. \tag{1.10}$$

We shall label this equivalence as the *extended B.E.N. principle,* since it generalizes an approach that was pioneered by Brezis and Ekeland [11] and by Nayroles [33] in 1976. More specifically, this combines the original B.E.N. principle (which assumes α to be cyclically mononotone) with the Fitzpatrick theorem; see the Examples 3.6 and 3.7 in Sect. 3.

A different formulation may also be introduced. By a simple translation, we may assume $u^0 \equiv 0$. Let us then define the real Banach space

$$X_0^p := \left\{ v \in L^p(0, T; V) \cap W^{1,p'}(0, T; V') : v(0) = 0 \right\} \quad (p' = p/(p-1)).$$

It is easily seen that the operator $D_t + \alpha : X_0^p \to \mathcal{P}((X_0^p)')$ is monotone, and may be represented by the function

$$f(v, v^*) := \int_0^T f_\alpha(v, v^* - D_t v)\, dt + \tfrac{1}{2}\|v(T)\|_H^2 \quad \forall (v, v^*) \in X_0^p \times L^{p'}(0, T; V'). \tag{1.11}$$

The system (1.9) (with $u^0 \equiv 0$) is thus equivalent to the *null-minimization* of the convex functional

$$v \mapsto \widetilde{J}(v, h) := f(v, h) - \int_0^T \langle h, v\rangle\, dt, \tag{1.12}$$

namely,

$$u \in X_0^p, \quad \widetilde{J}(u, h) = \inf_{v \in X_0^p} \widetilde{J}(v, h) = 0. \tag{1.13}$$

Whenever $\varphi : V \to \mathbf{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function(al) and $\alpha = \partial\varphi$, the problem (1.9) is a *gradient flow,* and may also be set in the form

$$\begin{cases} u \in L^p(0, T; V) \cap W^{1,p'}(0, T; V'), \quad u(0) = u^0, \\ \int_0^T [\varphi(u) + \langle D_t u - h, u\rangle]\, dt \le \int_0^T [\varphi(v) + \langle D_t u - h, v\rangle]\, dt \quad \forall v \in L^p(0, T; V). \end{cases} \tag{1.14}$$

This has the form "$\Phi_u(u) \le \Phi_u(v)$ for any v" (a *quasi-variational inequality);* this sort of variational structure does not need the use of the Fitzpatrick theory. It is natural to wonder whether this has consequences for the compactness and structural stability.

1.5 Plan of work

This is part of an ongoing research on the variational representation of (nonlinear) evolutionary P.D.E.s, and on the application of variational techniques to the analysis of their structural stability, see e.g. [42,46,48,49]. A somehow comparable program, based on the use of the Fitzpatrick theory, has been accomplished for the homogenization of quasilinear flows in [43–45]. By and large, homogenization might actually be regarded as a problem of structural stability, since it involves the asymptotic behavior of the operator and of the data. This is especially clear if one considers the passage from a two-scale to a single-scale formulation.

This article consists of three parts. The first one is an introduction to the Fitzpatrick theory. In Sect. 2 we recall the notion of representative function, and review some results of the related theory. In Sect. 3 we provide several examples of those functions; in Proposition 3.2 we also extend the B.E.N. principle.

The second part is devoted to the Γ-convergence of representative functions. In Sect. 4 we introduce some notions of weak convergence in the space $V \times V'$, that look suited for the study of representative functions. These include a nonlinear notion of convergence, that we name $\widetilde{\pi}$*-convergence,* see (4.1). We study the corresponding Γ-convergence, and provide a fairly general result of Γ-compactness in Theorem 4.4. In that section we extensively refer to the theory of Γ-convergence as developed in Dal Maso's monograph [18]. In Sect. 5 we show that the family of representative functions is closed under Γ-convergence with respect to weak-type topologies, see Theorem 5.1. We also briefly deal with some related issues: the convergence of Fenchel functions, the graph convergence of maximal monotone operators, and the Mosco-convergence. The two latter notions are especially appropriate for the structural stability of maximal monotone flows, like (1.9); however they miss the compactness properties, that are at the focus of this work. In Sect.6 we then address the representation of operators acting on time-dependent functions, and in Proposition 6.3 we mutually relate pointwise- and global-in-time convergence.

In the third part of this paper we apply the previous results to problems of the form (1.9). In Sect. 7 first we illustrate what we mean by compactness and structural stability in general. We then apply these concepts to (1.9), distinguishing some variants:

(i) global- and pointwise-in-time formulations of the time-periodic problem,
(ii) global- and pointwise-in-time formulations of the corresponding initial-value problem.

We consider this time-periodic problem since only in this case we are able to prove a result of compactness and structural stability without assuming compactness of the injection $V \to H$. (Notice that this hypothesis involves the structure of the problem, rather than just the regularity of the data.) By the results of Sects. 4, 5, 6, for each of these four formulations the representative functions Γ-converge to representative functions, up to extracting a subsequence. This allows us to prove the structural stability either of (1.9) or of the corresponding global-in-time formulation. More specifically, in Sect. 7 we address the periodic flow, whereas in Sect. 8 we deal with the corresponding initial-value problem. In Sect. 8 we also briefly illustrate how for gradient flows compactness and structural stability may be proved without using the Fitzpatrick theory.

The asymptotic analysis of the global formulation raises the question of the possible onset of long memory in the limit, that has been pointed out and studied for a linear problem by Tartar in [39–41]. In Sect. 9 we illustrate this issue, exhibit examples of monotone problems either with or without onset of long memory, and discuss the results of the two previous sections. (In discriminating between these two behaviors, the compactness of the injection $V \to H$ seems to play a more important role than the gradient structure of the operator.) Finally, we briefly revisit the examples of Sect. 3.

In the author's opinion, the main results of this work concern the Γ-compactness of representative functions with respect to a nonlinear weak-type topology (Sect. 4 and Theorem 5.1), and the ensuing structural stability of monotone flows (Sects. 7 and 8). This theory raises several further questions, that will be addressed apart. These include e.g. the extension of the results of Sects. 7 and 8 to nonmonotone operators, see [49]; a further analysis of the onset of long memory in monotone flows; the identification of the Γ-limit of sequences of representative functions; and so on.

2 Outline of the Fitzpatrick theory

In this section we briefly review the theory that was pioneered by Fitzpatrick [24] in 1988, and then developed by several other authors in the last ten years.

2.1 Fitzpatrick functions

Let V be a real Banach space. A (possibly multi-valued) operator $\alpha : V \to \mathcal{P}(V')$ with graph $A := \{(v, v^*) \in V \times V' : v^* \in \alpha(v)\}$ is said *monotone* if

$$v^* \in \alpha(v) \quad \Rightarrow \quad \langle v^* - v_0^*, v - v_0 \rangle \geq 0 \quad \forall (v_0, v_0^*) \in A. \tag{2.1}$$

(It will be convenient to include the empty operator, $\alpha \equiv \emptyset$, into the class of monotone operators.) Whenever the converse implication also holds, α is said *maximal monotone,* see e.g. [5,9,12,50]. We shall denote by $\mathcal{M}(V)$ the class of maximal monotone operators $V \to \mathcal{P}(V')$.

In [24] S. Fitzpatrick associated with any operator with graph $A \not\equiv \emptyset$ the function (now named the *Fitzpatrick function*)

$$\begin{aligned} f_\alpha(v, v^*) &:= \langle v^*, v\rangle + \sup\big\{\langle v^* - v_0^*, v_0 - v\rangle : (v_0, v_0^*) \in A\big\} \\ &= \sup\big\{\langle v^*, v_0\rangle - \langle v_0^*, v_0 - v\rangle : (v_0, v_0^*) \in A\big\} \quad \forall (v, v^*) \in V \times V'. \end{aligned} \tag{2.2}$$

Note that, even if α is nonmonotone,

$$\begin{aligned} &f_\alpha : V \times V' \to \mathbf{R} \cup \{+\infty\} \ \text{ is convex and lower semicontinuous,} \\ &f_\alpha(v, v^*) \geq \langle v^*, v\rangle \quad \text{if either} v \in \alpha^{-1}(V') \text{ or } v^* \in \alpha(V). \end{aligned} \tag{2.3}$$

The class of monotone operators will be of interest, because of the following result of Fitzpatrick.

Theorem 2.1 *[24] Let $\alpha : V \to \mathcal{P}(V')$, and denote its Fitzpatrick function by f_α. Then:*

(i) *α is monotone if and only if*

$$\forall (v, v^*) \in V \times V', \quad \langle v^*, v\rangle = f_\alpha(v, v^*) \quad \Leftarrow \quad v^* \in \alpha(v); \tag{2.4}$$

(ii) *$\alpha \in \mathcal{M}(V)$ ($\alpha \not\equiv \emptyset$) if and only if*

$$f_\alpha(v, v^*) \geq \langle v^*, v\rangle \quad \forall (v, v^*) \in V \times V', \tag{2.5}$$

$$f_\alpha(v, v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \tag{2.6}$$

Proof [(1)] First let us set

$$\pi(v, v^*) := \langle v^*, v\rangle \quad \forall (v, v^*) \in V \times V', \tag{2.7}$$

and note that, for any $(v, v^*) \in V \times V'$, by (2.2)

$$f_\alpha(v, v^*) \leq \langle v^*, v\rangle \quad \Leftrightarrow \quad \langle v^* - v_0^*, v - v_0\rangle \geq 0 \quad \forall (v_0, v_0^*) \in A. \tag{2.8}$$

Note that $f_\alpha \geq \pi$ on A by (2.2)$_2$, and that α is monotone if and only if $f_\alpha \leq \pi$ on A (namely, $f_\alpha = \pi$ on A). Part (i) is thus established.

Let us now assume that $\alpha \in \mathcal{M}(V)$. By (2.8), $f_\alpha(v, v^*) \leq \langle v^*, v\rangle$ then entails that $(v, v^*) \in A$. On the other hand, by part (i), $(v, v^*) \in A$ entails $f_\alpha(v, v^*) = \langle v^*, v\rangle$. In conclusion, $f_\alpha \geq \pi$ in the whole $V \times V'$; moreover, $f_\alpha(v, v^*) = \langle v^*, v\rangle$ entails $(v, v^*) \in A$, and conversely.

Finally, we show that (2.5) and (2.6) entail that $\alpha \in \mathcal{M}(V)$. By part (i), (2.6) entails that α is monotone. For any $(v, v^*) \in V \times V'$, if $\langle v^* - v_0^*, v - v_0\rangle \geq 0$ for any $(v_0, v_0^*) \in A$, then $f_\alpha(v, v^*) \leq \langle v^*, v\rangle$ by (2.8). As $f_\alpha \geq \pi$, we infer that $f_\alpha(v, v^*) = \langle v^*, v\rangle$, whence $(v, v^*) \in A$ by (2.6). We conclude that $\alpha \in \mathcal{M}(V)$. □

[1] This argument is based upon Theorems 3.4, 3.8 and 3.9 of [24], where V is assumed to be a Hausdorff locally convex space. The subsequent theory was then developed in Banach space, and we shall conform to this trend.
We display this proof here since this result plays a key role in this work, and the article [24] might not be easily available to the reader.

2.2 Representative functions

We shall denote by $\mathcal{F}(V)$ the class of the functions f such that

$$f : V \times V' \to \mathbf{R} \cup \{+\infty\} \text{ is convex and lower semicontinuous,} \tag{2.9}$$
$$f(v, v^*) \geq \langle v^*, v \rangle \quad \forall (v, v^*) \in V \times V'. \tag{2.10}$$

To any function $f \in \mathcal{F}(V)$ we shall then associate the operator α such that, for any $(v, v^*) \in V \times V'$,

$$v^* \in \alpha(v) \quad \Leftrightarrow \quad f(v, v^*) = \langle v^*, v \rangle. \tag{2.11}$$

The identically empty operator, $\alpha \equiv \emptyset$, is thus associated with the identically infinite function, $f \equiv +\infty$. (Any other operator and any other function will be said to be *proper).*

Whenever (2.11) holds, we shall say that $f \in \mathcal{F}(V)$ (variationally) *represents* α or that f is a *representative* of α, and that α is *representable.* We shall accordingly refer to $\mathcal{F}(V)$ as the class of *representative functions.* We shall also denote by $\mathcal{R}(V)$ the class of representable operators $V \to \mathcal{P}(V')$ and by $\mathcal{F}_\alpha(V)$ the subclass of the functions that represent some fixed operator $\alpha \in \mathcal{R}(V)$. Note that any function of $\mathcal{F}(V)$ represents just one operator of $\mathcal{R}(V)$, whereas an operator of $\mathcal{R}(V)$ may be represented by several functions of $\mathcal{F}(V)$. Let us denote by $\mathcal{I}$ the permutation operator

$$\mathcal{I} : V \times V' \to V' \times V : (v, v^*) \mapsto (v^*, v). \tag{2.12}$$

Obviously, if V is reflexive,

$$\begin{aligned} & g \in \mathcal{F}(V) \text{ represents an operator } \alpha : V \to \mathcal{P}(V') \\ \Leftrightarrow \quad & g \circ \mathcal{I}^{-1} \in \mathcal{F}(V') \text{ represents } \alpha^{-1} : V' \to \mathcal{P}(V). \end{aligned} \tag{2.13}$$

The next statement is also straightforward.

Proposition 2.2 *(i) The class $\mathcal{F}(V)$ is convex.*

(ii) For any nonempty family $\widehat{\mathcal{F}} \subset \mathcal{F}(V)$, the mapping $\widehat{f} : (v, v^*) \mapsto \sup\{f(v, v^*) : f \in \widehat{\mathcal{F}}\}$ is an element of $\mathcal{F}(V)$.

(iii) The properties (i) and (ii) hold also if $\mathcal{F}(V)$ is replaced by $\mathcal{F}_\alpha(V)$, for any $\alpha \in \mathcal{R}(V)$.

(iv) For any $f \in \mathcal{F}(V)$ and any convex and lower semicontinuous mapping $g : V \times V' \to \mathbf{R} \cup \{+\infty\}$, if $f \leq g$ pointwise, then $g \in \mathcal{F}(V)$ and (denoting by α_f and α_g the respective represented operators) $\text{graph}(\alpha_g) \subset \text{graph}(\alpha_f)$.

For any $f \in \mathcal{F}(V)$ and any closed convex set $K \subset V \times V'$, thus $f + I_K \in \mathcal{F}(V)$.

In Sect. 5 we shall also see that the class $\mathcal{F}(V)$ is stable by suitable notions of variational convergence. Here are some further properties:

(i) All representable operators are monotone, see Theorem 3.1 of [16]. (This would hold even if the representative functions were not assumed to be lower semicontinuous.)
(ii) Not all monotone operators are representable. E.g., the null mapping restricted to $V \setminus \{0\}$ is not representable. (This would fail even if the representative function were not required to be convex.)
(iii) All maximal monotone operators are representable. Actually, by part (ii) of Theorem 2.1, an operator is maximal monotone if and only if it is represented by its Fitzpatrick function.

(iv) Not all representable operators are maximal monotone. E.g., the trivial operator $\alpha \equiv \emptyset$ is represented by $f \equiv +\infty$ (an element of $\mathcal{F}(V)$). The nonmaximal monotone operator with graph $A = \{(0,0)\}$ is represented by $f_1 = I_{\{(0,0)\}}$ [2] and, if V is a Hilbert space, also by $f_2 : (u, v) \mapsto \|u + v\|^2/2$.

The representable operators thus form a strictly intermediate class between monotone and maximal monotone operators.

By the latter statement, for any nonempty subset S of $\mathbf{R}^N$, a monotone affine mapping $S \to \mathbf{R}^N$ is representable if and only if S is closed and convex.

Let us set $J(v, v^*) = f(v, v^*) - \langle v^*, v\rangle$ for any $(v, v^*) \in V \times V'$, and denote by ∂_v and ∂_{v^*} the partial subdifferential operators. By (2.9)–(2.11), for any $f \in \mathcal{F}(V)$ and any (v, v^*) we have

$$f(v, v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad J(v, v^*) = \inf J = 0 \quad \Rightarrow \quad \begin{cases} \partial_v f(v, v^*) \ni v^* \\ \partial_{v^*} f(v, v^*) \ni v. \end{cases} \tag{2.14}$$

After [24], the converse of the latter implication also holds whenever f represents a maximal monotone operator.

It is easy to see that the representable operators share the following properties with the maximal monotone operators.

Proposition 2.3 *([30]; Proposition 8) If $\alpha \in \mathcal{R}(V)$, then:*

(i) $\alpha(v)$ and $\alpha^{-1}(v^*)$ are closed and convex, for any $v \in V$ and any $v^* \in V'$;
(ii) for any sequence $\{(v_n, v_n^*)\}$ in A, [3]

$$v_n \rightharpoonup v \ \text{ in } V, \quad v_n^* \overset{*}{\rightharpoonup} v^* \ \text{ in } V', \quad \liminf_{n\to\infty} \langle v_n^*, v_n\rangle \le \langle v^*, v\rangle \quad \Rightarrow \quad (v, v^*) \in A. \tag{2.15}$$

Notice also that $\mathcal{F}(V)$ is an ordered space, equipped with the natural ordering of the functions $V \times V' \to \mathbf{R} \cup \{+\infty\}$: $f_1 \le f_2$ if and only if $f_1(v, v^*) \le f_2(v, v^*)$ for any $(v, v^*) \in V \times V'$. The next result mirrors the known statement that any monotone operator has a maximal monotone extension.

Theorem 2.4 *([24]) For any $g \in \mathcal{F}(V)$, the class $\{h \in \mathcal{F}(V) : h \le g\}$ has at least one minimal element with respect to the pointwise ordering. If V is reflexive, then any of these minimal elements is the Fitzpatrick function of a maximal monotone extension of the operator that is represented by g.*

2.3 Duality

In the remainder of this section we shall assume that the real Banach space V is reflexive, although for some statements this restriction may be dropped. Let us then identify the bidual space of V with V itself, denote by $g^* : V' \times V \to \mathbf{R} \cup \{+\infty\}$ the Fenchel-Legendre conjugate (if it exists) of any proper function $g : V \times V' \to \mathbf{R} \cup \{+\infty\}$, by g^{**} its double conjugate (assuming that g^* is proper, too), and by $[\cdot, \cdot]$ the canonical duality pairing between $V \times V'$ and $V' \times V$:

$$[(v, v^*), (v_0^*, v_0)] := \langle v_0^*, v\rangle + \langle v^*, v_0\rangle \quad \forall (v, v^*), (v_0, v_0^*) \in V \times V'. \tag{2.16}$$

[2] We denote by I_C the indicator function of any set C; that is, $I_C = 0$ in C and $I_C = +\infty$ outside C.

[3] We denote the strong, weak, and weak star convergence respectively by $\to$, $\rightharpoonup$, $\overset{*}{\rightharpoonup}$.

Note that by (2.6)

$$f(v, v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad (v^*, v) \in \partial f(v, v^*). \tag{2.17}$$

(This equality does not follow directly from (2.11), since the *joint* subdifferential might be strictly included into the product of the *partial* subdifferentials.) Let us denote the indicator function of the graph of α by I_α. As the definition (2.2) also reads

$$f_\alpha(v, v^*) = \sup\left\{[(v, v^*), (v_0^*, v_0)] - \langle v_0^*, v_0\rangle : v_0^* \in \alpha(v_0)\right\},$$

we have

$$f_\alpha(v, v^*) = (\pi + I_\alpha)^*(v^*, v) \quad \forall(v, v^*) \in V \times V', \tag{2.18}$$

that is, $f_\alpha = (\pi + I_\alpha)^* \circ \mathcal{I}$.

Theorem 2.5 *(Theorem 3.1 of [16], [29,38]) Let V be a reflexive Banach space. A function $g \in \mathcal{F}(V)$ represents a maximal monotone operator if and only if $g^* \in \mathcal{F}(V')$.*

For instance, as we pointed out above, the nonmaximal monotone operator with graph $A = \{(0, 0)\}$ is represented by $f = I_{\{(0,0)\}}$. Its convex conjugate reads $f^* \equiv 0$, which is not a representative function.

Corollary 2.6 *Let V be a reflexive Banach space, $g \in \mathcal{F}(V)$ and $g^* \in \mathcal{F}(V')$. If g represents an operator α, then g^* represents the inverse operator α^{-1}. (Thus g and $g^* \circ \mathcal{I}$ represent the same maximal monotone operator, if any.)*

Indeed, applying (2.11) and (2.17) to g and g^*,

$$\begin{aligned} v \in \alpha^{-1}(v^*) \quad &\Leftrightarrow \quad v^* \in \alpha(v) \quad \Leftrightarrow \quad (v^*, v) \in \partial f(v, v^*) \\ &\Leftrightarrow \quad (v, v^*) \in [(\partial f)^{-1}](v, v^*) = \partial g^*(v^*, v). \end{aligned} \tag{2.19}$$

Any $\alpha \in \mathcal{M}(V)$ is thus represented by both $(\pi + I_\alpha)^* \circ \mathcal{I}$ and $(\pi + I_\alpha)^{**}$, see also [15,34], besides several other functions, e.g. all convex combinations of these two functions. Moreover,

$$f_\alpha := (\pi + I_\alpha)^* \circ \mathcal{I} \le g \le s_\alpha := (\pi + I_\alpha)^{**} \quad \forall g \in \mathcal{F}_\alpha(V), \quad \forall \alpha \in \mathcal{M}(V). \tag{2.20}$$

(The function s_α was introduced in [15]; see also [34].) In passing note that the domain of the latter coincides with the closed convex hull of the graph of α in $V \times V'$.

The next result easily follows from the latter statements.

Corollary 2.7 *Let V be a reflexive Banach space. For any convex and lower semicontinuous function $g : V \times V' \to \mathbf{R} \cup \{+\infty\}$ and any $\alpha \in \mathcal{M}(V)$,*

$$g \in \mathcal{F}(V),\ g \text{ represents } \alpha \quad \Leftrightarrow \quad f_\alpha \le g \le s_\alpha, \tag{2.21}$$

$$g \text{ represents } \alpha \quad \Leftrightarrow \quad g^* \text{ represents } \alpha^{-1}. \tag{2.22}$$

Simple examples show that in general the set of the functions that represent a fixed maximal monotone operator is not totally ordered (with respect to the pointwise ordering).

Theorem 2.8 *([38]) Let V be a reflexive Banach space. Any $\alpha \in \mathcal{M}(V)$ may be represented by a function $g \in \mathcal{F}(V)$ such that $g^* = g \circ \mathcal{I}^{-1}$.*

Any maximal monotone operator may thus be represented by a (possibly nonunique) *self-dual* function. This class has extensively been studied by Ghoussoub and coworkers, see e.g. [25,26] and references therein.

Under suitable restrictions, the class $\mathcal{F}(V)$ is stable by linear combinations with non-negative scalars, and thus forms a convex subcone of the cone of lower semicontinuous convex mappings $V \times V' \to \mathbf{R} \cup \{+\infty\}$. Simple rules allow one to construct representative functions of cone combinations (i.e., linear combinations with nonnegative scalars) of representable operators, see e.g. Chap. 5 of [26]. For instance, the extended B.E.N. principle (see Proposition 3.2 ahead) follows from those rules.

3 Examples of representative functions

In this section we give some examples of representable operators and of their representative functions. Reference to these examples will only be done in Sect. 8, so that skipping this section would not impair the comprehension of the remainder.

Example 3.1 (Fenchel Function). Let V be a real Banach space. The subdifferential $\partial\varphi : V \to \mathcal{P}(V')$ of a proper, convex and lower semicontinuous function $\varphi : V \to \mathbf{R} \cup \{+\infty\}$ is represented by the function

$$f(v, v^*) = \varphi(v) + \varphi^*(v^*) \quad \forall (v, v^*) \in V \times V', \tag{3.1}$$

that we shall refer to as the *Fenchel function* of the operator $\partial\varphi$. This f is *self-dual,* that is, $f^* = f$ in the duality between $V \times V'$ and $V' \times V$. In this case the inequality (2.10) is a straightforward consequence of the definition of the convex conjugate function φ^*, and (2.11) (with $\alpha = \partial\varphi$) coincides with the classical *Fenchel inequality* (1.2) (as here the sign $=$ is equivalent to $\le$).

Next we show that, among the representative functions of maximal monotone operators, the Fenchel functions are the only ones that have an additive form.

Proposition 3.1 *A function $f \in \mathcal{F}(V)$ of the form*

$$f(v, v^*) = \varphi_1(v) + \varphi_2(v^*) \quad \forall (v, v^*) \in V \times V', \tag{3.2}$$

represents some operator $\alpha \in \mathcal{M}(V)$ (if and) only if φ_1 and φ_2 are mutually convex conjugate. In that case, $\alpha = \partial\varphi_1 = (\partial\varphi_2)^{-1}$.

Proof Defining π as in (2.7), setting $J := f - \pi$, and recalling (2.14), for any $(v, v^*) \in V \times V'$ we have

$$v^* \in \alpha(v) \quad \Leftrightarrow \quad \partial J(v, v^*) \ni 0 \quad \Leftrightarrow \quad \begin{cases} \partial\varphi_1(v) \ni v^* \\ \partial\varphi_2(v^*) \ni v; \end{cases} \tag{3.3}$$

the latter equivalence follows from the additive form of (3.2). Hence $\alpha \subset \partial\varphi_1$; more precisely $\alpha = \partial\varphi_1$, by the maximality of α. By (3.3) thus $(\partial\varphi_1)^{-1} = \alpha^{-1} \subset \partial\varphi_2$, whence $\partial\varphi_2 = (\partial\varphi_1)^{-1}$ by the maximality of α^{-1}. Hence $\varphi_2 = \varphi_1^* + C$ for some $C \in \mathbf{R}$. By (2.11) and (3.2) then

$$\varphi_1(v) + \varphi_1^*(v^*) + C = \langle v^*, v\rangle \quad \forall (v, v^*) \in \text{graph}(\partial\varphi_1) \ (\neq \emptyset). \tag{3.4}$$

By the classical Fenchel formula (1.2), we conclude that $C = 0$. □

Example 3.2 (Quasilinear Elliptic Operator). Let Ω be a bounded domain of $\mathbf{R}^N$ ($N > 1$), $p \in]1, +\infty[$, and set $V := W_0^{1,p}(\Omega)$. Let a maximal monotone mapping $\vec{\gamma} : \mathbf{R}^N \to \mathcal{P}(\mathbf{R}^N)$ be represented by a function $f \in \mathcal{F}(\mathbf{R}^N)$. If

$$\exists c_1, c_2 \in \mathbf{R}^+ : \ \forall \vec{w} \in \mathbf{R}^N, \quad \forall \vec{z} \in \gamma(\vec{w}), \qquad |\vec{z}| \le c_1 |\vec{w}|^{p'} + c_2, \tag{3.5}$$

the mapping $v \mapsto -\nabla \cdot \vec{\gamma}(\nabla v)$ then defines an operator $\widehat{\gamma} : V \to \mathcal{P}(V')$, which is promptly seen to be maximal monotone. Indeed, for any $\lambda > 0$ and any $v^* \in W^{-1,p'}(\Omega)$, the problem

$$v \in W_0^{1,p}(\Omega), \quad -\nabla \cdot \vec{\gamma}(\nabla v) - \lambda \nabla \cdot (|\nabla v|^{p-2} \nabla v) = v^* \quad \text{in } \mathcal{D}'(\Omega)$$

has a solution. As $v \mapsto -\nabla \cdot (|\nabla v|^{p-2} \nabla v)$ is the duality mapping $W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, by the classical Minty–Browder theorem (see e.g. [5]) we conclude that $\hat{\gamma}$ is maximal monotone. This obviously includes e.g. the case of the p-Laplacian operator, which corresponds to $\vec{\gamma}(\vec{v}) = |\vec{v}|^{p-2} \vec{v}$.

Let us assume that f is coercive in the sense that

$$\exists a, b > 0 : \forall (v, w) \in \mathbf{R}^2, \quad f(v, w) \ge a(|v|^p + |w|^{p'}) - b. \tag{3.6}$$

We claim that $\widehat{\gamma}$ is then represented by the following function $\psi : V \times V' \to \mathbf{R}$:

$$\psi(v, v^*) = \inf \left\{ \int_\Omega f(\nabla v, \vec{\eta}) \, dx : \vec{\eta} \in L^{p'}(\Omega)^N, -\nabla \cdot \vec{\eta} = v^* \ \text{ in } \mathcal{D}'(\Omega) \right\}, \tag{3.7}$$

for any $(v, v^*) \in V \times V'$. By the coerciveness of f, this infimum is attained at some $\vec{\xi}_{v^*} \in L^{p'}(\Omega)^N$. The function ψ is convex, lower semicontinuous (because of (3.6)), and

$$\psi(v, v^*) = \int_\Omega f(\nabla v, \vec{\xi}_{v^*}) \, dx \overset{f \in \mathcal{F}(\mathbf{R}^N)}{\ge} \int_\Omega \nabla v \cdot \vec{\xi}_{v*} \, dx = -\langle v, \nabla \cdot \vec{\xi}_{v^*} \rangle = \langle v, v^* \rangle. \tag{3.8}$$

Thus $\psi \in \mathcal{F}(V)$. Moreover, as $f(\nabla v, \vec{\xi}_{v^*}) \ge \nabla v \cdot \vec{\xi}_{v^*}$ pointwise in Ω, equality holds in (3.8) if and only if $f(\nabla v, \vec{\xi}_{v^*}) = \nabla v \cdot \vec{\xi}_{v^*}$ a.e. in Ω. As f represents $\vec{\gamma}$, this is equivalent to $\vec{\xi}_{v^*} \in \vec{\gamma}(\nabla v)$ a.e. in Ω, whence $v^* = -\nabla \cdot \vec{\xi}_{v^*} \in -\nabla \cdot \vec{\gamma}(\nabla v)$ in $W^{-1,p'}(\Omega)$. We thus conclude that ψ represents the operator $\widehat{\gamma}$. (We could not prescribe $\vec{\xi}_{v^*} = -\nabla \Delta^{-1} v^*$, since $\vec{\xi}_{v^*} \in \vec{\gamma}(\nabla v)$ need not be curl-free.)

Example 3.3 (Degenerate Quasilinear Elliptic Operator). Let Ω and $\vec{\gamma}$ be as in the latter example, with $N = 3$; let us define the Hilbert space V and the maximal monotone operator $\widehat{\gamma}$ as follows:

$$\begin{aligned} &V := \left\{ \vec{v} \in L^2(\Omega)^3 : \nabla \times \vec{v} \in L^2(\Omega)^3, \ \vec{\nu} \times \vec{v} = \vec{0} \text{ on } \partial\Omega \right\}, \\ &\widehat{\gamma} : V \to \mathcal{P}(V') : \vec{v} \mapsto \nabla \times \vec{\gamma}(\nabla \times \vec{v}). \end{aligned} \tag{3.9}$$

The dual space V' may thus be identified with $\left\{ \nabla \times \vec{v} : \vec{v} \in L^2(\Omega)^3 \right\}$. We claim that $\widehat{\gamma}$ is represented by the following function $\psi \in \mathcal{F}(V)$: for any $(\vec{v}, \vec{v}^*) \in V \times V'$,

$$\psi(\vec{v}, \vec{v}^*) = \inf \left\{ \int_\Omega f(\nabla \times \vec{v}, \vec{\eta}) \, dx : \vec{\eta} \in L^2(\Omega)^3, \nabla \times \vec{\eta} = \vec{v}^* \ \text{ in } \mathcal{D}'(\Omega)^3 \right\}. \tag{3.10}$$

By the coerciveness of f, this infimum is attained at some $\vec{\xi}_{\vec{v}^*} \in L^2(\Omega)^N$. We then have

$$\psi(\vec{v}, \vec{v}^*) = \int_\Omega f(\nabla\times\vec{v}, \vec{\xi}_{\vec{v}^*})\, dx \overset{f\in\mathcal{F}(\mathbf{R}^N)}{\geq} \int_\Omega \nabla\times\vec{v}\cdot\vec{\xi}_{\vec{v}^*}\, dx = \langle\vec{v}, \nabla\times\vec{\xi}_{\vec{v}^*}\rangle = \langle\vec{v}, \vec{v}^*\rangle; \quad (3.11)$$

as in the latter example, it is easily checked that $\psi \in \mathcal{F}(V)$. As $f(\nabla\times\vec{v}, \vec{\xi}_{\vec{v}^*}) \geq \nabla\times\vec{v}\cdot\vec{\xi}_{\vec{v}^*}$ a.e. in Ω, equality holds in (3.11) if and only if $f(\nabla\times\vec{v}, \vec{\xi}_{\vec{v}^*}) = (\nabla\times\vec{v})\cdot\vec{\xi}_{\vec{v}^*}$ a.e. in Ω. As f represents $\vec{\gamma}$, the latter equality is equivalent to $\vec{\xi}_{\vec{v}^*} \in \vec{\gamma}(\nabla\times\vec{v})$ a.e. in Ω, whence $\vec{v}^* \in \nabla\times\vec{\gamma}(\nabla\times\vec{v})$ in V', hence in the sense of distributions.

The degenerate operator $\hat{\gamma}$ is also maximal monotone, as it may easily be checked via the classical Minty–Browder theorem.

Example 3.4 (Time-Derivative). Here we shall refer to the Banach triplet (1.8), fix any $T > 0$, $p \in [2, +\infty[$, and set

$$\begin{aligned} &X_0^p := \left\{v \in L^p(0,T;V) \cap W^{1,p'}(0,T;V') : v(0) = 0\right\},\\ &\alpha(v) = D_t v \quad \text{a.e. in }]0,T[, \forall v \in X_0^p. \end{aligned} \tag{3.12}$$

The initial condition is meaningful, as any element of X_0^p may be identified with a continuous function $[0,T] \to H$. The restriction $v(0) = 0$ provides the monotonicity of the operator. This condition is not really restrictive, since it may be retrieved by simply shifting the unknown function; it might be dropped at the expense of replacing X_0^p by the corresponding affine space of the functions that attain a prescribed initial value.

The triplet (V, H, V') induces the triplet

$$\mathcal{V} := L^p(0,T;V) \subset \mathcal{H} := L^2(0,T;H) = \mathcal{H}' \subset \mathcal{V}' = L^{p'}(0,T;V') \tag{3.13}$$

with continuous and dense injections. We have

$$X_0^p \subset \mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}' \subset (X_0^p)', \tag{3.14}$$

with continuous and dense injections; but the characterization of the dual space $(X_0^p)'$ does not seem obvious. Let us denote by $[\cdot,\cdot]$ the duality pairing between $(X_0^p)'$ and X_0^p. The bounded linear operator $\alpha : X_0^p \to \mathcal{V}' \subset (X_0^p)'$ is monotone, as

$$[\alpha(v), v] = \int_0^T \langle D_t v, v\rangle\, dt = \tfrac{1}{2}\|v(T)\|_H^2 \geq 0 \quad \forall v \in X_0^p.$$

By the linearity this operator is then maximal monotone, although not cyclically monotone as it is not symmetric. It is represented by the function

$$\begin{aligned} f_\alpha(v, v^*) &= I_{D_t}(v, v^*) + [v^*, v] = I_{D_t}(v, v^*) + [D_t v, v]\\ &= I_{D_t}(v, v^*) + \tfrac{1}{2}\|v(T)\|_H^2 \quad \forall (v, v^*) \in X_0^p \times (X_0^p)'. \end{aligned} \tag{3.15}$$

One might also deal with a nonhomogeneous initial condition $v(0) = v^0$ (prescribed in H); in this case D_t would be monotone on the affine space $X_0^p + v^0$.

Example 3.5 (Time-Derivative with Periodicity). Let us fix any $T \in]0, +\infty]$, and set

$$\begin{aligned} &X_\sharp^p := \left\{v \in L^p(0,T;V) \cap W^{1,p'}(0,T;V') : v(0) = v(T)\right\},\\ &\alpha(v) = D_t v \quad \text{a.e. in }]0,T[, \quad \forall v \in X_\sharp^p. \end{aligned} \tag{3.16}$$

If $T = +\infty$, then for any $v \in X^p_\sharp$ it is easily seen that $v(t) \to 0$ in V' as $t \to +\infty$. The (degenerate) periodicity condition then reads $v(0) = \lim_{t\to+\infty} v(t) = 0$, so that

$$X^p_\sharp = \{v \in X^p : v(0) = 0\} \quad \text{if } T = +\infty; \tag{3.17}$$

Here the operator α is monotone and skew-symmetric, and we retrieve a limit case of Example 3.4.

If T is finite, an equivalent formulation might also be used. Let us first denote by $\mathcal{I}$ the circumference of $\mathbf{C}$ with center 0 and radius $T/2\pi$, and identify any T-periodic function on $\mathbf{R}$ with a function on $\mathcal{I}$, via the bijective mapping $\lambda :]0, T] \to \mathcal{I} : t \mapsto y = (T/2\pi)e^{2\pi it/T}$. One may then replace the operator α by

$$\widetilde{\alpha}(v) = D_t(v \circ \lambda^{-1}) \quad \text{a.e. in } \mathcal{I},$$
$$\forall v \in \widetilde{X}^p := L^p(\mathcal{I}; V) \cap W^{1,p'}(\mathcal{I}; V'). \tag{3.18}$$

The operator α is represented by the function

$$f_\alpha(v, v^*) = I_{D_t}(v, v^*) + [v^*, v] = I_{D_t}(v, v^*) + [D_t v, v] = I_{D_t}(v, v^*)$$
$$\forall (v, v^*) \in X^p_\sharp \times (X^p_\sharp)'.$$

Example 3.6 (Extended B.E.N. Principle).

Proposition 3.2 *([42]) Let V be a Banach space, and $L : V \to V'$ be a monotone, bounded, linear operator. If an operator $\alpha : V \to \mathcal{P}(V')$ is represented by a function $f_\alpha \in \mathcal{F}(V)$, then $\alpha + L$ is represented by the function*

$$f(v, v^*) = f_\alpha(v, v^* - Lv) + \langle Lv, v\rangle \quad \forall (v, v^*) \in V \times V'. \tag{3.19}$$

Proof The function $(v, v^*) \mapsto f_\alpha(v, v^* - Lv)$ is convex and lower semicontinuous, since it is the composition of the convex and lower semicontinuous function f_α with the linear and continuous transformation $(v, v^*) \mapsto (v, v^* - Lv)$. As the function $v \mapsto \langle Lv, v\rangle$ is also convex and lower semicontinuous, the same holds for f. It is straightforward to check that this function represents the operator $\alpha + L$. □

The latter result extends the B.E.N. principle of [11,33]; there α is assumed to be cyclic and maximal monotone, f_α is the Fenchel function (3.1), and L is the time-derivative. In this case Proposition 3.2 is applied to the triplet

$$\mathcal{V} := X^p_0 \subset \mathcal{H} := L^2(0, T; H) = \mathcal{H}' \subset \mathcal{V}',$$

with X^p_0 defined as in (3.12).

Example 3.7 (Abstract Quasilinear Parabolic Operator). In the functional framework of the Example 3.4, let us assume that

$$\alpha : V \to \mathcal{P}(V') \text{ is represented by a function } f_\alpha \in \mathcal{F}(V), \tag{3.20}$$

$$\exists C_1, C_2 > 0 : \quad \forall (v, v^*) \in \text{graph}(\alpha), \quad \|v^*\| \le C_1\|v\|^{p-1} + C_2, \tag{3.21}$$

and define $\mathcal{V} := L^p(0, T; V)$ as in (3.13). The operator $\alpha : V \to \mathcal{P}(V')$ is then canonically associated with an operator $\bar{\alpha} : \mathcal{V} \to \mathcal{P}(\mathcal{V}')$, which is represented by the functional

$$\bar{f}_\alpha : \mathcal{V} \times \mathcal{V}' \to \mathbf{R} \cup \{+\infty\} : (v, v^*) \mapsto \int_0^T f_\alpha(v, v^*)\, dt.$$

Dealing with initial-value problems for the operator $D_t+\alpha$, it seems convenient to use the space X_0^p, see (3.12). If α is as $\hat{\gamma}$ in the Example 3.2, then $D_t+\alpha$ is quasilinear parabolic. As $D_t:\mathcal{V}\to\mathcal{V}'$ is linear and bounded, if $\bar{\alpha}:\mathrm{Dom}(\bar{\alpha})\subset\mathcal{V}\to\mathcal{P}(\mathcal{V}')$ is maximal monotone then the same holds for $D_t+\bar{\alpha}$. However, here we shall be concerned with the representability rather than the maximal monotonicity.

It is easily seen that the restriction of $\bar{\alpha}$ to X_0^p, that we denote by $\widetilde{\alpha}:X_0^p\to\mathcal{P}(\mathcal{V}')\subset\mathcal{P}((X_0^p)')$, is represented by the function

$$\Phi_{\tilde{\alpha}}:X_0^p\times(X_0^p)'\to\mathbf{R}\cup\{+\infty\}:(v,v^*)\mapsto\begin{cases}\int_0^T f_\alpha(v,v^*)\,dt & \text{if } v^*\in\mathcal{V}',\\ +\infty & \text{otherwise.}\end{cases}\tag{3.22}$$

By the extended B.E.N. principle (see Proposition 3.2), $D_t+\widetilde{\alpha}$ is then represented by the function

$$\Psi:X_0^p\times(X_0^p)'\to\mathbf{R}\cup\{+\infty\}:(v,v^*)\mapsto\Phi_{\tilde{\alpha}}(v,v^*-D_tv)+\tfrac{1}{2}\|v(T)\|_H^2.\tag{3.23}$$

In a Hilbert space, this is easily extended to an operator of the form $\Lambda D_t+\widetilde{\alpha}$, provided that Λ is a self-adjoint operator.

Example 3.8 (Stefan Operator).This is a particular case of the latter example. Let us select $V=L^2(\Omega)$ and $H=H^{-1}(\Omega)$, and equip $H^{-1}(\Omega)$ with the scalar product $(w,v)=\langle-\Delta^{-1}w,v\rangle$ (in the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$), where $\theta=-\Delta^{-1}w$ is such that $\theta\in H_0^1(\Omega)$ and $-\Delta\theta=w$ in $\mathcal{D}'(\Omega)$. For any $\alpha\in\mathcal{M}(V)$ with affine growth, it is easily seen that the operator

$$X_0^2\to\mathcal{P}((X_0^2)'):v\mapsto D_tv-\Delta\alpha(v)\tag{3.24}$$

is representable and maximal monotone. If α is not strictly monotone, this operator is degenerate. For instance, an operator like this with α constant along an interval occurs in the weak formulation of the classical (scalar) Stefan problem; in this case α may also be assumed to be Lipschitz-continuous. This may easily be extended to the vector Stefan problem.

Example 3.9 (Time-Integral). Let V, H, p be as in the Example 3.4, and set

$$\begin{aligned}&Y_0^p:=W^{-1,p}(0,T;V)\cap L^{p'}(0,T;V')\ (=\{D_tv:v\in X_0^p\}),\\&\mathcal{J}v(t):=\int_0^t v(\tau)\,d\tau \text{ for }\quad \text{a.e. } t\in\,]0,T[,\ \forall v\in Y_0^p.\end{aligned}\tag{3.25}$$

This operator is maximal monotone, and is represented by the function

$$f_{\mathcal{J}}(v,v^*)=I_{\mathcal{J}}(v,v^*)+\tfrac{1}{2}\|\mathcal{J}v(T)\|_H^2\quad\forall(v,v^*)\in Y_0^p\times(Y_0^p)'.\tag{3.26}$$

This setting is closely related to Example 3.4, as $Y_0^p=D_t(X_0^p)$ (the image set of the operator D_t) and $\mathcal{J}:Y_0^p\to X_0^p$ is an isomorphism, with inverse D_t. For any linear invertible operator $\Lambda:V\to V$, the operator $v\mapsto\Lambda^*\mathcal{J}\Lambda u$ is maximal monotone, too.

Example 3.10 (Transport Operator). Let Ω be a domain of $\mathbf{R}^N$ $(N>1)$ of Lipschitz class, assume that

$$\begin{aligned}&\vec{\nu}\text{ is the outward-oriented unit normal vector-field on }\partial\Omega,\\&\vec{w}\in C^{0,1}(\bar{\Omega})^N\cap L^\infty(\Omega)^N,\quad\nabla\cdot\vec{w}\le 0\text{ a.e. in }\Omega,\end{aligned}\tag{3.27}$$

Omitting the trace operator, let us set

$$\begin{aligned} &\Gamma = \big\{x \in \partial\Omega : \vec{\nu}\cdot\overrightarrow{w} < 0\big\},\\ &V_0 := \big\{v \in H^1(\Omega) : v = 0 \ \text{ on } \Gamma\big\},\\ &\beta(v) := \overrightarrow{w}\cdot\nabla v \ \big(= \textstyle\sum_{i=1}^N w_i D_i v\big) \quad \text{a.e. in } \Omega, \quad \forall v \in V_0. \end{aligned} \tag{3.28}$$

The bounded linear operator $\beta : V_0 \to L^2(\Omega)$ is monotone as

$$\begin{aligned} \int_\Omega \beta(v) v\, dx &= \tfrac{1}{2}\int_\Omega \overrightarrow{w}\cdot\nabla(v^2)\, dx = \tfrac{1}{2}\int_\Omega [\nabla\cdot(\overrightarrow{w}\, v^2) - (\nabla\cdot\overrightarrow{w})v^2]\, dx\\ &= \tfrac{1}{2}\int_{\partial\Omega} \vec{\nu}\cdot\overrightarrow{w}\, v^2\, dx - \tfrac{1}{2}\int_\Omega (\nabla\cdot\overrightarrow{w})v^2\, dx \geq 0 \quad \forall v \in V_0, \end{aligned} \tag{3.29}$$

by $(3.27)_2$ and $(3.28)_1$. The operator β is then maximal monotone, but not cyclically monotone. Denoting by $I_{\overrightarrow{w}\cdot\nabla}$ the indicator function of the graph of the operator $\vec{v} \mapsto \overrightarrow{w}\cdot\nabla\vec{v}$, β itself is represented by

$$\begin{aligned} f_\beta(v, v^*) &= I_{\overrightarrow{w}\cdot\nabla}(v, v^*) + \int_\Omega (\overrightarrow{w}\cdot\nabla v) v\, dx\\ &= I_{\overrightarrow{w}\cdot\nabla}(v, v^*) + \tfrac{1}{2}\int_{\partial\Omega} \vec{\nu}\cdot\overrightarrow{w}\, v^2\, dx - \tfrac{1}{2}\int_\Omega (\nabla\cdot\overrightarrow{w})v^2\, dx, \end{aligned} \tag{3.30}$$

for any $(v, v^*) \in V_0 \times V_0'$. If Ω is an N-dimensional interval, this is easily extended to periodic boundary conditions.

On the basis of the Examples 3.2, 3.10 and of the extended B.E.N. principle (see Proposition 3.2), one may represent a large class of quasilinear second order elliptic operators, A, and the associated parabolic operators, $D_t + A$.

Several linear second order hyperbolic operators are also representable. For instance, this applies to $D_{tt} + A$ if A is a positive and self-adjoint operator on a Hilbert space.

Example 3.11 In particular the setting of the latter example applies if $\Omega =]0, T[$. This may also be extended to vector functions $v : [0, T] \to \mathbf{R}^M$ $(M \geq 1)$ Let $A : [0, T] \to \mathbf{R}^{M\times M}$ be Lipschitz continuous, symmetric and positive (semi)definite for any t, and such that $D_t A$ is negative (semi)definite for a.e. t. The operator

$$v : \{v \in H^1(0, T) : v(0) = 0\} \to L^2(0, T) : v \mapsto A\cdot D_t v \tag{3.31}$$

is bounded, linear and maximal monotone. This may also be extended to periodicity conditions for a domain of the form $\Omega =]a_1, b_1[\times\cdots\times]a_N, b_N[$.

Example 3.12 (Saddle Operator). Let B_1 and B_2 be two real Banach spaces, and at least one of them be reflexive. Let $E_i \subset B_i$ $(i = 1, 2)$ be nonempty, closed and convex sets, and let $L : E_1 \times E_2 \to \mathbf{R}$ be a saddle function such that

$$\begin{aligned} &L(\cdot, v_2) \text{ is convex and lower semicontinuous}, \quad \forall v_2 \in E_2,\\ &L(v_1, \cdot) \text{ is concave and upper semicontinuous}, \quad \forall v_1 \in E_1. \end{aligned} \tag{3.32}$$

Let us denote by $\partial_1 L(\partial_2 L$, resp.) the partial subdifferential (partial supdifferential, resp.) of L. The operator

$$\widetilde{\partial} L : E_1 \times E_2 \to \mathcal{P}(B_1') \times \mathcal{P}(B_2') : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} \partial_1 L(v_1, v_2) \\ -\partial_2 L(v_1, v_2) \end{pmatrix} \tag{3.33}$$

is then maximal monotone, but not cyclically monotone; see e.g. p. 137 [6], [35], p. 396 of [36]. Denoting by $\langle\langle \cdot, \cdot \rangle\rangle$ the duality pairing between $B_1 \times B_2$ and $B_1' \times B_2'$, after (1.5) the Fitzpatrick function of $\widetilde{\partial} L$ reads

$$\begin{aligned} &f_{\tilde{\partial} L}(\vec{v}, \vec{v}^*) := \\ &\quad \langle\langle \vec{v}^*, \vec{v} \rangle\rangle + \sup\left\{ \langle\langle \vec{v}^* - \vec{v}_0^*, \vec{v}_0 - \vec{v} \rangle\rangle : \vec{v}_0^* \in \widetilde{\partial} L(\vec{v}_0), \vec{v}_0 \in E_1 \times E_2 \right\} \\ &= \sup\left\{ \langle\langle \vec{v}^*, \vec{v}_0 \rangle\rangle - \langle\langle \vec{v}_0^*, \vec{v}_0 - \vec{v} \rangle\rangle : \vec{v}_0^* \in \widetilde{\partial} L(\vec{v}_0), \vec{v}_0 \in E_1 \times E_2 \right\} \\ &\forall \vec{v} = (v_1, v_2) \in E_1 \times E_2, \quad \forall \vec{v}^* = (v_1^*, v_2^*) \in B_1' \times B_2'. \end{aligned} \tag{3.34}$$

Example 3.13 (Saddle Flow). Here we combine the Example 3.4 with the latter one. In the functional framework of the Example 3.12, the operator

$$D_t + \widetilde{\partial} L : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} D_t v_1 \\ D_t v_2 \end{pmatrix} + \begin{pmatrix} \partial_1 L(v_1, v_2) \\ -\partial_2 L(v_1, v_2) \end{pmatrix} \tag{3.35}$$

is monotone, but not cyclically monotone. The equation $(D_t + \widetilde{\partial} L)(v_1, v_2) = (h_1, h_2)$ accounts for descent along the convex potential $v_1 \mapsto L(v_1, v_2) - h_1 \cdot v_1$, coupled with ascent along the concave potential $v_2 \mapsto L(v_1, v_2) + h_2 \cdot v_2$.

By the extended B.E.N. principle of Proposition 3.2 and by (3.34), the operator $D_t + \widetilde{\partial} L$ is easily represented.

3.1 Other classes

Further examples of representable operators may be built by combining the above ones. The next statement exhibits a further wide class of maximal monotone operators.

Proposition 3.3 *Let (V, H, V') be a Banach triplet as in (1.8), and an operator $\alpha \in \mathcal{M}(V)$ be strongly monotone, in the sense that*

$$\begin{aligned} &\exists C > 0 : \quad \forall (v_1, w_1), (v_2, w_2) \in \operatorname{graph}(\alpha), \\ &\langle w_1 - w_2, v_1 - v_2 \rangle \geq C \| v_1 - v_2 \|_V^2. \end{aligned} \tag{3.36}$$

If $\gamma : V \to V'$ is a Lipschitz-continuous operator with Lipschitz constant $L \leq C$, then $\alpha + \gamma \in \mathcal{M}(V)$.

Proof Let us denote by J the duality mapping $V \to V'$. The operator $\alpha + \gamma$ is obviously monotone, and for any $\lambda > C$, $\lambda J + \alpha + \gamma$ is coercive, hence onto V'. By the classical Minty–Browder theorem, we then infer that $\alpha + \gamma$ is also maximal monotone. □

The class of strongly monotone operators is thus stable under small Lipschitz perturbations. It is easily seen that this fails if we restrict this class to cyclically monotone operators.

3.2 Strong Monotonicity and Strict Convexity

Next we draw a useful consequence from the extended B.E.N. principle.

Proposition 3.4 *Let (V, H, V') be a Banach triplet as in (1.8), and $\alpha \in \mathcal{M}(V)$. If*

$$\begin{aligned} &\exists c > 0 : \quad \forall (v_1, w_1), (v_2, w_2) \in \operatorname{graph}(\alpha), \\ &\langle w_1 - w_2, v_1 - v_2 \rangle \geq c \| v_1 - v_2 \|_H^2. \end{aligned} \tag{3.37}$$

then α may be represented by a function f such that $f(\cdot, v^*)$ is strictly convex for any $v^* \in V'$.

(A stronger condition is obviously obtained if $\|v_1 - v_2\|_H^2$ is replaced by $\|v_1 - v_2\|_V^2$ in (3.37)*, namely if α is strongly monotone.)*

Proof Let us denote by L the canonic injection $V \to V'$, so that $\langle Lv, v\rangle = \|v\|_H^2$ for any $v \in V$. By (3.37) $\tilde{A} := A - cL$ is maximal monotone, hence it may be represented by a function $\tilde{f} \in \mathcal{F}(V)$. By the extended B.E.N. principle (see Proposition 3.2), $\tilde{A}$ is then represented by the function

$$f(v, v^*) = \tilde{f}(v, v^* - cLv) + c\|v\|_H^2 \quad \forall (v, v^*) \in V \times V', \tag{3.38}$$

As the function $v \mapsto \tilde{f}(v, v^* - cLv)$ is convex, the thesis follows. □

4 Γ-Compactness of representative functions

In the remainder of this work we shall be concerned with the structural stability of maximal monotone operators and related equations, via the variational representation that we introduced above. In this section we deal with the variational convergence and compactness of families of representative functions, via De Giorgi's notion of Γ-convergence.

4.1 Some linear and nonlinear topologies

We assume throughout that V is a real Banach space. We shall denote by $\widetilde{\pi}$ the coarsest among the topologies of $V \times V'$ that are finer than the product of the weak topology of V by the weak star topology of V', and that make continuous the mapping π (defined in (2.7)). For any sequence $\{(v_n, v_n^*)\}$ in $V \times V'$, thus

$$\begin{aligned} &(v_n, v_n^*) \underset{\widetilde{\pi}}{\to} (v, v^*) \quad \text{in } V \times V' \quad \Leftrightarrow \\ &v_n \rightharpoonup v \quad \text{in } V, \quad v_n^* \overset{*}{\rightharpoonup} v^* \quad \text{in } V', \quad \langle v_n^*, v_n\rangle \to \langle v^*, v\rangle, \end{aligned} \tag{4.1}$$

and similarly for any net. This topology defines a nonlinear convergence: a linear combination of two converging sequences need not converge. We shall also use the following linear topologies:

ω is the product of the weak topology of V by the weak star topology of V',
ws is the product of the weak topology of V by the strong topology of V',
$sw*$ is the product of the strong topology of V by the weak star topology of V',
s is the strong topology of $V \times V'$.

These convergences, with the only exception of ω, are especially appropriate for the analysis of representable operators. For instance, by (2.11) the graph of any representable operator is closed with respect to the convergences ws, $sw*$ and s, but not with respect to ω.

4.2 Metrizability and sequential characterization

We shall say that a topology τ on a Banach space B is *locally metrizable* if B may be equipped with a metrizable topology, that has the same restriction as τ to any norm-bounded subset of B.

Lemma 4.1 *(Local Metrizability) If V' is separable, then the topologies ω, $\widetilde{\pi}$, ws and $sw*$ are* locally metrizable.

Proof As V' is separable the same holds for V. The space V equipped with the weak topology is then locally metrizable, and the same holds for V' equipped with the weak star topology; see e.g. Sect. III.6 of [10]. Therefore $(V\times V', \omega)$ (namely, $V\times V'$ equipped with the topology ω) is also locally metrizable. The same holds for the product topologies ws and $sw*$. We are left with the proof of this property for the topology $\widetilde{\pi}$.

Let us equip $V\times V'\times \mathbf{R}$ with the topology $\bar{\omega}$, that is defined as the product of the weak topology of V, the weak star topology of V', and the ordinary topology of $\mathbf{R}$. This product space is locally metrizable. Let us define the mapping

$$\theta : V\times V' \to V\times V'\times \mathbf{R} : (v, v^*) \mapsto (v, v^*, \langle v^*, v\rangle), \tag{4.2}$$

and note that $(V\times V', \widetilde{\pi})$ is homeomorphic to the image set $\theta(V\times V')$ equipped with the topology induced by $\bar{\omega}$, As this set is locally metrizable, we conclude that the topology $\widetilde{\pi}$ is also locally metrizable. □

It may be noticed that the mapping θ establishes a one-to-one correspondence between the nonlinear convergence $\widetilde{\pi}$ in the linear space $V\times V'$ and a linear convergence in the nonlinear subset $\theta(V\times V')$ of the linear topological space $V\times V'\times \mathbf{R}$ (equipped with the topology $\bar{\omega}$). Note also that bounded subsets of $V\times V'$ need not be relatively compact with respect to the topology $\widetilde{\pi}$ (nor ws, $sw*$ and s, obviously).

Lemma 4.2 *(p. 54 of [18]) For any pair of topologies $\tau_1 \subset \tau_2$ over a set X, and any sequence $\{f_n\}$ of functions $X \to \mathbf{R}\cup\{+\infty\}$,*[4]

$$\Gamma\tau_1 \liminf_{n\to\infty} f_n \le \Gamma\tau_2 \liminf_{n\to\infty} f_n, \quad \Gamma\tau_1 \limsup_{n\to\infty} f_n \le \Gamma\tau_2 \limsup_{n\to\infty} f_n. \tag{4.3}$$

The same then holds for the Γ-limits, whenever they exist.

We remind the reader that, for functions defined on a topological space, the definition of Γ-convergence involves the filter of the neighborhoods of each point; see e.g. p. 25–27 of [2], p. 38 of [18]. If the space is first-countable (in particular, if it is metrizable), that notion may equivalently be formulated in terms of the family of converging sequences, but this does not apply in general; see e.g. p. 270 of [2], Chap. 8 of [18]. We shall refer to these two notions as *topological* and *sequential* Γ-convergence, respectively. (If it is not otherwise specified, reference to the topological notion should be understood.)

Proposition 4.3 *Let V' be separable, and τ be any of the topologies $\omega, \widetilde{\pi}$, ws and $sw*$ of $V\times V'$. Let σ be a metrizable topology on $V\times V'$ that is locally equivalent to τ (by Lemma 4.1 such a topology exists). Let $\{\psi_n\}$ be a sequence of functions $V\times V' \to \mathbf{R}\cup\{+\infty\}$ that is* equi-coercive, *in the sense that*

$$\forall C \in \mathbf{R}, \ \sup_{n\in\mathbf{N}} \Big\{\|v\|_V + \|v^*\|_{V'} : (v, v^*) \in V\times V', \psi_n(v, v^*) \le C\Big\} < +\infty. \tag{4.4}$$

Then ψ_n $\Gamma\tau$-converges topologically if and only if it $\Gamma\sigma$-converges sequentially, hence if and only if it $\Gamma\tau$-converges sequentially, that is,

$$\begin{aligned} &\forall (v, v^*) \in V\times V', \quad \forall \textit{ sequence } \{(v_n, v_n^*)\} \textit{ in } V\times V', \\ &(v_n, v_n^*) \xrightarrow[\tau]{} (v, v^*) \ \textit{ in } V\times V' \ \Rightarrow \ \liminf_{n\to\infty} \psi_n(v_n, v_n^*) \ge \psi(v, v^*), \end{aligned} \tag{4.5}$$

[4] By $\Gamma\tau$ lim we shall denote the Γ limit with respect to a topology τ.

$$\begin{aligned}&\forall(v,v^*)\in V\times V',\ \exists \textit{ sequence } \{(v_n,v_n^*)\} \textit{ in } V\times V':\\&(v_n,v_n^*)\ \underset{\tau}{\to}\ (v,v^*) \ \textit{ and } \ \psi_n(v_n,v_n^*)\to\psi(v,v^*).\end{aligned} \tag{4.6}$$

Proof After p. 353 of [1], at p. 93 of [18] this result is proved for the weak topology; see also p. 285 of [2]. That argument may be extended verbatim to any locally metrizable topology τ as above. □

Remark For Proposition 4.3 to hold, the assumption (4.4) is in order when the topologies ω and $\widetilde{\pi}$ are considered. Dealing with the topology ws, (4.4) might be replaced by the weaker condition

$$\forall C\in\mathbf{R},\ \sup_{n\in\mathbf{N},v^*\in V'}\Big\{\|v\|_V : v\in V,\ \psi_n(v,v^*)\le C\Big\}<+\infty, \tag{4.7}$$

since the strong topology V' is already a metric topology. An analogous statement holds for the topology $sw*$, if (4.7) is modified by exchanging the roles of V and V'. Dealing with the topology s, (4.4) may be dropped.

This remark will also apply to Theorem 4.4, and to all other results that rest upon Proposition 4.3. □

4.3 Compactness

The next result will play a key role in the remainder of this work.

Theorem 4.4 *Let V' be separable, and τ be any of the topologies ω, $\widetilde{\pi}$, ws and $sw*$ of $V\times V'$. Let $\{\psi_n\}$ be a sequence of functions $V\times V'\to\mathbf{R}\cup\{+\infty\}$ that fulfills (4.4). Then, up to extracting a subsequence, ψ_n $\Gamma\tau$-converges to some function ψ both topologically and sequentially.*

The same holds for the topology s, and

$$\begin{aligned}&\Gamma\omega\lim\psi_n\le\Gamma\widetilde{\pi}\lim\psi_n\le\min\{\Gamma ws\lim\psi_n,\Gamma sw*\lim\psi_n\},\\&\max\{\Gamma ws\lim\psi_n,\Gamma sw*\lim\psi_n\}\le\Gamma s\lim\psi_n.\end{aligned} \tag{4.8}$$

Proof We adapt the argument of Corollary 8.12 of p. 95 of [18] to the present more general setting. By Lemma 4.1, the space $V\times V'$ can be equipped with a metrizable topology σ that is locally equivalent to τ. By the assumption of separability of V', $(V\times V',s)$ is separable, so that the same holds for any ball B of this product space. As the topology τ is coarser than s, (B,τ) is also separable, and the same thus holds for (B,σ). As $V\times V'$ is an increasing and countable union of balls, the metrizable space $(V\times V',\sigma)$ is an increasing and countable union of separable subspaces. $(V\times V',\sigma)$ is then also separable. As any separable metric space has a countable basis, see e.g. p. 111 of [17], we conclude that $(V\times V',\sigma)$ has a countable basis.

By Lemma 4.5 below, $\{\psi_n\}$ then has a $\Gamma\sigma$-convergent subsequence, which is also sequentially $\Gamma\sigma$-convergent as σ is a metric topology. By Proposition 4.3, this subsequence then $\Gamma\tau$-converges topologically and sequentially. By (4.4), the Γ-limit does not attain the value $-\infty$. The thesis is thus established for the topology τ.

For the metric topology s the thesis is a direct consequence of Lemma 4.5; in this case (4.4) is only used to exclude that the Γ-limit attain the value $-\infty$, and might be replaced by a simpler requirement. (4.8) directly follows from Lemma 4.2. □

Lemma 4.5 *(p. 152 of [2], p. 90 of [18]) If a topological space X has a countable basis, then every sequence $\{f_n\}$ of functions $X\to\mathbf{R}\cup\{\pm\infty\}$ has a Γ-convergent subsequence.*

Remark (i) As we pointed out above, dealing with the topologies ws and $sw*$, in Theorem 4.4 the hypothesis (4.4) might be weakened. In the case of the topology s, it may be dropped, at the expense of allowing the Γ-limit to attain the value $-\infty$.

(ii) It is known that a sequence $\{A_n\}$ of subsets of a topological space X converges in the sense of Kuratowski to a (closed) set A with respect to a topology τ if and only if $I_{A_n} \xrightarrow{\Gamma} I_A$ with respect to τ; see e.g. [2,18]. A compactness result analogous to Theorem 4.4 might then be formulated in terms of the Kuratowski convergence of the indicator functions. For a sequence $\{A_n\}$ of subsets of $V \times V'$, the sequence $\{I_{A_n}\}$ however fulfills the hypothesis (4.4) only if these sets are uniformly bounded. This excludes any sequence of maximal monotone operators. If the A_ns are the graphs of a sequence of equi-coercive operators, this might be remedied; but we shall not delve on this issue here, since a function I_A is convex only if A is linear. □

5 Stability properties of representative functions

In this section we provide sufficient conditions for the stability of the class of representative functions and of the class of representable operators, with respect to the notions of weak convergence that we introduced in the previous section. We then discuss some related concepts: the graph convergence of maximal monotone operators, the convergence of Fenchel functions, and the Mosco-convergence.

We remind the reader that we denote by $\mathcal{F}(V)$ the class of the representative functions $f : V \times V' \to \mathbf{R} \cup \{+\infty\}$, namely, the convex and lower semicontinuous functions such that $f(v, v^*) \geq \langle v^*, v\rangle$ for any $(v, v^*) \in V \times V'$. We denote by $\mathcal{R}(V)$ the class of representable operators $V \to \mathcal{P}(V')$, and by $\mathcal{M}(V)$ the subclass of maximal monotone operators.

Theorem 5.1 *Let V' be separable, and τ be any of the topologies $\widetilde{\pi}$, ws, $sw*$ and s (but not ω) of $V \times V'$. Let $\{\psi_n\}$ be a sequence in $\mathcal{F}(V)$ that sequentially $\Gamma\tau$-converges to a function ψ. Then:*

- *(i)* $\psi \in \mathcal{F}(V)$.
- *(ii)* *If α_n (α, resp.) is the operator that is represented by ψ_n (ψ, resp.), then for any sequence $\{(v_n, v_n^*)\}$ in $V \times V'$*

$$v_n^* \in \alpha_n(v_n) \quad \forall n, \quad (v_n, v_n^*) \underset{\tau}{\to} (v, v^*) \quad \Rightarrow \quad v^* \in \alpha(v). \tag{5.1}$$

(This second part may be compared with part (ii) of Proposition 2.3.)

Proof (i) The function ψ is convex and sequentially lower semicontinuous with respect to the topology τ, since both properties are preserved by passage to the upper Γ-limit, respectively in any vector space and in any topological space; see e.g. p. 264 of [2] and p. 57, 126 of [18].

For any $(v, v^*) \in V \times V'$, by (4.6) there exists a sequence $\{(v_n, v_n^*) \in V \times V'\}$ such that $(v_n, v_n^*) \underset{\tau}{\to} (v, v^*)$ and $\psi_n(v_n, v_n^*) \to \psi(v, v^*)$; thus $\langle v_n^*, v_n\rangle \to \langle v^*, v\rangle$. Therefore

$$\langle v^*, v\rangle = \lim_{n\to\infty} \langle v_n^*, v_n\rangle \overset{\psi_n\in\mathcal{F}(V)}{\leq} \lim_{n\to\infty} \psi_n(v_n, v_n^*) = \psi(v, v^*). \tag{5.2}$$

Thus $\psi \in \mathcal{F}(V)$.

(ii) For any sequence $\{(v_n, v_n^*) \in A_n\}$ such that $(v_n, v_n^*) \underset{\tau}{\to} (v, v^*)$,

$$\psi(v, v^*) \overset{(4.5)}{\leq} \liminf_{n\to\infty} \psi_n(v_n, v_n^*) \overset{v_n^*\in\alpha_n(v_n)}{=} \liminf_{n\to\infty} \langle v_n^*, v_n\rangle = \langle v^*, v\rangle. \tag{5.3}$$

Thus $v^* \in \alpha(v)$, as ψ represents α. The implication (5.1) is thus established. □

An alternative proof of this theorem proceeds as follows. As π is continuous with respect to the topology τ,

$$\psi_n - \pi \xrightarrow{\Gamma} \psi - \pi \quad \text{sequentially w.r.t. } \tau. \tag{5.4}$$

It is known that this entails that the limit of any converging sequence of minimizers of $\psi_n - \pi$ is a minimizer of $\psi - \pi$, see e.g. p. 78 of [18]. The same then holds for null-minimizers. Thus $\psi \in \mathcal{F}(V)$ and the implication (5.1) is fulfilled.

In the above theorem, a priori any of the functions ψ_n and ψ might represent the empty set. However, in (5.1) $v_n^* \in \alpha_n(v_n)(v^* \in \alpha(v)$, resp.) clearly entails that $\alpha_n \not\equiv \emptyset(\alpha \not\equiv \emptyset$, resp.).

Theorems 4.4 and 5.1 yield the next statement.

Corollary 5.2 *Let V' be separable, and τ be any of the topologies $\widetilde{\pi}$, ws, $sw*$ and s (but not ω) of $V \times V'$. Let $\{\psi_n\}$ be a sequence in $\mathcal{F}(V)$ that fulfills (4.4). Then there exists $\psi \in \mathcal{F}(V)$ such that, up to extracting a subsequence, $\psi_n \xrightarrow{\Gamma} \psi$ topologically and sequentially with respect to τ. The property (5.1) then holds.*

Next we exhibit a condition that guarantees that a Γ-limit of representative functions is strictly convex with respect to its first argument; this will entail uniqueness of the solution v of the associated equation $\alpha(v) \ni v^*$ for a given v^*. Let us denote by L the canonic injection $V \to V'$, so that $\langle Lv, v\rangle = \|v\|_H^2$ for any $v \in V$.

Proposition 5.3 *Let (V, H, V') be a Banach triplet as in (1.8), and the canonic injection $V \to H$ be compact. Let $\{\alpha_n\}$ be a sequence in $\mathcal{M}(V)$ such that*

$$\begin{aligned} &\exists c > 0: \quad \forall n, \quad \forall (v_1, w_1), (v_2, w_2) \in \mathrm{graph}(\alpha_n), \\ &\langle w_1 - w_2, v_1 - v_2\rangle \geq c\|v_1 - v_2\|_H^2. \end{aligned} \tag{5.5}$$

For any n, let $\tilde{f}_n$ represent the maximal monotone operator $\alpha_n - cL$, and set

$$f_n(v, v^*) = \tilde{f}_n(v, v^* - cLv) + c\|v\|_H^2 \quad \forall (v, v^*) \in V \times V', \quad \forall n. \tag{5.6}$$

(By the extended B.E.N. principle, see Proposition 3.2, $f_n \in \mathcal{F}(V)$ and f_n represents α_n.) Let τ be any of the topologies ω, $\widetilde{\pi}$, ws, $sw*$ and s of $V \times V'$. If f_n either topologically or sequentially $\Gamma\tau$-converges to f, then $f(\cdot, v^*)$ is strictly convex for any $v^* \in V'$.

Proof Let us set $g_n(v, v^*) = \tilde{f}_n(v, v^* - cLv)$ and $q(v, v^*) = c\|v\|_H^2$, so that (5.6) also reads $f_n = g_n + q$. By the assumption of compactness, the function q is continuous on $(V \times V', \tau)$. The either topological or sequential $\Gamma\tau$-convergence of $\{f_n\}$ is then tantamount to that of $\{g_n\}$. Each function g_n is convex, since it is the composition of $\tilde{f}_n$ with the linear transformation $(v, v^*) \mapsto (v, v^* - cLv)$. The function $g = \Gamma\tau \lim_{n\to\infty} g_n$ is then convex, too. By the τ-continuity of q, we have $f = g + q$. As q is strictly convex, the thesis then follows. □

Remark (i) A stronger condition is obviously obtained if in (5.5) $\|v_1 - v_2\|_H^2$ is replaced by $\|v_1 - v_2\|_V^2$, namely if the operators α_ns are equi-strongly-monotone.

(ii) The strict convexity obviously entails the uniqueness of the null-minimizer of the function

$$V \to \mathbf{R} \cup \{+\infty\} : v \mapsto f(v, v^*) - \langle v, v^*\rangle.$$

□

5.1 Graph convergence of representable operators

Let V' be separable, $\{\alpha_n : V \to \mathcal{P}(V')\}$ be a sequence in $\mathcal{R}(V)$, and $\alpha \in \mathcal{R}(V)$; let us denote by A_n and A the respective graphs. Let τ be any of the topologies $\widetilde{\pi}$, ws, $sw*$, s (but not ω) of $V \times V'$, and define the sequential convergence in the sense of Kuratowski:

$$
\begin{aligned}
&A = K\tau \lim_{n\to\infty} A_n \ \text{(sequentially)} \quad \Leftrightarrow\\
&\text{(i) } \forall (v, v^*) \in A, \exists \text{ sequence}\{(v_n, v_n^*) \in A_n\} : (v_n, v_n^*) \underset{\tau}{\to} (v, v^*), \text{ and}\\
&\text{(ii) } (v_n, v_n^*) \in A_n \quad \forall n, \quad (v_n, v_n^*) \underset{\tau}{\to} (v, v^*) \quad \Rightarrow \quad (v, v^*) \in A.
\end{aligned}
\tag{5.7}
$$

The two latter properties respectively read

$$
A \subset K\tau \liminf_{n\to\infty} A_n, \quad K\tau \limsup_{n\to\infty} A_n \subset A \quad \text{(both sequentially).}
$$

This is equivalent to $I_{A_n} \xrightarrow{\Gamma} I_A$ sequentially with respect to the topology τ; this may be proved along the lines of p. 43 of [18].

Part (ii) of Theorem 5.1 also reads $K\tau \limsup_{n\to\infty} A_n \subset A$ (sequentially). This yields the next statement, that extends to representable operators a property of maximal monotone operators, see e.g. p. 361 of [2], and raises the question of deriving sufficient conditions for getting $A \subset K\tau \liminf_{n\to\infty} A_n$.

Corollary 5.4 *Let the hypotheses of Theorem 5.1 be fulfilled, and denote by A_n and A the respective graphs of α_n and α. If $A \subset K\tau \liminf_{n\to\infty} A_n$, namely,*

$$
\forall (v, v^*) \in A, \exists \textit{ sequence } \{(v_n, v_n^*) \in A_n\} : (v_n, v_n^*) \underset{\tau}{\to} (v, v^*), \tag{5.8}
$$

then $A = K\tau \lim_{n\to\infty} A_n$ sequentially.

5.2 Γ-Compactness of Fenchel functions

Next we deal with the properties of Γ-convergence of the class of Fenchel functions, cf. (3.1). Let us still assume that V' is separable. For any n, let $\varphi_n : V \to \mathbf{R} \cup \{+\infty\}$ be convex, lower semicontinuous and proper, and ψ_n be its Fenchel function:

$$
\psi_n(v, v^*) = \varphi_n(v) + \varphi_n^*(v^*) \quad \forall (v, v^*) \in V \times V', \quad \forall n. \tag{5.9}
$$

We shall assume the following condition of V-equi-coerciveness:

$$
\forall C \in \mathbf{R}, \ \sup_{n\in\mathbf{N}} \big\{\|v\|_V : v \in V, \varphi_n(v) \le C\big\} < +\infty. \tag{5.10}
$$

Theorems 4.4 and 5.1 yield the next statement.

Corollary 5.5 *Let V' be separable, and τ be any of the topologies $\widetilde{\pi}$, ws, $sw*$, s (but not ω) of $V \times V'$. Let $\{\varphi_n\}$ be a sequence in $\mathcal{F}(V)$ such that both $\{\varphi_n\}$ and $\{\varphi_n^*\}$ fulfill (5.10) respectively in V and V'. Then there exists ψ_τ such that, up to extracting a subsequence,*

$$
\varphi_n(v) + \varphi_n^*(v^*) \xrightarrow{\Gamma} \psi_\tau(v, v^*) \quad \textit{topologically and sequentially w.r.t. } \tau. \tag{5.11}
$$

This entails that $\psi_\tau \in \mathcal{F}(V)$.

By (4.8), defining $\psi_{\tilde{\pi}}$, ψ_{ws}, ψ_{sw*}, ψ_s in an obvious way, we have

$$
\psi_{\tilde{\pi}} \le \min\{\psi_{ws}, \psi_{sw*}\}, \quad \max\{\psi_{ws}, \psi_{sw*}\} \le \psi_s. \tag{5.12}
$$

Next we see that the Fenchel class is stable for Γws- and $\Gamma sw*$-convergence.

Corollary 5.6 *Let V be separable and reflexive, and $\{\varphi_n\}$ be a sequence of convex, lower semicontinuous and proper functions $V \to \mathbf{R} \cup \{+\infty\}$, that fulfill (5.10). Then there exists a convex, lower semicontinuous and proper function φ such that, possibly extracting a subsequence,*

$$\varphi_n + \varphi_n^* \xrightarrow{\Gamma} \varphi + \varphi^* \quad \text{topologically and sequentially w.r.t. the topology } ws. \tag{5.13}$$

A dual statement holds for the topology sw, if the sequence $\{\varphi_n^*\}$ fulfills the dual formulation of the equi-coerciveness condition (5.10).

Proof By Proposition 8.10 of p. 93 of [18], $\varphi_n \xrightarrow{\Gamma} \varphi$ topologically and sequentially with respect to the weak topology, up to extracting a subsequence. By Lemma 5.7 here below, then $\varphi_n^* \xrightarrow{\Gamma} \varphi^*$ strongly. The two latter statements clearly entail (5.13). □

The next statement follows from p. 283 of [2] and Proposition 4.3 above.

Lemma 5.7 *Let W be a separable and reflexive Banach space, and $\{\theta_n\}$ be a sequence of convex, lower semicontinuous and proper functions $W \to \mathbf{R} \cup \{+\infty\}$ that fulfills (5.10). Then*

$$\begin{aligned} &\theta_n \xrightarrow{\Gamma} \theta \ \text{ topologically and sequentially weakly in } W \\ \Leftrightarrow \quad &\theta_n^* \xrightarrow{\Gamma} \theta^* \ \text{ strongly in } W'. \end{aligned} \tag{5.14}$$

At variance with ψ_{ws} and ψ_{sw*}, the functions ψ_ω and ψ_s need not be in the Fenchel class: we shall see a counterexample ahead in this section. To this author it is not clear whether this property does or does not hold for $\psi_{\tilde{\pi}}$.

5.3 Mosco-convergence

Let us us assume that V is reflexive, and denote by $\psi_n \xrightarrow{M} \psi$ the convergence in the sense of Mosco of a sequence $\{\psi_n\}$ of functions $V \times V' \to \mathbf{R} \cup \{+\infty\}$—that is, the Γ-convergence of ψ_n to ψ with respect to both the sequential-weak and the strong topology of $V \times V'$, see e.g. [2].

Proposition 5.8 *Let V be separable and reflexive, and $\{\varphi_n\}$ be a sequence of functions $V \times V' \to \mathbf{R} \cup \{+\infty\}$. If*

$$\psi_n \xrightarrow{M} \psi \quad \text{in } V \times V', \tag{5.15}$$

then

$$\psi_n \xrightarrow{\Gamma} \psi \quad \text{sequentially w.r.t. each of the topologies } \tilde{\pi}, ws, sw. \tag{5.16}$$

If $\psi_n \in \mathcal{F}(V)$ for any n, then $\psi \in \mathcal{F}(V)$.

Proof As the topologies $\tilde{\pi}, ws, sw$ are intermediate between the weak and the strong topology of $V \times V'$, by Lemma 4.2 we infer that $\psi_\omega \le \psi_\tau \le \psi_s$ for any $\tau \in \{\tilde{\pi}, ws, sw\}$. By (5.15), $\psi_\omega = \psi_s$, and (5.16) follows. The final statement stems from Theorem 5.1. □

The Mosco-convergence is especially interesting for Fenchel functions. Let us consider a sequence $\{\varphi_n\}$ of convex and lower semicontinuous functions $V \to \mathbf{R} \cup \{+\infty\}$. If V is reflexive, then after p. 295 of [2]

$$\varphi_n \xrightarrow{M} \varphi \quad \text{in } V \quad \Leftrightarrow \quad \varphi_n^* \xrightarrow{M} \varphi^* \quad \text{in } V'. \tag{5.17}$$

It follows that $\varphi_n \xrightarrow{M} \varphi$ if and only if $\varphi_n + \varphi_n^* \xrightarrow{M} \varphi + \varphi^*$; this is easily checked, as the two addenda act on different variables. For convex and lower semicontinuous functions, still assuming that the space V is reflexive, we thus have

$$\begin{aligned}
&\varphi_n \xrightarrow{M} \varphi \quad \text{in } V \quad \Leftrightarrow\\
&\varphi_n + \varphi_n^* \xrightarrow{M} \varphi + \varphi^* \quad \text{in } V \times V' \quad \Rightarrow \\
&\varphi_n + \varphi_n^* \xrightarrow{\Gamma} \varphi + \varphi^* \quad \text{in } V \times V', \text{ sequentially w.r.t. } \widetilde{\pi}, ws, sw.
\end{aligned} \tag{5.18}$$

On the other hand, by p. 283 of [2], if $\varphi_n + \varphi_n^*$ sequentially $\Gamma\omega$-converges to $\varphi + \varphi^*$ then it also Γ-converges strongly; thus

$$\begin{aligned}
&\varphi_n + \varphi_n^* \xrightarrow{\Gamma} \varphi + \varphi^* \quad \text{in } V \times V', \text{ sequentially w.r.t. } \omega \quad \Rightarrow \\
&\varphi_n + \varphi_n^* \xrightarrow{M} \varphi + \varphi^* \quad \text{in } V \times V'.
\end{aligned} \tag{5.19}$$

Remark The notion of Mosco-convergence is especially appropriate for the structural stability of gradient flows in Hilbert spaces, and entails the convergence in the sense of Kuratowski of the subdifferentials p. 373 of [2]. It is a rather strong property, that much simplifies the passage to the limit in the analysis of several nonlinear P.D.E.s; see e.g. [37]. This notion however misses the compactness property that we pointed out for instance for the $\Gamma\widetilde{\pi}$-convergence, see Theorem 4.4. For this reason in this work we rather use the latter notion.

Dealing with maximal monotone operators, a similar point may be done for the graph convergence, i.e., the convergence of the graphs in the sense of Kuratowski. This also misses the compactness property, that is at the focus of this work. □

5.4 An example

Next we briefly illustrate a simple example which displays some of the above features, and provides some counterexamples. Let $\{\bar{v}_n\}$ be a weakly vanishing sequence of unit elements of a Hilbert space H that we identify with its dual, and set

$$f_n(v) := \tfrac{1}{2}\|v - \bar{v}_n\|^2 = \tfrac{1}{2}\|v\|^2 - (v, \bar{v}_n) + \tfrac{1}{2} \quad \forall v \in H, \quad \forall n. \tag{5.20}$$

This represents the cyclical and maximal monotone operator $\partial f_n : v \mapsto v + \bar{v}_n$. We have

$$f_n^*(v^*) = \tfrac{1}{2}\|v^*\|^2 + (v^*, \bar{v}_n) = \tfrac{1}{2}\|v^* + \bar{v}_n\|^2 - \tfrac{1}{2} \quad \forall v^* \in H, \forall n. \tag{5.21}$$

Both sequences $\{f_n\}$ and $\{f_n^*\}$ fulfill (5.10), and

$$f_n(v) \xrightarrow{\Gamma} \tfrac{1}{2}\|v\|^2 =: f(v) \quad \text{weakly in } H, \tag{5.22}$$

$$f_n(v) \xrightarrow{\Gamma} \tfrac{1}{2}\|v\|^2 + \tfrac{1}{2} =: g(v) \quad \text{strongly in } H, \tag{5.23}$$

$$f_n^*(v^*) \xrightarrow{\Gamma} \tfrac{1}{2}\|v^*\|^2 - \tfrac{1}{2} = g^*(v^*) \quad \text{weakly in } H, \tag{5.24}$$

$$f_n^*(v^*) \xrightarrow{\Gamma} \tfrac{1}{2}\|v^*\|^2 = f^*(v^*) \quad \text{strongly in } H; \tag{5.25}$$

hence

$$f_n + f_n^* \xrightarrow{\Gamma} f + g^* \quad \text{weakly in } H^2, \tag{5.26}$$

$$f_n + f_n^* \xrightarrow{\Gamma} f + f^* \quad \text{weakly-strongly in } H^2, \tag{5.27}$$

$$f_n + f_n^* \xrightarrow{\Gamma} g + g^* \quad \text{strongly-weakly in } H^2, \tag{5.28}$$

$$f_n + f_n^* \xrightarrow{\Gamma} g + f^* \quad \text{strongly in } H^2, \tag{5.29}$$

and $f + f^* = g + g^*$. All of these Γ-convergences are both topological and sequential, by Proposition 4.3.

No one of the sequences $\{f_n\}$, $\{f_n^*\}$ and $\{f_n + f_n^*\}$ converges in the sense of Mosco, nor from any of these three sequences a Mosco-convergent subsequence may be extracted. This confirms the restrictiveness of the Mosco-convergence and of the Mosco-compactness, even for sequences of convex and lower semicontinuous functions.

The function $f + g^*$ is not representative, as $f(0) + g^*(0) < \langle 0, 0 \rangle$. The Fenchel function $f + f^* = g + g^*$ represents the identity operator. The function $g + f^* = (f + g^*)^*$ is representative, but it represents the empty operator, as $f(v) + g^*(v^*) > \langle v, v^* \rangle$ for any $(v, v^*) \in H^2$; thus this is no Fenchel function. These statements are all consistent with part (i) of Theorem 5.1.

By Theorem 4.4, there exists a function $\psi : H^2 \to \mathbf{R} \cup \{+\infty\}$ such that

$$\psi_n := f_n + f_n^* \xrightarrow{\Gamma} \psi \quad \text{topologically and sequentially w. r. t. the topology } \widetilde{\pi}, \tag{5.30}$$

and by part (i) of Theorem 5.1 ψ is representative. One may even be more specific, and show that

$$\psi = f + f^* \ (= g + g^*). \tag{5.31}$$

In order to prove this equality,[5] let us select any $(v, v^*) \in H^2$, and any sequence $\{(\xi_n, \xi_n^*)\}$ in H^2 such that $(v + \xi_n, v^* + \xi_n^*) \underset{\widetilde{\pi}}{\to} (v, v^*)$; that is, $\xi_n \rightharpoonup 0$ in H, $\xi_n^* \overset{*}{\rightharpoonup} 0$ in H, and $(\xi_n^*, \xi_n) \to 0$. Note that

$$\begin{aligned} &f_n(v + \xi_n) + f_n^*(v_n^* + \xi_n^*) - f(v) - f^*(v^*) \\ &= (v, \xi_n) + \tfrac{1}{2}\|\xi_n\|^2 - (\xi_n, \bar{v}_n) + (v^*, \xi_n^*) + \tfrac{1}{2} + \tfrac{1}{2}\|\xi_n^*\|^2 + (\xi_n^*, \bar{v}_n) + o(1) \\ &= \tfrac{1}{2}\|\xi_n - \xi_n^*\|^2 - (\xi_n^*, \xi_n) - (\xi_n - \xi_n^*, \bar{v}_n) + \tfrac{1}{2} + o(1) \\ &= \tfrac{1}{2}\|\xi_n - \xi_n^* - \bar{v}_n\|^2 + o(1) \ge o(1); \end{aligned} \tag{5.32}$$

therefore $\liminf_{n\to\infty}[f_n(v) + f_n^*(v^*)] \ge f(v) + f^*(v^*)$. This corresponds to (4.5) for the topology $\tau = \widetilde{\pi}$; by (5.22) and (5.25), the condition (4.6) holds, too. (5.31) is thus established.

5.5 Other examples

Next we display a sequence that converges in $V \times V'$ with respect to the topology $\widetilde{\pi}$, but neither with respect to ws nor $sw*$. If V is an infinite-dimensional Hilbert space and is identified with its dual, it suffices to fix an orthonormal sequence $\{e_n\}$ of unit elements, and to consider the sequence $\{(e_n, e_{n+1})\}$ in $V^2 = V \times V'$. Denoting by τ_n the translation operator $v \mapsto v(\cdot - n)$, for any $v \in H^1(\mathbf{R})$ and any $n \in \mathbf{N}$, the sequence $\{(\tau_n v, D_t \tau_n v)\}$ is another example.

6 Representation in spaces of time-dependent functions

In this section we extend some of the previous developments to time-dependent functions, in view of the analysis of monotone flows in the next section.

Let us fix any finite $T > 0$, any $p \in]1, +\infty[$ and set $\mathcal{V} := L^p(0, T; V)$. Let us define the topology $\widetilde{\pi}$ in $\mathcal{V} \times \mathcal{V}'$ as in (4.1), by replacing the space V by $\mathcal{V}$ and the associated duality

[5] The following argument was pointed out by the anonymous reviewer.

pairing $\langle\cdot,\cdot\rangle$ by

$$\langle\langle v^*, v\rangle\rangle := \int_0^T \langle v^*(t), v(t)\rangle\, dt \quad \forall (v, v^*) \in \mathcal{V}\times\mathcal{V}'.$$

Definitions and results of Sects. 4 and 5 take over to time-dependent operators and to their time-integrated representative functions, simply by replacing the space V by $\mathcal{V}$.

Proposition 6.1 *Let a function $\psi \in \mathcal{F}(V)$ be such that*

$$\forall C \in \mathbf{R},\ \sup\left\{\|v\|_V + \|v^*\|_{V'} : (v, v^*) \in V \times V', \psi(v, v^*) \le C\right\} < +\infty. \tag{6.1}$$

Then the functional

$$\Psi(v, v^*) := \int_0^T \psi(v(t), v^*(t))\, dt \quad \forall (v, v^*) \in \mathcal{V}\times\mathcal{V}' \tag{6.2}$$

is an element of $\mathcal{F}(\mathcal{V})$. Moreover, ψ represents an operator $\alpha : V \to \mathcal{P}(V')$ if and only if Ψ represents the operator

$$\widehat{\alpha} : \mathcal{V} \to \mathcal{P}(\mathcal{V}') : v \mapsto \alpha(v(\cdot)). \tag{6.3}$$

(The same clearly applies for the double time integration: $(v, v^*) \mapsto \int_0^T (T-t)\psi(v(t), v^*(t))\, dt$.)

Proof We just check the "if" part of the last assertion, the remainder being pretty obvious. For any $(v, v^*) \in \mathcal{V}\times\mathcal{V}'$,

$$\int_0^T \psi(v(t), v^*(t))\, dt \ge \langle\langle v^*, v\rangle\rangle, \ \ \psi(v(t), v^*(t)) \ge \langle v^*(t), v(t)\rangle \quad \text{for a.e. } t \in]0, T[.$$

Whenever the first inequality is reduced to an equality, the same then applies to the second one. The function Ψ then represents the operator $\widehat{\alpha}$ only if ψ represents α. □

We shall relate the $\widetilde{\pi}$-convergence in $V \times V'$ a.e. in $]0, T[$ with the $\widetilde{\pi}$-convergence in $\mathcal{V}\times\mathcal{V}'$. First we state an auxiliary result.

Lemma 6.2 *Let $\varepsilon \in]0, 1[$ and $p \in]1, +\infty[$. For any $\sigma > \varepsilon$, there exists a constant $C_{\sigma,\varepsilon} > 0$ such that for any $(v, v^*) \in W^{\sigma,p}(0, T; V) \times W^{\sigma,p'}(0, T; V')$,*

$$\|\langle v^*, v\rangle\|_{W^{\varepsilon,1}(0,T)} \le C_{\sigma,\varepsilon}\|v^*\|_{W^{\sigma,p'}(0,T;V')}\|v\|_{W^{\sigma,p}(0,T;V)}. \tag{6.4}$$

Proof Setting $f := \langle v^*, v\rangle$ a.e. in $]0, T[$, we have

$$\|f\|_{W^{\varepsilon,1}(0,T)} = \int_0^T |f(t)|\, dt + \iint_{]0,T[^2} \frac{|f(t) - f(\tau)|}{|t-\tau|^{1+\varepsilon}}\, dt d\tau, \tag{6.5}$$

$$\begin{aligned} |f(t) - f(\tau)| &\le |\langle v^*(t) - v^*(\tau), v(t)\rangle| + |\langle v^*(\tau), v(t) - v(\tau)\rangle| \\ &\le \|v^*(t) - v^*(\tau)\|_{V'}\|v(t)\|_V + \|v^*(\tau)\|_{V'}\|v(t) - v(\tau)\|_V \\ &=: I_1(t,\tau) + I_2(t,\tau) \text{ for a.e. } t, \tau \in]0, T[. \end{aligned}$$

For any $\sigma > \varepsilon$, let us set $p' := p/(p-1)$ and $q' := 1/(\sigma - \varepsilon + 1/p')$, so that $\sigma = \varepsilon + 1/q' - 1/p'$ and $q := q'/(q'-1) > p$. We then have

$$\begin{aligned}
&\iint\limits_{]0,T[^2} \frac{I_1(t,\tau)}{|t-\tau|^{1+\varepsilon}}\,dt d\tau = \iint\limits_{]0,T[^2} \frac{\|v^*(t)-v^*(\tau)\|_{V'}}{|t-\tau|^{1/q'+\varepsilon}} \cdot \frac{\|v(t)\|_V}{|t-\tau|^{1/q}}\,dt d\tau \\
&\le \left\| \frac{\|v^*(t)-v^*(\tau)\|_{V'}}{|t-\tau|^{1/q'+\varepsilon}} \right\|_{L^{p'}(]0,T[^2)} \left\| \frac{\|v(t)\|_V}{|t-\tau|^{1/q}} \right\|_{L^p(]0,T[^2)} \\
&= \left\| \frac{\|v^*(t)-v^*(\tau)\|_{V'}}{|t-\tau|^{1/p'+\sigma}} \right\|_{L^{p'}(]0,T[^2)} \left(\int_0^T \|v(t)\|_V^p \left(\int_0^T \frac{1}{|t-\tau|^{p/q}}\,d\tau \right) dt \right)^{1/p} \\
&\le C_{1,\sigma,\varepsilon} \|v^*\|_{W^{\sigma,p'}(0,T;V')} \|v\|_{L^p(0,T;V)},
\end{aligned} \tag{6.6}$$

with $C_{1,\sigma,\varepsilon} := \sup_{t\in]0,T[} \left(\int_0^T \frac{1}{|t-\tau|^{p/q}}\,d\tau\right)^{1/p} < +\infty$. Similarly, exchanging the roles of v, V, p with that of v^*, V', p' (resp.), it is easily seen that there exists a finite constant $C_{2,\sigma,\varepsilon}$ such that

$$\iint\limits_{]0,T[^2} \frac{I_2(t,\tau)}{|t-\tau|^{1+\varepsilon}}\,dt d\tau \le C_{2,\sigma,\varepsilon} \|v^*\|_{L^{p'}(0,T;V')} \|v\|_{W^{\sigma,p}(0,T;V)}. \tag{6.7}$$

Finally, (6.5)–(6.7) yield (6.4). □

Proposition 6.3 *Let $p \in]1, +\infty[$, and $\{(v_n, v_n^*)\}$ be a bounded sequence in $W^{\sigma,p}(0,T;V) \times W^{\sigma,p'}(0,T;V')$ for some $\sigma > 0$ and a finite $T > 0$. If*

$$(v_n, v_n^*) \underset{\pi}{\rightrightarrows} (v, v^*) \quad \text{in } V \times V', \text{ a.e. in}]0,T[, \tag{6.8}$$

then

$$(v_n, v_n^*) \underset{\pi}{\rightrightarrows} (v, v^*) \quad \text{in } \mathcal{V} \times \mathcal{V}'. \tag{6.9}$$

On the other hand (6.9) does not entail (6.8), not even for a subsequence.

Proof (i) First we prove the implication "(6.8) ⇒ (6.9)". Let us denote by $[\cdot,\cdot]$ the canonic duality pairing between $V \times V'$ and $V' \times V$. For any $(\xi^*, \xi) \in V' \times V$, by (6.8)

$$[(v_n, v_n^*), (\xi^*, \xi)] \to [(v, v^*), (\xi^*, \xi)] \quad \text{a.e. in }]0,T[.$$

By the assumption of boundedness, the sequence $\{[(v_n, v_n^*), (\xi^*, \xi)]\} = \{\langle v_n, \xi^* \rangle + \langle v_n^*, \xi \rangle\}$ is bounded in $W^{\sigma,1}(0,T)$, and this space has compact injection into $L^1(0,T)$ (as T is finite). Thus

$$[(v_n, v_n^*), (\xi^*, \xi)] \to [(v, v^*), (\xi^*, \xi)] \quad \text{strongly in } L^1(0,T).$$

Hence $(v_n, v_n^*) \to (v, v^*)$ weakly in $\mathcal{V} \times \mathcal{V}'$. By (6.8), $\langle v_n^*, v_n \rangle \to \langle v^*, v \rangle$ a.e. in $]0,T[$. Moreover, by Lemma 6.2 the sequence $\{\langle v_n^*, v_n \rangle\}$ is compact in $L^1(0,T)$. Therefore

$$\int_0^T \langle v_n^*, v_n \rangle\,dt \to \int_0^T \langle v^*, v \rangle\,dt, \qquad \text{that is,} \quad \langle\langle v_n^*, v_n \rangle\rangle \to \langle\langle v^*, v \rangle\rangle.$$

This yields (6.9).

(ii) Next we show by a counterexample that "(6.8) $\not\Rightarrow$ (6.9)" for $V = H$ and $p = 2$. Let $\{h_n\}$ be an orthonormal basis of H, and set

$$v_n(t) = h_n \sin(2\pi t/T), \quad v_n^*(t) = h_n \sin(4\pi t/T) \quad \forall t \in]0, T[, \quad \forall n. \tag{6.10}$$

Thus

$$(v_n, v_n^*) \underset{\pi}{\Rightarrow} (v, v^*) \quad \text{in } L^2(0, T; H)^2 = \mathcal{V} \times \mathcal{V}',$$

although

$$(v_n(t), v_n^*(t)) \underset{\pi}{\not\Rightarrow} (0, 0) \quad \text{in } H^2 = V \times V', \forall t \in]0, T[.$$

□

Remark If $\{(v_n, v_n^*)\}$ is just bounded in $\mathcal{V} \times \mathcal{V}'$, then the implication "(6.9) $\Rightarrow$ (6.8)" fails. Here is a simple counterexample for $V = \mathbf{R}$. Let us set

$$v_n^* = v_n = 0 \quad \text{in }]0, T - \tfrac{1}{n}], \quad v_n^* = v_n = \sqrt{n} \quad \text{in }]T - \tfrac{1}{n}, T[, \forall n. \tag{6.11}$$

Hence $(v_n, v_n^*) \to (0, 0)$ in $\mathbf{R}^2$ and $v_n^* \cdot v_n \to 0$ for any $t \in]0, T[$; but $\int_0^T v_n^*(t) \cdot v_n(t)\, dt = 1$ for any n. Thus $(v_n, v_n^*) \underset{\pi}{\Rightarrow} (0, 0)$ in $\mathbf{R}^2$ a.e. in $]0, T[$, but not in $L^2(0, T; \mathbf{R}^2)$, not even for a subsequence. □

Despite of the counterexample (6.10), next we derive (6.8) from (6.9) for $\{v_n^*\} = \{D_t v_n\}$, under the assumption that the canonic injection $V \to H$ is compact.

Lemma 6.4 *Let (V, H, V') be a Banach triplet as in (1.8), with compact injection $V \to H$; let $p \in]1, +\infty[$ and $T > 0$ be finite. If, for some $\sigma > 0$,*

$$v_n \rightharpoonup v \quad \text{in } W^{\sigma,p}(0, T; V) \cap W^{1+\sigma,p'}(0, T; V'), \tag{6.12}$$

then

$$\langle D_t v_n, v_n \rangle \to \langle D_t v, v \rangle \quad \text{in } L^1(0, T). \tag{6.13}$$

Proof By the compactness hypothesis and by the classical Rellich compactness theorem, $v_n \to v$ in $W^{\sigma/2,p}(0, T; H)$. Hence, possibly extracting a further subsequence, $\|v_n\|_H^2 \to \|v\|_H^2$ in $L^1(0, T)$. (For $\sigma = 0$ this would fail.) Therefore

$$\langle D_t v_n, v_n \rangle = \tfrac{1}{2} D_t \|v_n(t)\|_H^2 \to \tfrac{1}{2} D_t \|v(t)\|_H^2 = \langle D_t v, v \rangle \quad \text{in } \mathcal{D}'(0, T). \tag{6.14}$$

On the other hand, by Lemma 6.2 the sequence $\{\langle D_t v_n, v_n \rangle\}$ is bounded in $W^{\sigma,1}(0, T)$; (6.13) then follows. □

7 Compactness and structural stability of periodic monotone flows

In the next two sections we use the results of Sects. 4, 5, 6 to study the compactness and the structural stability of the variational formulation of flows of the form $D_t u + \alpha(u) \ni h$, with α maximal monotone. In this section we deal with periodic flow, and in the next one with the initial-value problem.

7.1 Operator compactness and structural stability

Let us first illustrate a fairly general framework. For a given problem, let us denote

by $\mathcal{D}$ the set of the admissible data (e.g., the source term of a P.D.E.),
by $\mathcal{O}$ the set of the admissible (either linear or nonlinear) operator(s),
by $\mathcal{S}$ the set of the admissible solutions of the problem.

Let us assume that each of these sets is equipped with a topology and that a (possibly multi-valued) *resolution operator* $R : \mathcal{D}\times\mathcal{O} \to \mathcal{S}$ is defined.

We shall say that the class of admissible operators is (sequentially) compact if

$$\text{any sequence } \{o_n\} \text{ in } \mathcal{O} \text{ accumulates at some } o \in \mathcal{O}, \tag{7.1}$$

and that the problem is *structurally stable* if the resolution operator R is (sequentially) closed, namely, for any sequence $\{(d_n, o_n, s_n)\}$ in $\mathcal{D}\times\mathcal{O}\times\mathcal{S}$,

$$s_n \in R(d_n, o_n)\ \forall n, \quad (d_n, o_n, s_n) \to (d, o, s) \quad \Rightarrow \quad s \in R(d, o). \tag{7.2}$$

Certainly, it would also be desirable that

$$\text{any element } s \in R(\mathcal{D}, \mathcal{O}) \text{ may be retrieved as in (7.2).} \tag{7.3}$$

If so, no spurious solution might occur by passage to the limit, so that the set of the limits of solutions would coincide with that of the solutions of the asymptotic problem. If (7.2) holds, then the property (7.3) is trivially fulfilled whenever the limit problem has only one solution; otherwise it looks harder to be proved.

7.2 Continuous dependence of solution

The property (7.2) concerns the stability of the solutions with respect to perturbations. This extends the notion of well-posedness in the sense of Hadamard, by including variations of the operators. For minimization problems, this similarly extends Tychonov's generalized notion of well-posedness, see e.g. [21].

The structural stability may be compared with the property of (sequential) continuous dependence of the solution on operators and data. By this we mean that the resolution operator R is single-valued, and that for any sequence $\{(d_n, o_n)\}$ in $\mathcal{D}\times\mathcal{O}$,

$$o_n \to o,\ d_n \to d \quad \Rightarrow \quad s_n := R(d_n, o_n) \to R(d, o) =: s. \tag{7.4}$$

If R is single-valued and maps bounded sets to (sequentially) compact sets, then it is clear that the structural stability (7.2) entails the continuous dependence (7.4). The structural stability somehow surrogates this continuous dependence when the uniqueness of the solution fails.

The structural stability and (in case of uniqueness of the solution) the continuous dependence on operators and data look as natural requirements for the applicative soundness of a model. The finite-dimensional approximability of infinite-dimensional operators looks also related to these properties: this latter issue is of obvious relevance e.g. for numerical analysis.

7.3 Representable operators

The above program of compactness and structural stability may be applied as follows to the class of representable operators acting on a reflexive and separable Banach space V. In this case, the convergence of the operators may be replaced by the Γ-convergence of the respective representative functions.

For any n, let an operator $\alpha_n \in \mathcal{R}(V)$ be represented by a function $\varphi_n \in \mathcal{F}(V)$. If this sequence of functions is equi-coercive in the sense of (4.4), and if τ is any of the topologies $\widetilde{\pi}$, ws, $sw*$, s, then by the compactness Theorem 4.4 φ_n topologically and sequentially $\Gamma\tau$-converges to some φ, up to extracting a subsequence. By Theorem 5.1, φ then represents an operator $\alpha : V \to \mathcal{P}(V')$. This provides (7.1), i.e. the compactness of the class of operators. Whenever $v_n^* \in \alpha_n(v_n)$ for any n and $(v_n, v_n^*) \underset{\tau}{\to} (v, v^*)$, by (5.1) we then conclude that $v^* \in \alpha(v)$; (7.2) is thus established. If we regard v^* as the datum and v as the solution, this represents the structural stability of the problem.

In the above scheme the selection of the topology is of course crucial. If the data converge strongly, i.e. $v_n^* \to v^*$ in V', then the sequence $\{v_n^*\}$ is bounded. If the inverse operators α_n^{-1} are equibounded, then the sequence $\{v_n\}$ is bounded, too; up to extracting a subsequence, it then weakly converges to some v in V. Thus $(v_n, v_n^*) \to (v, v^*)$ in the ws-topology of $V \times V'$. Next we illustrate some examples associated with evolutionary equations, in which instead the topology $\widetilde{\pi}$ arises naturally.

7.4 Abstract quasilinear parabolic operators

Let V and H be Hilbert spaces, with V separable and

$$V \subset H = H' \subset V' \text{ with continuous and dense injections.} \tag{7.5}$$

Let us assume that we are given a sequence $\{\alpha_n\}$ of operators and one $\{h_n\}$ of functions, such that

$$\forall n,\ \alpha_n : V \to \mathcal{P}(V') \text{ is maximal monotone,} \tag{7.6}$$

$$\exists a, b > 0 : \forall n, \forall (v, v^*) \in \operatorname{graph}(\alpha_n), \quad \langle v^*, v\rangle \ge a\|v\|_V^2 - b, \tag{7.7}$$

$$\exists C_1, C_2 > 0 : \forall n, \forall (v, v^*) \in \operatorname{graph}(\alpha_n), \quad \|v^*\|_{V'} \le C_1\|v\|_V + C_2, \tag{7.8}$$

$$\forall n,\ h_n \in L^2(0, T; V'). \tag{7.9}$$

For instance, let Ω be a Lipschitz domain of $\mathbf{R}^N$ ($N \ge 1$). If $\{\vec{\gamma}_n\}$ is a sequence of maximal monotone mappings $\mathbf{R}^N \to \mathcal{P}(\mathbf{R}^N)$, one may take

$$V = H_0^1(\Omega), \quad H = L^2(\Omega), \quad \alpha_n(v) = -\nabla\cdot\vec{\gamma}_n(\nabla v) \quad \text{in } \mathcal{D}'(\Omega). \tag{7.10}$$

If $N = 3$, as in the Example 3.3 of Sect. 3, one may also deal with

$$\begin{aligned} &V = \left\{\vec{v} \in L^2(\Omega)^3 : \nabla\times\vec{v} \in L^2(\Omega)^3, \quad \vec{\nu}\times\vec{v} = \vec{0} \text{ in } H^{-1/2}(\partial\Omega)^3\right\},\\ &H = L^2(\Omega)^3, \quad \overrightarrow{\alpha}_n(\vec{v}) = \nabla\times\vec{\gamma}_n(\nabla\times\vec{v}) \quad \text{in } \mathcal{D}'(\Omega)^3, \forall\vec{v} \in V; \end{aligned} \tag{7.11}$$

here by $\vec{\nu}$ we denote the outward-oriented unit normal vector-field on $\partial\Omega$. If Ω is also bounded, then in (7.10) the inclusion $V \subset H$ is compact, at variance with (7.11). However, in the next section we shall see that a different selection of the pivot space provides the compactness.

We shall consider the structural stability of a time-periodic problem, since in this case we can prove a result of compactness and structural stability without assuming compactness of the injection $V \to H$. On the other hand, in the next section addressing the initial-value problem we shall be able to achieve analogous results only assuming the compactness of that injection.

Let $T \in]0, +\infty]$ ($T = +\infty$ included), and set

$$\begin{aligned} &X := L^2(0, T; V) \cap H^1(0, T; V'),\\ &H_\sharp^1(0, T; V') := \left\{v \in H^1(0, T; V') : v(0) = v(T)\right\},\\ &X_\sharp := L^2(0, T; V) \cap H_\sharp^1(0, T; V') \end{aligned} \tag{7.12}$$

(here we set $v(+\infty) := \lim_{t\to+\infty} v(t)$). Next we shall deal with the flow

$$u_n \in X_\sharp, \quad D_t u_n + \alpha_n(u_n) \ni h_n \quad \text{in } V', \text{a.e. in }]0, T[\ (n \in \mathbf{N}). \tag{7.13}$$

Note that, after (3.17),

$$X_\sharp = \{v \in X : v(0) = 0\} \quad \text{if } T = +\infty. \tag{7.14}$$

In this way the initial-value problem with vanishing Cauchy datum may be regarded as a periodic problem with infinite period. The condition $u_n(0) = 0$ is not really restrictive, since it may be retrieved by shifting the unknown function u. More specifically, in order to deal with the initial condition $u_n(0) = u_n^0$ (a prescribed element of V), it suffices to replace u_n by $\tilde{u}_n := u_n - u_n^0$, and α_n by $\tilde{\alpha}_n := \alpha_n(\cdot + u_n^0)$. This preserves the properties (7.6)–(7.8). Next we review an existence and boundedness result.

Lemma 7.1 *Let (7.5)–(7.9) be fulfilled and* $0 < T \le +\infty$. *Then:*

(i) *For any n, problem* (7.13) *has a solution. This is unique if either* $T = +\infty$ *or* α_n *is strictly monotone. If moreover*

$$\sup_{n\in\mathbf{N}} \|h_n\|_{L^2(0,T;V')} < +\infty, \tag{7.15}$$

then the sequence $\{u_n\}$ *is bounded in* $X_\sharp$.

(ii) *For any n, if* $h_n \in H^1_\sharp(0, T; V')$ *and*

$$\exists\tilde{a} > 0 : \forall n, \forall(v_1, v_1^*), (v_2, v_2^*) \in \text{graph}(\alpha_n), \\ \langle v_1^* - v_2^*, v_1 - v_2\rangle \ge \tilde{a}\|v_1 - v_2\|_V^2 , \tag{7.16}$$

then $u_n \in H^1_\sharp(0, T; V)$. *If moreover*

$$\sup_{n\in\mathbf{N}} \|h_n\|_{H^1_\sharp(0,T;V')} < +\infty, \tag{7.17}$$

then $\{u_n\}$ *is bounded in* $H^1_\sharp(0, T; V)$.

(iii) *If moreover the sequence* $\{\alpha_n : V \to V'\}$ *is equi-Lipschitz-continuous, i.e.,*

$$\exists L > 0 : \forall n, \forall v_1, v_2 \in V, \quad \|\alpha_n(v_1) - \alpha_n(v_2)\|_{V'} \le L\|v_1 - v_2\|_V, \tag{7.18}$$

then the sequence $\{u_n\}$ *is also bounded in* $H^2(0, T; V')$.

7.5 Outline of the proof

Part (i) stems from the classical theory, see e.g. [9]. Part (ii) may also be proved by applying the time-incremental-ratio operator δ_h to the equation (7.13), and then multiplying it by $\delta_h u_n$. This yields a uniform estimate for u_n in $W^{1,\infty}(0, T; H) \cap H^1(0, T; V)$. By (7.18) we have

$$\|\delta_h\alpha_n(u_n)\|_{L^2(0,T;V')} \le L\|\delta_h u_n\|_{L^2(0,T;V)} \quad \forall n \in \mathbf{N}.$$

By comparing the terms of the equation (7.13), a uniform estimate for u_n in $H^2(0, T; V')$ then follows. □

Remark (i) Parts (ii) and (iii) of the above lemma take over to fractional derivatives. More specifically, for any $s \in]0, 1[$, if (7.16) is fulfilled and the sequence $\{h_n\}$ is bounded in $H^s_\sharp(0, T; V')$, then $\{u_n\}$ is bounded in $H^s_\sharp(0, T; V)$. If (7.18) is also fulfilled, then $\{u_n\}$ is bounded in $H^{1+s}(0, T; V')$. The results of this section may easily be extended to these fractional spaces, too.

(ii) The assumptions of this lemma are consistent with a number of relevant problems. But (7.16) excludes for instance the weak formulation of the classical Stefan problem, see the Example 3.8 of Sect. 3. □

7.6 Variational formulations

Next we provide two different variational formulations of the inclusion (7.13). Let us first set $\mathcal{V} := L^2(0, T; V)$ and define the duality pairings $\langle\cdot,\cdot\rangle$ and $\langle\langle\cdot,\cdot\rangle\rangle$ as in Sect. 6. For any n, let us denote by $\widehat{\alpha}_n$ the operator $\mathcal{V} \to \mathcal{P}(\mathcal{V}')$ that is canonically associated with the mapping α_n as in (6.3). The pointwise-in-time problem (7.13) clearly entails the global-in-time formulation

$$u_n \in X_\sharp, \quad D_t u_n + \widehat{\alpha}_n(u_n) \ni h_n \quad \text{in } \mathcal{V}'. \tag{7.19}$$

Three representable operators may be singled out from these two equation:

(i) the pointwise-in-time operator $\alpha_n : V \mapsto \mathcal{P}(V')$, see (7.13);
(ii) the global-in-time operator $\widehat{\alpha}_n : \mathcal{V} \mapsto \mathcal{P}(\mathcal{V}')$, see (7.19);
(iii) the global-in-time operator $D_t + \widehat{\alpha}_n : X_\sharp \mapsto \mathcal{P}(X_\sharp')$, see (7.19).

A variational representation is indeed associated with each of these operators; for the third one, this stems from the extended B.E.N. principle, see Sect. 3. Here we shall just deal with the first two formulations. Difficulties instead seem to arise in addressing (iii), since in this case it is not clear how the equi-coerciveness property (4.4) might be established.

Let the sequence $\{\alpha_n\}$ fulfill the conditions (7.5)–(7.8). This entails that

$$\langle v^*, v\rangle \geq a\|v\|_V^2 - b \geq aC_1^{-2}(\|v^*\|_{V'} - C_2)^2 - b \quad \forall(v, v^*) \in V\times V', v^* \in \alpha_n(v), \quad \forall n.$$

The sequence $\{\pi + I_{\alpha_n}\}$ is then $V\times V'$-equi-coercive, in the sense of (6.1). The same then applies to the sequence of the Svaiter functions $\{\psi_n := (\pi + I_{\alpha_n})^{**}\}$; thus

$$\forall C \in \mathbf{R}, \ \sup_{n\in\mathbf{N}} \left\{\|v\|_V + \|v^*\|_{V'} : (v, v^*) \in V\times V', \ \psi_n(v, v^*) \leq C\right\} < +\infty. \tag{7.20}$$

By Proposition 6.1, each $\widehat{\alpha}_n$ is then represented by the time-integrated functional $\Psi_n \in \mathcal{F}(\mathcal{V})$:

$$\Psi_n(v, v^*) := \int_0^T \psi_n(v(t), v^*(t))\, dt \quad \forall(v, v^*) \in \mathcal{V}\times\mathcal{V}', \quad \forall n, \tag{7.21}$$

and this is $\mathcal{V}\times\mathcal{V}'$-equi-coercive, in the sense that

$$\forall C \in \mathbf{R}, \ \sup_{n\in\mathbf{N}} \left\{\|v\|_{\mathcal{V}} + \|v^*\|_{\mathcal{V}'} : (v, v^*) \in \mathcal{V}\times\mathcal{V}', \ \Psi_n(v, v^*) \leq C\right\} < +\infty. \tag{7.22}$$

Remark The assumptions (7.5)–(7.8) do not entail the $V \times V'$-equi-coerciveness of the sequence of Fitzpatrick functions $\{f_{\alpha_n} := (\pi + I_{\alpha_n})^*\}$. A trivial counterexample is provided by the identity mapping for V, which is associated with the Svaiter function

$$s(x, y) = \begin{cases} \|x\|_V^2 & \text{if } x = y \\ +\infty & \text{if } x \neq y \end{cases} \quad \forall(x, y) \in V^2. \tag{7.23}$$

This corresponds to the Fitzpatrick function

$$f(x, y) = s^*(x, y) = \tfrac{1}{4}\|x + y\|_V^2 \quad \forall(x, y) \in V^2, \tag{7.24}$$

which indeed is not coercive. □

Because of the time-periodicity, $D_t : X_\sharp \to \mathcal{V}'$ is skew-adjoint:

$$\langle\langle D_t w, v\rangle\rangle = \int_0^T \langle D_t w, v\rangle \, dt = -\int_0^T \langle w, D_t v\rangle \, dt = -\langle\langle w, D_t v\rangle\rangle \quad \forall w, v \in X_\sharp$$

(this applies for any positive $T \le +\infty$). Thus

$$\langle\langle D_t v, v\rangle\rangle = 0 \quad \forall v \in X_\sharp, \tag{7.25}$$

and we get the next statement.

Proposition 7.2 *For any n, the problems* (7.13) *and* (7.19) *are respectively equivalent to*

$$u_n \in X_\sharp, \quad \psi_n(u_n, h_n - D_t u_n) \le \langle u_n, h_n - D_t u_n\rangle \quad \textit{a.e. in }]0, T[, \tag{7.26}$$

$$u_n \in X_\sharp, \quad \Psi_n(u_n, h_n - D_t u_n) \le \langle\langle u_n, h_n\rangle\rangle. \tag{7.27}$$

Each of these inequalities is tantamount to the corresponding equality. In particular, (7.27) *is equivalent to the null-minimization (i.e., $\Phi_n(u_n) = \inf \Phi_n = 0$) of the functional*

$$\Phi_n : X_\sharp \to \mathbf{R} \cup \{+\infty\} : v \mapsto \Psi_n(v, h_n - D_t v) - \langle\langle v, h_n\rangle\rangle. \tag{7.28}$$

7.7 (i) Global formulation

We shall deal with the dependence of the solution of problem (7.19) on the source term h_n and on the operator α_n, assuming that

$$h_n \to h \quad \text{in } \mathcal{V}'. \tag{7.29}$$

We shall express the compactness of the operators α_ns indirectly, via the compactness of a sequence of associated representative functionals.

Lemma 7.3 *(Compactness of $\{\Psi_n\}$) Let a sequence $\{\alpha_n\}$ of operators fulfill* (7.5)–(7.9) *(for any positive $T \le +\infty$). For any n, let us define the Svaiter functions $\psi_n := (\pi + I_{\alpha_n})^{**}$ ($\in \mathcal{F}(V)$) and Ψ_n ($\in \mathcal{F}(\mathcal{V})$) as in* (7.21)*. Then there exist $\psi \in \mathcal{F}(V)$ and $\Psi \in \mathcal{F}(\mathcal{V})$ such that, up to extracting subsequences,*

$$\psi_n \xrightarrow{\Gamma} \psi \quad \textit{sequentially w.r.t. the topology } \widetilde{\pi} \textit{ of } V \times V', \tag{7.30}$$

$$\Psi_n \xrightarrow{\Gamma} \Psi \quad \textit{sequentially w.r.t. the topology } \widetilde{\pi} \textit{ of } \mathcal{V} \times \mathcal{V}'. \tag{7.31}$$

Proof By Theorem 4.4, there exist ψ and Ψ that fulfill (7.30) and (7.31), up to extracting subsequences. By Theorem 5.1, $\psi \in \mathcal{F}(V)$ and $\Psi \in \mathcal{F}(\mathcal{V})$. □

Remark In spite of (7.21), the function Ψ need not be the definite integral of ψ. We shall see a counterexample in the next section. □

Theorem 7.4 *(Structural Stability of (7.19)) For any n, let* (7.5)–(7.9) *be fulfilled, and (for any positive $T \le +\infty$) u_n be a solution of problem* (7.13) *[which exists by Lemma 7.1], hence also of problem* (7.9)*. Let $\{\Psi_n\}$ and Ψ be as in Lemma 7.3, and $\widehat{\alpha} : \mathcal{V} \to \mathcal{P}(\mathcal{V}')$ be the operator that is represented by Ψ.*

If (7.29) *is fulfilled, then there exists $u \in X_\sharp$ such that, possibly extracting a subsequence,*

$$u_n \rightharpoonup u \quad \textit{in } X_\sharp, \tag{7.32}$$

$$u \in X_\sharp, \quad D_t u + \widehat{\alpha}(u) \ni h \quad \textit{in } \mathcal{V}'. \tag{7.33}$$

If the sequence $\{\psi_n\}$ *is equi-strongly-monotone in the sense of* (5.5)*, then the solutions of* (7.19) *and of* (7.33) *are unique, and* (7.32) *holds for the whole sequence.*

Proof By part (i) of Lemma 7.1, the sequence $\{u_n\}$ is bounded in $X_\sharp$. Hence there exists $u \in X_\sharp$ such that, up to extracting a subsequence, (7.32) holds. By (7.25) and (7.29),

$$\langle\langle u_n, h_n - D_t u_n\rangle\rangle = \langle\langle u_n, h_n\rangle\rangle \to \langle\langle u, h\rangle\rangle = \langle\langle u, h - D_t u\rangle\rangle.$$

Hence

$$(u_n, h_n - D_t u_n) \underset{\pi}{\rightrightarrows} (u, h - D_t u) \quad \text{in } \mathcal{V}\times\mathcal{V}'. \tag{7.34}$$

(In passing notice that this sequence need not converge with respect to the topologies ws or sw.) Therefore

$$\begin{aligned}\Psi(u, h - D_t u) &\overset{(7.31),(7.34)}{\le} \liminf_{n\to\infty} \Psi_n(u_n, h_n - D_t u_n)\\ &\overset{(7.27)}{\le} \liminf_{n\to\infty} \langle\langle u_n, h_n - D_t u_n\rangle\rangle \overset{(7.34)}{=} \langle\langle u, h - D_t u\rangle\rangle.\end{aligned} \tag{7.35}$$

As Ψ represents the operator $\widehat{\alpha}$, (7.33) is thus established.

If (5.5) is also fulfilled, then by Proposition 5.3 the functional Ψ is strictly convex with respect to its first argument. The null-minimizer of the asymptotic functional $v \mapsto \Psi(v, h - D_t v) - \langle\langle v, h - D_t v\rangle\rangle$ is then unique, and (7.32) holds for the whole sequence. □

7.8 (ii) Pointwise formulation

The next statement rests upon the compactness of the canonic injection $V \to H$, and the assumptions (7.16) of equi-coerciveness and (7.18) of equi-Lipschitz-continuity.

Theorem 7.5 *(Structural Stability of* (7.13)*) For any n, let* (7.5)–(7.9), (7.16)–(7.18) *be fulfilled, and* u_n *be the solution of problem* (7.13) *[which exists and is unique by Lemma 7.1].*

Then there exists $u \in H^1_\sharp(0, T; V)$ *such that, possibly extracting a subsequence,*

$$u_n \rightharpoonup u \quad \text{in } H^1_\sharp(0, T; V) \cap H^2(0, T; V'). \tag{7.36}$$

Let ψ *be as in Lemma 7.3, and* $\alpha : V \to \mathcal{P}(V')$ *be the operator that is represented by* ψ*. If*

$$\text{the canonic injection } V \to H \text{ is compact,} \tag{7.37}$$

$$h_n \to h \quad \text{in } \mathcal{V}', \tag{7.38}$$

then

$$D_t u + \alpha(u) \ni h \quad \text{in } V', \text{ a.e. in }]0, T[. \tag{7.39}$$

If (5.5) *is also fulfilled, then the solutions of* (7.19) *and of* (7.39) *are unique in the space* $H^1_\sharp(0, T; V)$*, and* (7.36) *holds for the whole sequence.*

Proof By Lemma 7.1, the sequence $\{u_n\}$ is bounded in $H^1_\sharp(0, T; V) \cap H^2(0, T; V')$. Hence there exists an element u of this space such that (7.36) holds up to extracting a subsequence.

By Lemma 6.4, (7.36) and (7.37),

$$\langle u_n, D_t u_n\rangle \to \langle u, D_t u\rangle \quad \text{in } L^1_{\mathrm{loc}}(0, T). \tag{7.40}$$

By (7.38) then, up to extracting a further subsequence, $\langle u_n, h_n - D_t u_n\rangle \to \langle u, h - D_t u\rangle$ a.e. in $]0, T[$. Thus

$$(u_n, h_n - D_t u_n) \underset{\pi}{\Rightarrow} (u, h - D_t u) \quad \text{in } V \times V', \text{ a.e. in }]0, T[. \tag{7.41}$$

Therefore

$$\begin{aligned} \psi(u, h - D_t u) &\overset{(7.30),(7.41)}{\leq} \liminf_{n\to\infty} \psi_n(u_n, h_n - D_t u_n) \\ &\overset{(7.26)}{\leq} \liminf_{n\to\infty} \langle u_n, h_n - D_t u_n\rangle \overset{(7.41)}{=} \langle u, h - D_t u\rangle \quad \text{a.e. in }]0, T[. \end{aligned} \tag{7.42}$$

(7.39) is thus established.

If (5.5) is also fulfilled, then by Proposition 5.3 the functional ψ is strictly convex with respect to its first argument. The null-minimizer of the asymptotic functional

$$V \to \mathbf{R} \cup \{+\infty\} : v \mapsto \psi(v, h - D_t v) - \langle v, h - D_t v\rangle$$

is then unique a.e. in $]0, T[$, and (7.36) holds for the whole sequence. □

8 Compactness and structural stability of initial-value monotone flows

In this section we study the compactness and the structural stability of the initial-value problem for flows of the form $D_t u + \alpha(u) \ni h$ on a bounded time interval $]0, T[$.

We shall still assume that $u^0 \equiv 0$. Let us first define the space

$$X_0 := \{v \in L^2(0, T; V) \cap H^1(0, T; V') : v(0) = 0\}, \tag{8.1}$$

and formulate the initial-value problem

$$u_n \in X_0, \quad D_t u_n + \alpha_n(u_n) \ni h_n \quad \text{in} V', \text{ a.e. in }]0, T[. \tag{8.2}$$

Next we assess the existence and boundedness of the solution of this problem, amending Lemma 7.1.

Lemma 8.1 *Let (7.5)–(7.9) be fulfilled. Then:*

(i) *For any n, problem (8.2) has one and only one solution. If moreover the sequence $\{h_n\}$ is bounded in $L^2(0, T; V')$, then $\{u_n\}$ is bounded in X_0.*

(ii) *If the condition (7.16) of equi-strict-monotonicity holds, the sequence $\{h_n\}$ is bounded in $H^1(0, T; V')$, and*

$$\forall n, \exists w_n \in H : \ w_n \in h_n(0) - \alpha_n(0), \quad \sup \|w_n\|_H < +\infty, \tag{8.3}$$

then $\{u_n\}$ is also bounded in $W^{1,\infty}(0, T; H) \cap H^1(0, T; V)$.

(iii) *If the condition (7.18) of equi-Lipschitz-continuity is also fulfilled, then the sequence $\{u_n\}$ is also bounded in $H^2(0, T; V')$.*

We omit this standard argument, that may be found e.g. in [9], and just point out that the condition (8.3) provides the boundedness of $\{D_t u_n(0)\}$ in H.

For any n, let the operator α_n be represented by $\psi_n \in \mathcal{F}(V)$. The initial-value problem (8.2) is thus equivalent to

$$u_n \in X_0, \quad \psi_n(u_n, h_n - D_t u_n) \leq \langle u_n, h_n - D_t u_n\rangle \quad \text{a.e. in }]0, T[. \tag{8.4}$$

8.1 (i) Global Formulation

Here it is in order to modify the functional framework. Let us set

$$\widetilde{\mathcal{H}} := \Big\{ v :]0, T[\to H \text{ measurable: } \int_0^T (T-t)\|v(t)\|_H^2 \, dt < +\infty \Big\},$$
$$\mathcal{W} := \Big\{ v \in \widetilde{\mathcal{H}} : \int_0^T (T-t)\|v(t)\|_V^2 \, dt < +\infty \Big\}, \tag{8.5}$$

which are Hilbert spaces equipped with the respective graph norms. Identifying $\widetilde{\mathcal{H}}$ with its dual space, we get the Hilbert triplet

$$\mathcal{W} \subset \widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}' \subset \mathcal{W}', \quad \text{with continuous and dense injections.} \tag{8.6}$$

For any n, let us define

$$J_n(v, v^*) := \int_0^T (T-t)\psi_n(v(t), v^*(t)) \, dt \quad \forall (v, v^*) \in \mathcal{W} \times \mathcal{W}'; \tag{8.7}$$

this is a representative function in $\mathcal{W}$, that is $J_n \in \mathcal{F}(\mathcal{W})$; in particular,

$$J_n(v, v^*) \geq \int_0^T (T-t)\langle v(t), v^*(t)\rangle \, dt \quad \forall (v, v^*) \in \mathcal{W} \times \mathcal{W}',$$

as the latter integral coincides with the duality pairing in $\mathcal{W} \times \mathcal{W}'$. The function J_n indeed represents an operator $\widetilde{\alpha}_n : \mathcal{W} \to \mathcal{P}(\mathcal{W}')$, that is defined by the condition

$$v^* \in \widetilde{\alpha}_n(v) \quad \Leftrightarrow \quad J_n(v, v^*) = \int_0^T (T-t)\langle v(t), v^*(t)\rangle \, dt. \tag{8.8}$$

By double time integration, for any n the problem (8.4) entails the global-in-time formulation

$$u_n \in X_0, \quad J_n(u_n, h_n - D_t u_n) \leq \int_0^T (T-t)\langle u_n, h_n - D_t u_n\rangle \, dt, \tag{8.9}$$

that is,

$$u \in X_0, \quad D_t u_n + \widetilde{\alpha}_n(u_n) \ni h \quad \text{in } \mathcal{W}'. \tag{8.10}$$

Theorem 8.2 *(Structural Stability of* (8.10)*) Let* (7.5)–(7.9) *be fulfilled, the sequence* $\{h_n\}$ *be bounded in* $L^2(0, T; V')$*, and for any n* u_n *be the solution of problem* (8.2) *[which exists and is unique by Lemma 8.1].*

Then there exists $u \in X_0$ *such that, possibly extracting a subsequence,*

$$u_n \rightharpoonup u \quad \text{in } X_0. \tag{8.11}$$

Moreover, there exists $J \in \mathcal{F}(\mathcal{W})$ *such that*

$$J_n \xrightarrow{\Gamma} J \quad \text{sequentially w.r.t. the topology } \widetilde{\pi} \text{ of } \mathcal{W} \times \mathcal{W}'. \tag{8.12}$$

If

$$\text{the canonic injection } V \to H \text{ is compact,} \tag{8.13}$$

$$h_n \to h \quad \text{in } \mathcal{W}', \tag{8.14}$$

then, denoting by $\widetilde{\alpha} : \mathcal{W} \to \mathcal{P}(\mathcal{W}')$ *the operator that is represented by* J,

$$u \in X_0, \quad D_t u + \widetilde{\alpha}(u) \ni h \quad \text{in } \mathcal{W}'. \tag{8.15}$$

If the sequence $\{\psi_n\}$ *is equi-strongly-monotone in the sense of* (5.5)*, then the solutions of* (8.9) *and* (8.15) *are unique, and* (8.11) *holds for the whole sequence.*

Proof By Lemma 8.2, the sequence $\{u_n\}$ is bounded in X_0. Hence there exists $u \in X_0$ such that, up to extracting a subsequence, (8.11) holds.

By (8.13) the injection $X_0 \to L^2(0, T; H)$ is compact. By (8.11) then

$$\int_0^T (T-t)\langle D_t u_n, u_n\rangle\, dt = \frac{1}{2}\int_0^T \|u_n(t)\|_H^2\, dt$$
$$\to \tfrac{1}{2}\int_0^T \|u(t)\|_H^2\, dt = \int_0^T (T-t)\langle D_t u, u\rangle\, dt. \tag{8.16}$$

By (8.14), we then have

$$(h_n - D_t u_n, u_n) \underset{\pi}{\rightrightarrows} (h - D_t u, u) \quad \text{in } \mathcal{W}\times\mathcal{W}'. \tag{8.17}$$

As in Lemma 7.3, it is readily seen that there exists $J \in \mathcal{F}(\mathcal{W})$ as in (8.12). Therefore

$$J(u, h - D_t u) \overset{(8.12),(8.17)}{\leq} \liminf_{n\to\infty} J_n(u_n, h_n - D_t u_n)$$
$$\overset{(8.9)}{\leq} \liminf_{n\to\infty} \int_0^T (T-t)\langle u_n, h_n - D_t u_n\rangle\, dt$$
$$\overset{(8.17)}{=} \int_0^T (T-t)\langle u, h - D_t u\rangle\, dt \quad \text{a.e. in }]0, T[. \tag{8.18}$$

(8.15) is thus established.

If (5.5) is also fulfilled, then by Proposition 5.3 the functional J is strictly convex with respect to its first argument. The null-minimizer of the asymptotic functional

$$v \mapsto J(v, h - D_t v) - \int_0^T (T-t)\langle v, h - D_t v\rangle\, dt$$

is then unique, and (8.11) holds for the whole sequence. □

8.2 (ii) Pointwise formulation

Next we deal with the problem (8.2), which is tantamount to (8.4).

Theorem 8.3 *(Structural Stability of* (8.2)*) Let* (7.5)–(7.9) *and* (7.16), (7.18) *and* (8.3) *be fulfilled, the sequence* $\{h_n\}$ *be bounded in* $H^1(0,T;V')$*, and for any n let* u_n *be the solution of problem* (8.2) *[which exists and is unique by Lemma 8.1].*

Then there exists $u \in X_0 \cap H^1(0,T;V) \cap H^2(0,T;V')$ *such that, possibly extracting a subsequence,*

$$u_n \rightharpoonup u \quad \text{in } H^1(0,T;V) \cap H^2(0,T;V'). \tag{8.19}$$

Moreover, there exists $\psi \in \mathcal{F}(V)$ *such that*

$$\psi_n \xrightarrow{\Gamma} \psi \qquad \text{sequentially w.r.t. the topology } \widetilde{\pi} \text{ of } V \times V'. \tag{8.20}$$

Let $\alpha : V \to \mathcal{P}(V')$ *be the operator that is represented by* ψ*. If* (8.13) *and* (8.14) *are fulfilled, then u is the unique solution of the problem*

$$u \in X_0, \quad D_t u + \alpha(u) \ni h \quad \text{in } V', \text{ a.e. in }]0,T[, \tag{8.21}$$

and (8.19) *holds for the whole sequence.*

Proof By part (ii) of Lemma 8.2, the sequence $\{u_n\}$ is bounded in $X_0 \cap H^1(0,T;V) \cap H^2(0,T;V')$. Hence there exists u in this space such that, up to extracting a subsequence, (8.19) holds. By (8.13) and Lemma 6.4, then $\langle D_t u_n, u_n\rangle \to \langle D_t u, u\rangle$ in $L^1(0,T)$. By (8.14), possibly extracting a further subsequence, we thus have

$$\langle h_n - D_t u_n, u_n\rangle \to \langle h - D_t u, u\rangle \quad \text{a.e. in }]0,T[, \tag{8.22}$$

hence

$$(h_n - D_t u_n, u_n) \underset{\widetilde{\pi}}{\rightarrow} (\langle h - D_t u, u) \quad \text{in } V \times V', \text{ a.e. in }]0,T[. \tag{8.23}$$

By Lemma 7.3, there exists $\psi \in \mathcal{F}(V)$ such that (8.20) holds. Therefore

$$\begin{aligned} \psi(u, h - D_t u) &\overset{(8.20),(8.23)}{\le} \liminf_{n\to\infty} \; \psi_n(u_n, h_n - D_t u_n) \\ &\overset{(8.4)}{\le} \liminf_{n\to\infty} \langle u_n, h_n - D_t u_n\rangle \overset{(8.23)}{\le} \langle u, h - D_t u\rangle \quad \text{a.e. in }]0,T[. \end{aligned} \tag{8.24}$$

As the function ψ represents the operator α, (8.21) is thus established.

By the monotonicity of α, the solution of this problem is unique, and (8.19) thus holds for the whole sequence. □

8.3 Two degenerate operators

Theorem 7.4 holds for both the elliptic operator (7.10) and the degenerate elliptic operator (7.11). On the other hand, Theorems 7.5, 8.2 and 8.3 apparently only apply to (7.10), because the canonic injection $V \to H$ is compact just in this case (provided that the domain Ω is bounded and, e.g., of Lipschitz class).

The formulation (7.11) may however be amended as follows. Let us first define the Hilbert spaces

$$\widetilde{H} := \big\{\vec{v} \in L^2(\Omega)^3 : \nabla\cdot\vec{v} = 0 \text{ in } \mathcal{D}'(\Omega)\big\}, \quad \widetilde{V} := \widetilde{H} \cap H_0^1(\Omega)^3. \tag{8.25}$$

It is known that $\widetilde{H} = \nabla \times H^1(\Omega)^3$, under suitable restrictions on the domain Ω; this holds e.g. if Ω is homeomorphic to a convex set. Identifying $\widetilde{H}$ with its dual $\widetilde{H}'$, we get the Hilbert triplet

$$\widetilde{V} \subset \widetilde{H} = \widetilde{H}' \subset \widetilde{V}' \text{ with continuous and dense injections.} \tag{8.26}$$

In this framework the injection $\widetilde{V} \to \widetilde{H}$ is compact, and we may replace (7.11) by

$$\widetilde{\vec{\alpha}}_n : \widetilde{V} \to \widetilde{V}' : \vec{v} \mapsto \nabla \times \vec{\gamma}_n(\nabla \times \vec{v}). \tag{8.27}$$

We may thus apply Theorems 7.5, 8.2 and 8.3 to this operator, too.

The Stefan operator (3.24) is also degenerate, see the Example 3.8 of Sect. 3. Theorem 7.4 and 8.2 may be applied to the weak formulation of the Stefan problem. On the other hand, this operator does not fulfill (7.16), and this excludes the application of Theorems 7.5 and 8.3.

8.4 Gradient flows

For any n, let $\varphi : V \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function(al), let $2 \le p < \infty$, $p' = p/(p-1)$, $h_n \in L^{p'}(0, T; V')$, and $u_n^0 \in H$. The Cauchy problem

$$\begin{cases} D_t u_n + \partial\varphi_n(u_n) \ni h_n, \\ u_n(0) = u_n^0 \end{cases} \tag{8.28}$$

may also be set pointwise in time as

$$\begin{cases} u_n \in L^p(0, T; V) \cap W^{1,p'}(0, T; V'), \quad u_n(0) = u_n^0, \\ \varphi_n(u_n) + \langle D_t u_n - h_n, u_n \rangle \le \varphi_n(v) + \langle D_t u_n - h_n, v \rangle \quad \forall v \in V, \text{ for a.e. } t \in]0, T[. \end{cases} \tag{8.29}$$

Setting $\Phi_n := \int_0^T \varphi_n(\cdot)\, dt$ for any n, the problem (8.28) may also be reformulated globally in time as

$$\begin{cases} u_n \in L^p(0, T; V) \cap W^{1,p'}(0, T; V'), \quad u_n(0) = u_n^0, \\ \Phi_n(u_n) + \frac{1}{2}\|u_n(T)\|_H^2 - \frac{1}{2}\|u_n(0)\|_H^2 - \int_0^T (T - t)\langle u_n, h_n \rangle\, dt \\ \le \Phi_n(v) + \int_0^T (T - t)\langle v, h_n + D_t u_n \rangle\, dt \quad \forall v \in L^p(0, T; V). \end{cases} \tag{8.30}$$

Compactness and structural stability of these problems may be addressed without using the Fitzpatrick theory, as here we just outline. Let us assume that the sequence $\{\varphi_n\}$ is equi-coercive on V, in the sense that

$$\forall C \in \mathbf{R}, \ \sup_{n \in \mathbf{N}} \left\{ \|v\|_V : v \in V, \varphi_n(v) \le C \right\} < +\infty; \tag{8.31}$$

the sequence $\{\Phi_n\}$ is then also equi-coercive on $L^p(0, T; V)$, i.e.,

$$\forall C \in \mathbf{R}, \ \sup_{n \in \mathbf{N}} \left\{ \|v\|_{L^p(0,T;V)} : v \in L^p(0, T; V), \Phi_n(v) \le C \right\} < +\infty. \tag{8.32}$$

By Corollary 8.12 of p. 95 of [18], φ_n (Φ_n, resp.) then weakly Γ-converges to some function $\varphi : V \to \mathbf{R} \cup \{+\infty\}$ ($\Phi : L^p(0, T; V) \to \mathbf{R} \cup \{+\infty\}$, resp.), up to extracting a subsequence. The functions φ and Φ are both convex and lower semicontinuous; under natural restrictions on $\{\varphi_n\}$ they are also proper (i.e., $\not\equiv +\infty$). In the next section we shall see that in general however $\Phi \ne \int_0^T \varphi(\cdot)\, dt$.

The structural stability of the global-in-time formulation (8.30) may be established as follows, assuming that $p = 2$ for consistency with the above developments. By part (i) of Lemma 7.1, under natural hypotheses on the data the sequence $\{u_n\}$ is bounded in $L^2(0, T; V) \cap H^1(0, T; V')$. There exists then u such that $u_n \rightharpoonup u$ in this space, up to extracting a subsequence. By an obvious identification,

$$L^2(0, T; V) \cap H^1(0, T; V') \subset C_w^0([0, T]; H)$$

(the space of weakly continuous functions $[0, T] \to H$). Therefore $u_n(T) \rightharpoonup u(T)$ in H, whence $\liminf_{n\to\infty} \|u_n(T)\|_H^2 \geq \|u(T)\|_H^2$. The form of the global-in-time problem (8.30) is then preserved in the limit.

An analogous conclusion may be attained for the local-in-time formulation (8.29). Under further regularity assumptions on the data, by part (iii) of Lemma 7.1 the sequence $\{u_n\}$ is also bounded in $H^1(0, T; V) \cap H^2(0, T; V')$; hence $u_n \rightharpoonup u$ in this space, up to extracting a subsequence. If the injection $V \to H$ is compact, by Lemma 6.4 $\langle D_t u_n, u_n\rangle \to \langle D_t u, u\rangle$ in $L^1(0, T)$. The form of the local-in-time problem (8.29) is then preserved in the limit, too.

There is an obvious analogy between the statements and the arguments of Theorems 7.4, 8.2 and 8.3, and these properties of compactness and structural stability for gradient flows. One might wonder whether *doubling the variables* and using representative functions might here provide more precise results. For instance, one may represent the operator $\alpha_n = \partial\varphi_n$ by the Fenchel function $\varphi_n + \varphi_n^*$. But in the case of gradient flows this author does not see any significative advantage in using the Fitzpatrick approach.

9 Long memory and examples

In this section we discuss the possible onset of long memory in the limit for flows of the form $D_t u + \alpha(u) \ni h$, and briefly revisit the examples of Sect. 3.

9.1 Onset of long memory

The global-in-time formulation (7.33) is weaker than the pointwise-in-time problem (7.39), because at any instant t a priori $[\widehat{\alpha}(u)](\cdot, t)$ ($\in \mathcal{V}'$) might depend not only on $u(\cdot, t)$ but also on $u|_{\Omega\times]0,T[}$. This corresponds to the possible onset of *long memory* through the limit procedure. In the case of the initial-value problem of Sect. 8, this also raises the issue of the causality of the operator $\widehat{\alpha}$.

Next we review a simple example that was pointed out by Tartar; see [39,40], Chap. 23 of [41] and references therein. Let us select a bounded sequence $\{a_n\}$ of $L^\infty(\Omega)$, with Ω a bounded Lipschitz domain of $\mathbf{R}^N$. For any n, the short-memory equation

$$D_t u_n + a_n(x) u_n = 0 \quad \text{in } \Omega\times]0, T[\tag{9.1}$$

may be interpreted in two different ways: either as an ordinary differential equation parameterized by $x \in \Omega$, or (nonequivalently) as an equation in a space of functions $\Omega \to \mathbf{R}$. Assuming the second point of view, the Eq. (9.1) is associated with the linear semigroup

$$S_n(t) : L^p(\Omega) \to L^p(\Omega) : v(x) \mapsto \exp\{-a_n(x)t\}\, v(x) \quad \forall p \in]1, +\infty[, \forall n. \tag{9.2}$$

If the sequence a_n converges in $L^1(\Omega)$ *strongly,* the semigroup S_n converges to a semigroup. In this case the exponential form is preserved in the limit, and with it the first-order form of the equation Eq. (9.1). If instead a_n converges in $L^1(\Omega)$ just *weakly,* then the exponential form is necessarily lost in the limit, and a long memory effect occurs. The precise form of the limit equation may be found in, Chap. 23 of [41]. For the equation (9.1) indeed there is no way to pass to the limit in the product $a_n u_n$, since both sequences converge weakly (and there is no property of compensated compactness).

For $p = 2$, this fits the setup of Sects. 7 and 8 for $V = H = L^2(\Omega)$. For any n, the positive linear operator

$$\alpha_n : L^2(\Omega) \to L^2(\Omega) : v \mapsto a_n(x)v \tag{9.3}$$

is variationally represented e.g. by the Fenchel function f_{α_n} or by the Svaiter function s_{α_n}:

$$f_{\alpha_n}(u, v) = \frac{1}{2} \int_\Omega \left[a_n(x)\, u(x)^2 + a_n(x)^{-1}\, v(x)^2\right] dx, \tag{9.4}$$

$$s_{\alpha_n}(u, v) := (I_{\alpha_n} + \pi)^{**}(u, v) = \begin{cases} \int_\Omega a_n(x)\, u(x)^2\, dx & \text{if } v = a_n u \text{ a.e. in } \Omega \\ +\infty & \text{otherwise,} \end{cases} \tag{9.5}$$

for any $u, v \in L^2(\Omega)$. Both sequences are $V \times V'$-equi-coercive, at variance with the sequence of the Fitzpatrick functions of the form (7.24). Theorem 7.4 thus applies. On the other hand, here Theorems 7.5, 8.2 and 8.3 do not apply, because the canonic injection $V \to H$ is not compact.

9.2 No onset of long memory

Next we display a linear equation in which no long memory arises in the limit. Let us select a bounded sequence $\{a_n\}$ of $L^\infty(\Omega)$ with the a_ns equi-bounded from below by a positive constant, and for any n consider the linear equation

$$D_t u_n - \nabla \cdot [a_n(x) \nabla u_n] = 0 \quad \text{in } \mathcal{D}'(\Omega), \text{ a.e. in }]0, T[. \tag{9.6}$$

The positive linear operator

$$\tilde{\alpha}_n : H_0^1(\Omega) \to H^{-1}(\Omega) : v \mapsto -\nabla \cdot [a_n(x) \nabla v] \tag{9.7}$$

may be variationally represented e.g. by the Fenchel and Svaiter functions:

$$f_{\tilde{\alpha}_n}(u, v^*) = \inf \Big\{ \ \frac{1}{2} \int_\Omega a_n(x) \big(|\nabla u(x)|^2\, dx + |\nabla \eta(x)|^2\big)\, dx : \\ \eta \in H_0^1(\Omega), -\nabla \cdot [a_n(x) \nabla \eta] = v^* \text{ in } \mathcal{D}'(\Omega) \Big\}, \tag{9.8}$$

$$s_{\tilde{\alpha}_n}(u, v^*) := (I_{\tilde{\alpha}_n} + \pi)^{**}(u, v^*) \\ = \begin{cases} \int_\Omega a_n(x)\, |\nabla u(x)|^2\, dx & \text{if } v^* = -\nabla \cdot [a_n(x) \nabla u] \text{ in } \mathcal{D}'(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{9.9}$$

for any $(u, v^*) \in H_0^1(\Omega) \times H^{-1}(\Omega)$. Both sequences are $H_0^1(\Omega) \times H^{-1}(\Omega)$-equi-coercive. In this case the canonic injection $V = H_0^1(\Omega) \to H = L^2(\Omega)$ is compact (provided that Ω is regular enough). Theorems 7.5 and 8.3 may thus be applied, and this excludes the onset of long memory in the limit. On the other hand, it is indeed possible to pass to the limit in the Eq. (9.6), via Murat and Tartar's *compensated compactness*.

The same conclusion is easily attained if the scalars $a_n(x)$ are replaced by (possibly asymmetric) matrices $A_n(x)$, under standard restrictions.

9.3 Discussion

Tartar's example (9.1) provides a basis for discussing the application of the results of Sects. 7 and 8.

(i) Let us consider the global-in-time formulation of the periodic problem (7.19), and define $\psi_n, \Psi_n, \psi, \Psi$ as in (7.21), (7.30), (7.31).

As Tartar's example (9.1) fits the assumptions of Theorem 7.4, we claim that

$$\text{the hypotheses of Theorem 7.4} \quad \not\Rightarrow \quad \Psi = \int_0^T \psi(\cdot)\, dt. \tag{9.10}$$

Indeed otherwise the pointwise-in-time formulation would also be preserved in the limit, and this would exclude the onset of long memory.

(ii) It is not obvious that an analogous conclusion holds for the global-in-time formulation of the initial-value problem (8.2). Theorem 8.2 indeed assumes the compactness of the injection $V \to H$, and this hypothesis fails in (9.1).

This rather looks as a weakness of the present theory, since there is no apparent reason why replacing the time-periodicity by the initial condition should make a difference as for the onset of long memory. A result of compactness and structural stability for this global-in-time problem without that compactness hypothesis is obtained in [49], via a quite different approach.

(iii) For any n, the solution of the pointwise-in-time problem (7.13) solves the corresponding global-in-time periodic problem (7.19). By Theorem 7.4 we know that under the hypothesis (5.5) the asymptotic global-in-time formulation has a unique solution. Under the stronger hypotheses of Theorem 7.5 (in particular, the compactness of the injection $V \to H$ and enhanced regularity of the data), we then infer that the two asymptotic problems (7.33) and (7.39) are also mutually equivalent. It follows that

$$\text{the hypotheses of Theorem 7.5} \quad \Rightarrow \quad \Psi = \int_0^T \psi(\cdot)\, dt. \tag{9.11}$$

An analogous conclusion may be attained for the initial-value problems (8.15) and (8.21).

(iv) With reference to the discussion of the asymptotic behavior of the gradient flow (8.28), let us set $\varphi_n(v) = \frac{1}{2}\int_\Omega a_n(x)v(x)^2\, dx$ for any $v \in L^2(\Omega)$, and denote by φ and Φ the respective Γ-limit of the sequences $\{\varphi_n\}$ and $\{\Phi_n\}$ in the weak L^2-topology.

As Tartar's example also fits this setup, one may conclude that

$$\text{for the example (9.1)}, \ \ \Phi \neq \int_0^T \varphi(\cdot)\, dt. \tag{9.12}$$

9.4 About the examples of Sect. 3

The selection of the topology is crucial in the analysis of compactness and structural stability of monotone equations. Let us distinguish two classes of monotone equations:

(i) Equations of the form

$$\alpha(u) \ni h \quad \text{with a single operator } \alpha \in \mathcal{M}(V); \tag{9.13}$$

see e.g. the Examples 3.2– 3.5, 3.9– 3.12 of Sect. 3. If α is represented by a function f_α, then this inclusion is equivalent to $f_\alpha(u,h) = \langle h, u\rangle$. Here compactness and structural stability may be studied by using either of the topologies $\widetilde{\pi}$ or ws.

(ii) Equations of the form

$$\begin{aligned} &\alpha(u) + \Lambda u \ni h \\ &\text{with } \alpha \in \mathcal{M}(V) \text{ and } \Lambda : V \to V' \text{ linear, bounded and positive;} \end{aligned} \tag{9.14}$$

see e.g. the Examples 3.6, 3.7, 3.8, 3.13 of Sect. 3. On the basis of the extended B.E.N. principle of Sect. 3, the operator $\alpha + \Lambda$ may be represented by the function

$$f_{\alpha+\Lambda}(u,h) = f_\alpha(u, h - \Lambda u) + \langle \Lambda u, u\rangle \quad \forall (u,h) \in V \times V'. \tag{9.15}$$

If the operator Λ is compact, then compactness and structural stability of (9.14) may be studied by using either of the topologies $\widetilde{\pi}$ or ws, along the lines of Sects. 7 and 8.

Under further regularity hypotheses, the Γ-compactness with respect to the topology s may also be used; this excludes the onset of long memory.

9.5 Further monotone flows

The analysis of Sect. 7 may be extended to several other monotone equations. For instance, by the extended B.E.N. principle, if $\alpha : V \to \mathcal{P}(V')$ is maximal monotone, $L : H \to H$ is positive and self-adjoint and $a > 0$, then the integro-differential inclusion

$$L \int_0^t u(\tau)\, d\tau + a D_t u + \alpha(u) \ni h \quad \text{in } V', \text{ a.e. in }]0, T[\tag{9.16}$$

may be reformulated as the null-minimization of the functional

$$\begin{aligned} F(v, h) \;:= & \int_0^T \Big[f_\alpha\Big(v, h - L\int_0^t v(\tau)\, d\tau - a D_t v\Big) - \langle h, v\rangle \Big]\, dt \\ & + \frac{1}{2}\Big\| L^{1/2} \int_0^T v(\tau)\, d\tau \Big\|_H^2 + \frac{a}{2}\|v(T)\|_H^2 - \frac{a}{2}\|v(0)\|_H^2 \\ & \forall v \in X_\sharp, \forall h \in L^2(0, T; V'). \end{aligned} \tag{9.17}$$

Of course (9.16) might also be written either as a second-order differential inclusion for the function $w(t) := \int_0^t u(\tau)\, d\tau$, or as a first-order system for the pair (u, w).

Acknowledgements This research was partially supported by the P.R.I.N. project "Phase transitions, hysteresis and multiscaling" of Italian M.I.U.R.. The author gratefully acknowledges a useful talk with Ulisse Stefanelli, and several helpful remarks of the anonymous Reviewer.

References

1. Ambrosetti, A., Sbordone, C.: Γ-convergenza e G-convergenza per problemi non lineari di tipo ellittico. Boll. Un. Mat. Ital **13A**(5), 352–362 (1976)
2. Attouch, H.: Variational Convergence for Functions and Operators. Pitman, Boston (1984)
3. Auchmuty, G.: Saddle-points and existence-uniqueness for evolution equations. Differ. Integral Equ. **6**, 1161–1171 (1993)
4. Barbu, V.: Existence theorems for a class of two point boundary problems. J. Differ. Equ. **17**, 236–257 (1975)
5. Barbu, V.: Nonlinear Differential Equations of Monotone Types in Banach Spaces. Springer, Berlin (2010)
6. Barbu, V., Precupanu, T.: Convexity and Optimization in Banach Spaces. Editura Academiei, Bucuresti (1978)
7. Braides, A.: Γ-Convergence for Beginners. Oxford University Press, Oxford (2002)
8. Braides, A., Defranceschi, A.: Homogenization of Multiple Integrals. Oxford University Press, Oxford (1998)
9. Brezis, H.: Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert. North-Holland, Amsterdam (1973)
10. Brezis, H.: Analyse Fonctionelle. Théorie et Applications. Masson, Paris (1983)
11. Brezis, H., Ekeland, I.: Un principe variationnel associé à certaines équations paraboliques. I. Le cas indépendant du temps and II. Le cas dépendant du temps. C. R. Acad. Sci. Paris Sér. A-B **282** (1976) 971–974, and ibid. 1197–1198

12. Browder, F.: Nonlinear operators and nonlinear equations of evolution in Banach spaces. Proc. Symp. Pure Math., **XVIII** Part II. A.M.S., Providence (1976)
13. Buliga, M., Saxcé, G.de , Vallée, C.: Existence and construction of bipotentials for graphs of multivalued laws. J. Convex. Anal. **15**, 87–104 (2008)
14. Buliga, M., Saxcé, G.de , Vallée, C.: Bipotentials for non-monotone multivalued operators: fundamental results and applications. Acta Appl. Math. **110**, 955–972 (2010)
15. Burachik, R.S., Svaiter, B.F.: Maximal monotone operators, convex functions, and a special family of enlargements. Set Valued Anal. **10**, 297–316 (2002)
16. Burachik, R.S., Svaiter, B.F.: Maximal monotonicity, conjugation and the duality product. Proc. Am. Math. Soc **131**, 2379–2383 (2003)
17. Checcucci, V., Tognoli, A., Vesentini, E.: Lezioni di Topologia Generale. Feltrinelli, Milano (1972)
18. Dal Maso, G.: An Introduction to Γ-Convergence. Birkhäuser, Boston (1993)
19. De Giorgi, E., Franzoni, T.: Su un tipo di convergenza variazionale. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur **58**(8), 842–850 (1975)
20. De Giorgi, E., Spagnolo, S.: Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine. Boll. Un. Mat. Ital **8**, 391–411 (1973)
21. Dontchev, A.L., Zolezzi, T.: Well-Posed Optimization Problems. Springer, Berlin (1993)
22. Ekeland, I., Temam, R.: Analyse Convexe et Problèmes Variationnelles. Dunod Gauthier-Villars, Paris (1974)
23. Fenchel, W.: Convex Cones, Sets, and Functions. Princeton University, Princeton (1953)
24. Fitzpatrick, S.: Representing monotone operators by convex functions. Workshop/Miniconference on Functional Analysis and Optimization (Canberra, 1988), 59–65, Proc. Centre Math. Anal. Austral. Nat. Univ., 20, Austral. Nat. Univ., Canberra, (1988)
25. Ghoussoub, N.: A variational theory for monotone vector fields. J. Fixed Point Theory Appl. **4**, 107–135 (2008)
26. Ghoussoub, N.: Self-Dual Partial Differential Systems and their Variational Principles. Springer, Berlin (2009)
27. Ghoussoub, N., Tzou, L: A variational principle for gradient flows. Math. Ann. **330**, 519–549 (2004)
28. Hiriart-Urruty, J.-B., Lemarechal, C.: Convex Analysis and Optimization Algorithms. Springer, Berlin (1993)
29. Marques Alves, M., Svaiter, B.F.: Brøndsted-Rockafellar property and maximality of monotone operators representable by convex functions in non-reflexive Banach spaces. J. Convex Anal. **15**, 693–706 (2008)
30. Martinez-Legaz, J.-E., Svaiter, B.F.: Monotone operators representable by l.s.c. convex functions. Set Valued Anal. **13**, 21–46 (2005)
31. Martinez-Legaz, J.-E., Svaiter, B.F.: Minimal convex functions bounded below by the duality product. Proc. Am. Math. Soc **136**, 873–878 (2008)
32. Martinez-Legaz, J.-E., Théra, M.: A convex representation of maximal monotone operators. J. Nonlinear Convex Anal. **2**, 243–247 (2001)
33. Nayroles, B.: Deux théorèmes de minimum pour certains systèmes dissipatifs. C. R. Acad. Sci. Paris Sér. A-B **282** (1976) A1035–A1038
34. Penot, J.-P.: The relevance of convex analysis for the study of monotonicity. Nonlinear Anal. **58**, 855–871 (2004)
35. Rockafellar, R.T.: A general correspondence between dual minimax problems and convex programs. Pacific J. Math. **25**, 597–611 (1968)
36. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1969)
37. Stefanelli, U.: The Brezis-Ekeland principle for doubly nonlinear equations. SIAM J. Control Optim **8**, 1615–1642 (2008)
38. Svaiter, B.F.: Fixed points in the family of convex representations of a maximal monotone operator. Proc. Am. Math. Soc **131**, 3851–3859 (2003)
39. Tartar, L.: Nonlocal effects induced by homogenization. In: Colombini, F., Marino, A., Modica, L., Spagnolo, S. (eds.) Partial Differential Equations and the Calculus of Variations, vol. II, pp. 925–938. Birkhäuser, Boston (1989)
40. Tartar, L.: Memory effects and homogenization. Arch. Rational Mech. Anal. **111**, 121–133 (1990)
41. Tartar, L.: The General Theory of Homogenization. A Personalized Introduction. Springer, Berlin; UMI, Bologna, (2009)
42. Visintin, A.: Extension of the Brezis-Ekeland-Nayroles principle to monotone operators. Adv. Math. Sci. Appl. **18**, 633–650 (2008)
43. Visintin, A.: Scale-transformations in the homogenization of nonlinear magnetic processes. Arch. Rat. Mech. Anal. **198**, 569–611 (2010)

44. Visintin, A.: Homogenization of a parabolic model of ferromagnetism. J. Differ. Eq. **250**, 1521–1552 (2011)
45. Visintin, A.: Scale-transformations and homogenization of maximal monotone relations, and applications. Arch. Rational Mech. Anal. **198**(2), 569–611 (2010)
46. Visintin, A.: Structural stability of doubly nonlinear flows. Boll. Un. Mat. Ital **IV**, 363–391 (2011)
47. Visintin, A.: On the structural stability of monotone flows. Boll. Un. Mat. Ital **IV**, 471–479 (2011)
48. Visintin, A.: Structural stability of rate-independent nonpotential flows. (in press)
49. Visintin, A.: Compactness and structural stability of pseudo-monotone equations. (in progress)
50. Zeidler, E.: Nonlinear Functional Analysis and Its Applications. II/B. Nonlinear Monotone Operators. Springer, New York (1990)