Notes on Black Holes:
From Classical to Quantum Aspects

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1 Introduction

Since Hawking's(1974) discovery of black hole evaporation[1] there have been many independent derivation of black hole radiance [2, 3, 4, 5, 6, 7] with special emphasis on the Schwarzschild solution. The two main parameters associated by the classical theory to a spherically symmetric black hole are

1. the Schwarzschild radius and area

$$r_g = \frac{2MG}{c^2} \simeq 3 \times 10^5 \frac{M}{M_\odot} cm, \qquad A = 4\pi r_g^2 = 16\pi \frac{G^2 M^2}{c^4} \tag{1.1}$$

   which characterizes the size of the black hole, $M_\odot$ being the solar mass and M the black hole mass,

2. the surface gravity

$$k = \frac{c^4}{4MG} \simeq 1.5 \times 10^{15} \frac{M_\odot}{M} cm \cdot sec^{-2} \tag{1.2}$$

   which characterizes the strength of gravity at the Schwarzschild radius.

Moreover, between M, A and k one has the differential identity $dM = kdA/8\pi$. On the other hand, the main outcomes of Hawking's discovery were

1. the precise identification of the surface gravity with the temperature of the black hole, according to the formula

$$T = \frac{\hbar k}{2\pi c} \simeq 0.6 \times 10^{-23} \frac{M_\odot}{M} erg \tag{1.3}$$

   which characterizes the black body spectrum of the quantum emission,

2. the precise identification of the area with the entropy of the black hole

$$S = \frac{A}{4L_P^2} = \frac{4\pi M^2 G}{\hbar c} \simeq 1.12 \times 10^{76} \left(\frac{M}{M_\odot}\right)^2 \tag{1.4}$$

   whose microscopic characterization is somewhat less clear but having a well defined thermodynamic meaning ($L_P = \sqrt{\hbar G c^{-3}}$ is the Planck length).

The differential identity now becomes $dM = TdS$, and therefore it receives a thermodynamic basis. The main theme of these lectures is to provide an understanding of these results, which involve the use of quantum field theory in a curved background. The above identifications were in fact proposed earlier on the basis of the classical properties endowed by the surface gravity and the horizon area. In particular, the horizon area as a measure of black hole entropy was systematically investigated by Bekenstein [8, 9, 10], but it was not until Hawking's discovery that the proposal got a consistent foundation.
With appropriate modifications, these results also apply to rotating and charged black hole solutions, although not many explicit calculations have been done in this case (see however Ref.[11, 12] where black hole evaporation is investigated from the point of view of the equivalence principle for both rotating and non rotating black holes). Certainly part of the reason for this is to be found in the very intricate form of the Kerr metric and part in the fact that we have no analogue of the Birkhoff theorem in non spherical gravitational collapse. In any case, the

self-consistency of the Hawking treatment is greatly reinforced by its validity for rotating black holes, as it includes beautifully the superradiance phenomenon and the thermal emission in a unique formula. This radiation formula is an asymptotic result, concerning the high frequency part of the quantum field whose vacuum state describes the empty space outside the collapsing matter, long before the onset of collapse. Indeed, one expects that the Fourier components of the field having wavelength less than $r_g \simeq M$ will interact significantly with the black hole, i.e only frequencies greater than M^{-1} will be important. In this case, geometric optics is a sufficiently accurate approximation near the black hole. However, as the geometry is also assumed to be unaffected by the emitted particles, consistency is maintained only if the back reaction on the metric is negligible. This happens for black holes which emit quanta of much smaller energy than the black hole mass. The typical energy of these particles is of order of the black hole temperature $T \simeq \hbar c^3 G^{-1} M^{-1}$, which is much less than Mc^2 for black holes whose mass is much larger than the Planck mass. Therefore it seems that to derive the quantum emission of black holes a minimal set of assumptions might be

1. that the geometry outside the collapsing body settles down to a nearly stationary state. By the black hole uniqueness theorems [13, 14] this must be then a metric in the Kerr-Newman sequence.

2. That quantum field theory in a curved background be valid up to sufficiently high energies.

Notice that, a priori, the evolution of black holes cannot be followed up to the evaporation end point by using the semiclassical Hawking result. We will not touch this rather subtle and difficult problem (excellent reviews on quantum black holes have been published, see Ref. [15, 16, 17, 18] and references therein) and we just stop where the real problems begin.
The rest of the paper is divided in two main parts. In the first we discuss the definition and some properties of black holes according to Einstein general relativity. The exposition of the classical part is not self-contained, being intended as a guide to the more sophisticated treatment which exist in the literature. In particular, the proof of various statements will be only informal, but the references to the literature will be given as we proceed along. In the second part we review the mechanism of black hole radiance, after a brief introduction of the connection between quantum field theory, acceleration and thermal effects has been given. Detailed calculations for the rotating and charged black holes are presented and explicit expressions for the Bogoljubov coefficients are given. On the other hand many relevant topics are left apart, in particular stress tensor calculations, one-loop effects of quantum fields, entropy corrections, Euclidean field theory with conical singularities which is relevant for black holes, dilatonic and $2D$-dimensional black holes, etc. These additional advancements would in fact cover another course. Nor we report about astrophysical consequences of black hole emission.
Unless otherwise stated we shall use dimensionless units, in which $c = G = \hbar = 1$. The units of length, mass and time are then respectively the Planck length $L_P = (G\hbar/c^3)^{1/2}$, the Planck mass $M_P = L_P c^2/G$ and the Planck time $T_P = L_P/c$. Our curvature conventions will be the same as in the influential book of Hawking and Ellis (referred to as HE herefrom) [19], and Wald [20]. So the signature of the metric tensor will be 2 (one negative eigenvalue), the curvature tensor is defined by

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc} \tag{1.5}$$

The Ricci tensor will be $R_{ab} = R^c_{acb}$ and the curvature scalar $R = g^{ab}R_{ab}$. The covariant derivative operator will be denoted ∇_a and space-time singularities in Penrose diagrams will be represented by dashed heavy lines. A Killing vector field ξ^a is one that satisfies the equation

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{1.6}$$

Hence translations along the integral curves of ξ are symmetries of the metric tensor, i.e. the Killing fields represent the symmetries of the gravitational field.
For general results in semi-Riemannian geometry, see the book of O'Neil [21] (referred to as ON).

2 General Relativity

General relativity is A. Einstein's theory of the gravitational interaction. Under his head, it has come to be a theory of spacetime itself, a fact which makes quantization a rather difficult task. It also encompasses the special status of inertial reference frames, which was perhaps the main motivation , for Einstein, in developing his theory. But general relativity predicts the occurrence of spacetime singularities under quite general conditions. Quantization of matter fields is not enough to cure the predicted singular behaviour, and therefore one has to quantize gravity itself. However, the marriage of general relativity with quantum mechanics is not yet consistently formulated, and as such it remains one of the main issues of modern physics.

2.1 Gravity

Gravity is a very weak, long range attractive interaction. It obeys the weak equivalence principle (universality of free fall) very precisely, since the ratios of inertial masses (m_i) to passive gravitational masses (m_g) was found to be independent of the matter content of freely falling bodies with high accuracy, as can be seen from the figure

$$\left(\frac{m_i}{m_g}\right)_{wood} - \left(\frac{m_i}{m_g}\right)_{platinum} \leq 10^{-9} \tag{2.1}$$

from Eötvös (1899), and more recently from Dicke and collaborators (1967) with improved accuracy. These remarkable properties led Einstein to the theory of general relativity, according to which

(i) a pair (M, g) is a mathematical model of space-time if M is a non compact, paracompact (see HE or ON) and connected 4-manifold (by the word "manifold" we always mean one that satisfies the Hausdorf topological condition) and g is a Lorentz metric on M, i.e. a metric with signature 2. It is taken non compact, for otherwise it contains closed timelike curves [19]. The metric tensor g is identified with the gravitational field.

(ii) To each matter field there is associated a symmetric, second order tensor field T_{ab} which describes the energy-momentum content of matter and which obeys the balance equation

$$\nabla_a T^a_b = 0 \tag{2.2}$$

One normally assumes that $T_{ab} = 0$ implies vanishing matter fields, i.e. that all matter fields carry energy. Some energy conditions must in general be imposed to T_{ab} to insure its physical validity. These are:

The weak energy condition (WEC): for any causal vector (i.e. non spacelike) k^a, $T_{ab}k^a k^b \geq 0$. The meaning is obvious.

The dominant energy condition (DEC): for any future directed causal vector k^a, $-T^a_b k^b$ is a future directed causal vector (which implies that the speed of matter is always less than or equal to the speed of light).

The strong energy condition (SEC): for any unit timelike vector k^a

$$T_{ab}k^a k^b \geq -\frac{T}{2}, \qquad T = g^{ab}T_{ab} \tag{2.3}$$

By Einstein's equations, this is equivalent to $R_{ab}k^a k^b \geq 0$. This is the condition appearing in the singularity theorems of general relativity.
To see the meaning of the energy conditions, consider an isotropic perfet fluid, with energy density ρ, pressure p and stress tensor $T_{ab} = (\rho + p)u_a u_b + pg_{ab}$. Then WEC holds if $\rho \geq 0$ and $\rho + p \geq 0$, DEC holds if $\rho \geq |p|$ and SEC holds if $\rho + 3p \geq 0$ and $\rho + p \geq 0$.

(iii) The gravitational field itself, i.e. the metric tensor, obeys Einstein field equations

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{2.4}$$

Observationally, the cosmological constant Λ is very small. It is so tiny to become a real problem for physics (see [22] for an illuminating review), since it must be less than 10^{-120} units of Planck length (for otherwise the rate of expansion of the galaxies might be different from the presently observed one).

The requirement *(ii)* can be met by deriving the field equations from an invariant, local Lagrangian density $\mathcal{L}$. Then the stress tensor can be obtained from the matter part of $\mathcal{L}$ by functional differentiation w.r.t. the metric

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_M}{\delta g^{ab}} \tag{2.5}$$

and therefore it is automatically symmetric and conserved (see any modern textbook in general relativity). The matter field equations derived from a local Lagrangian density, with a Lorentz metric, usually are also of hyperbolic type, making possible to satisfy the postulate of local causality, among other things.
Remark 1 It may seem obvious that on a given manifold a Lorentz metric exists, but in fact it is not. A necessary condition is the following (see HE or ON for further details): since M is paracompact, it admits a Riemannian metric k_{ab}. If there is also a Lorentz metric g_{ab}, then the equation

$$g_{ab}\chi^b = \lambda k_{ab}\chi^b \tag{2.6}$$

admits one negative eigenvalue and three positive's. The eigenvector with the negative eigenvalue is determined up to a sign and is defined everywhere, so it determines a timelike line element field throughout M (a line element field is a map $p \in M \to (V, -V)$, where V is a vector tangent to M at p which is nowhere vanishing).

Conversely, suppose there is a line element field $p \to (V, -V)$ over M. Then take a Riemann metric k_{ab} and define the new metric

$$g_{ab} = k_{ab} - 2V^{-2}V_aV_b, \qquad V^2 \equiv k_{ab}V^aV^b \tag{2.7}$$

It is straightforward to verify that this is a Lorentz metric. Thus the existence of a Lorentz metric is equivalent to the existence of a nowhere vanishing line element field.
If M is compact, it is a standard topological theorem that existence of a line element field implies the vanishing of the Euler characteristic. Thus all odd dimensional spheres admit a Lorentz metric, the even dimensional spheres do not. The torus also admits a Lorentz metric and, more generally, all compact surfaces with genus one admit a Lorentz metric.
Remark 2 We notice that the energy conditions, as well as many other concepts in relativity, depend crucially on the existence of a well defined notion of time orientability. Locally, it is a daily life fact that future and past are unambiguously defined, which means that locally a timelike vector field exists giving the direction of time.
So we define a local orientation at $p \in M$ as a pair $\{\mathcal{U}, \tau\}$, where $\mathcal{U}$ is a neighbourhood of p and τ a timelike vector field on $\mathcal{U}$. If there is a local orientation at p, we call the time cone at $q \in \mathcal{U}$ containing the vector $\tau(q)$, the *future time cone at* q. Similarly, the cone containing $-\tau(q)$ will be called the *past time cone at* q.
The spacetime manifold is said to be time orientable if there is a covering by local orientations, $\{\mathcal{U}_\alpha, \tau_\alpha\}$, such that on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ the two orientations agree, i.e. τ_α and τ_β both belong to the same time cone.
Clearly, if there is a globally defined timelike vector field, the manifold will be time orientable. Conversely, we may choose a partition of unity $\{f_\alpha\}$ subordinate to the covering $\{\mathcal{U}_\alpha\}$ (see ON for definitions) and define $K = \sum_\alpha f_\alpha \tau_\alpha$. As time cones are convex, this is a timelike vector field on M.
The notion of time orientability is so obvious that one can hardly believe it is not always satisfied. However, take the open strip $0 < X < 1$ in $2D$ Minkowski space with metric $ds^2 = -dT^2 + dX^2$. Then glueing the two boundaries after a twist of the stright line $X = 1$, gives a non time orientable manifold (it is an infinite Mobius band with a Lorentz metric, so it is also non orientable as a manifold. However, it is space orientable).

2.2 Asymptotically flat spaces

The black holes forms as a result of catastrophic gravitational collapse of isolated bodies, when they are so massive that the battle of matter and radiation pressure against gravity is lost. When a black hole forms the escape velocity of a particle from the surface of the black hole, reaches the velocity of light and therefore the body cannot be seen at infinity. To define precisely the notion of a black hole in general relativity, one needs a precise notion of infinity. This is provided by Penrose's definition of asymptotic simplicity.
In a time orientable space-time one can define the future and the past of any set. It is customary to denote $I^+(p)$ the set of points which can be reached from p by a future directed timelike curve. Thus $I^+(p)$ is called the chronological future of p. The chronological past is similarly defined and denoted $I^-(p)$. The chronological future or past of any set S is then defined by $I^\pm(S) = \cup_{p \in S} I^\pm(p)$. The causal future of p, denoted $J^+(p)$, is the union of p with the set of events that can be reached from p by a future directed causal (i.e. non-spacelike) curve and for a set S, one defines $J^\pm(S) = \cup_{p \in S} J^\pm(p)$.

Now $I^+(p)$ is always an open set and the closure $\bar{I}^+(p)$ is always equal to the closure of $J^+(p)$. However, $J^+(p)$ is not always closed as can be seen by removing a point from Minkowski space. To remove such artificial situations one requires $J^\pm(K)$ to be a closed set for any compact set K, in which case the space-time is said to be *causally simple.*
Neither time orientability nor causal simplicity are sufficient to exclude certain causal pathologies like the existence of closed causal curves.
As an example, take the cylinder $S^1 \times \mathbb{R}$ with the Lorentz metric $ds^2 = -2dz\, d\phi$, with z running along the axis and ϕ along the circle. This spacetime is causally simple and time orientable, but any circle with fixed z clearly is a closed null geodesic.
For this reason space-time is required to be *strongly causal* everywhere, where one says that

(i) Strong causality holds at $p \in M$ if every neighbourhood of p contains a neighbourhood of p which no causal curve intersects more than once. This means that there are no causal curves returning arbitrarily close to p, in particular there are no closed timelike or null curves. If the condition holds everywhere the space-time is called strongly causal.

In a strongly causal space-time, an event is distinguished by its chronological future and/or past, namely $I^\pm(p) = I^\pm(q)$ if and only if $p = q$.
By opening slightly the null cones in the cylinder spacetime above, one introduces timelike curves. Although this example is not a strongly causal spacetime, the phenomenon may still appear even in a strongly causal spacetime. If by opening the light cones the spacetime remains strongly causal as before, then one says that it is a *stably causal spacetime.* Mathematically, to open the null cones one changes the metric g_{ab} into $\tilde{g}_{ab} = g_{ab} - t_a t_b$, where t^a is a timelike vector field. So stable causality means that there is a timelike t^a such that $\tilde{g}_{ab}$ has no closed causal curves.
Another important requirement is the possibility to predict the future evolution of matter fields from the Cauchy data. In this context, a partial Cauchy surface Σ, is a three dimensional spacelike hypersurface without edge, which no causal curve intersects more than once. The future Cauchy development of Σ, denoted $D^+(\Sigma)$, is the region of space-time that can be predicted from Σ. It is defined as the set of points p such that every past directed and inextendible causal curve from p intersects Σ.
The past Cauchy development, $D^-(\Sigma)$, is then defined similarly. If a partial Cauchy surface is such that $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma) = \mathcal{M}$, then Σ is called a *Cauchy surface* and one says that space-time is globally hyperbolic. We refer the reader to HE or Wald's book for a detailed study of global hyperbolicity and its implications, of which we only mention one. This is that the topology of a globally hyperbolic space-time is that of $\mathcal{M} = \mathbb{R} \times \Sigma$, where Σ is a Riemannian three-manifold which is also a Cauchy surface for $\mathcal{M}$.
If spacetime is not globally hyperbolic, then $D(\Sigma)$ is a proper region and should have a boundary. The boundary of $D(\Sigma)$ is called the *cauchy horizon* of Σ. Beyond this boundary, the state of matter fields or of the gravitational field is not uniquely determined by the initial data on Σ. Such horizons occur in many black holes spacetime.
[Def.1] Space-time is asymptotically simple and empty if it can be embedded in a strongly causal space-time $(\tilde{M}, \tilde{g})$ as a manifold with boundary, denoted $\mathcal{I}$, in such a way that

(a) $\tilde{g}_{ab} = \Omega g_{ab}$ in $\tilde{M} \cap M$, where Ω is a differentiable, positive function.

(b) On $\mathcal{I}$, $\Omega = 0$ and $d\Omega \neq 0$.

(c) Every null geodesic in M has two end points on $\mathcal{I}$.

(d) $R_{ab} = 0$ in an open neighbourhood of $\mathcal{I}$ in $M \cap \tilde{M}$.

Remark This definition is motivated by the following mathematical fact[19, 20]: let $\tilde{g}_{ab}$ be the metric of the Einstein static universe and η_{ab} the Minkowski metric. Then there exists a conformal isometry of Minkowski space into an open region of the Einstein static universe, where $\tilde{g}_{ab} = \Omega\eta_{ab}$. The boundary of this region is the union $\mathcal{I}^+ \cup \mathcal{I}^- \cup i^+ \cup i^- \cup i^0$, where $i^\pm$ are points representing timelike future and past infinity, in the sense that every timelike geodesic begins at i^- and ends at i^+, i^0 is a point representing spacelike infinity and $\mathcal{I}^\pm$ are two null hypersurfaces representing null future and past infinity. Moreover the conformal factor vanishes when restricted to the boundary. This is why this boundary can be thought as giving a precise definition of "infinity" in Minkowski space. One can see this writing the Minkowski metric in the form

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.8}$$

and defining $u = t - r = \tan[(T - R)/2]$, $v = t + r = \tan[(T + R)/2]$, where $-\pi < T + R < \pi$, $-\pi < T - R < \pi$, $0 \leq R \leq \pi$. Then one finds

$$ds^2 = \frac{1}{4}(\cos[(T + R)/2]\cos[(T - R)/2])^{-2}(-dT^2 + dR^2 + \sin^2 R(d\theta^2 + \sin^2\theta d\phi^2)) \tag{2.9}$$

The metric on $\mathbb{R} \times S^3$

$$\tilde{ds}^2 = -dT^2 + dR^2 + \sin^2 R(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.10}$$

is just the Einstein static metric, except that the coordinates are restricted by the above inequalities.
The 3-surface $\mathcal{I}^+$ is the set $T + R = \pi$, i^0 is the point $T = 0$, $R = \pi$, $\mathcal{I}^-$ is the set $T - R = -\pi$ and $i^\pm$ are the points $R = 0$, $T = \pm\pi$.
An asymptotically simple space is similar to Minkowski space at infinity since it enjoys the following properties (the proof is in [19] or [20]:

(i) Every null geodesic in M reaching $\mathcal{I}$ has unbounded affine parameter and $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$, where each component $\mathcal{I}^\pm$ is a null, connected hypersurface with topology $\mathbb{R} \times S^2$. Each component is generated by a congruence of null geodesics without end points in $\tilde{M}$.

(ii) (M, g) is causally simple, globally hyperbolic and topologically $\mathbb{R}^4$.

(iii) $\tilde{\nabla}_a \partial_b \Omega = 0$ on $\mathcal{I}$.

(iv) The metric tensor of M approaches the flat Minkowski metric near $\mathcal{I}$.

By property (i) $\mathcal{I}$ represents null infinity of M, with $\mathcal{I}^+$ the set of all future points of null geodesics and $\mathcal{I}^-$ the set of all past points of null geodesics. By property (ii), the Cauchy surface of (M, g) has the topology of $\mathbb{R}^3$. By property (iii), the null geodesics generating the boundary are non expanding and non rotating. By property (iv), the space is flat at null infinity, in particular the Weyl tensor vanishes at $\mathcal{I}^\pm$ reaching zero value with various powers of the affine parameter along null geodesics (the so called peeling properties).
The condition (c) in the definition is very strong, since it excludes many space-times of interest. Then the notion of a weakly asymptotically simple space is introduced by replacing (c) with

(e) There is an open region $\mathcal{U}$ of M which is isometric to an open neighbourhood $\mathcal{U}'$ of $\mathcal{I}'$ in some asymptotically simple and empty space M'.

This notion of asymptotic (null) flatness can be considerably extended to include spatial infinity in the conformal manifold $(\tilde{M}, \tilde{g})$ [23, 20]. However, most results can be understood without a precise notion of spatial infinity, so we prefer not to embark in this rather technical definition.

2.3 Black Holes

The gravitational collapse of matter or radiation is expected to produce space-time singularities which means, by definition, that there are incomplete causal curves, i.e curves which are inextendible in at least one direction but having finite affine length (for a detailed discussion of this concept of "singularity", see [19, 20]). If there were a naked singularity then there would be past directed causal curves from $\mathcal{I}^+$ which do not intersect Σ because they end in the singularity before meeting Σ. In other words, one does not know what comes out from the singularity. However, in a spherically symmetric collapse the following properties hold:

(1) The metric outside the star is the Schwarzschild solutions (Birkhoff theorem). However, the surface of the star passes inside the Schwarzschild radius $r = 2M$, after which there will be closed trapped surfaces around the star. A closed trapped surface is a spacelike 2-surface such that both families of null geodesics orthogonal to it are converging.

(2) There is a space-time singularity.

(3) The region containing the singularity is not visible to observers that remain outside the Schwarzschild radius.

Hence the incomplete curves enter in a region of space-time that cannot be seen at future null infinity (i.e. at $\mathcal{I}^+$). This is the black hole region of space-time. The same conclusion can in fact be drawn for small departures from spherical symmetry. This is why one expects that what happens in the space-time at large distances from the collapsing body is predictable from a partial (at least) Cauchy surface, or that the region outside the black hole is predictable. This idea of predictability is captured by the following definition.
[Def.2] A weakly asymptotically simple and empty space-time is future asymptotically predictable if there exists a partial Cauchy surface Σ such that $\mathcal{I}^+$ is contained in the closure of $D^+(\Sigma)$ in the manifold $\tilde{M}$ (we have to take the closure because $\mathcal{I}^+$ is not in $D^+(\Sigma) \cap M$).
[Def.3] A future asymptotically predictable space contains a black hole if $B = M - J^-(\mathcal{I}^+)$ is non empty, in which case one calls B the black hole region of space-time. The boundary of $J^-(\mathcal{I}^+)$, denoted H, is called the event horizon of the black hole. It is the boundary of the region from which particles or photons can escape to infinity.
The black hole region B is a closed set, because if $q \in \mathcal{I}^+$ and $p \in J^-(q) \cap M$, then $p \in I^-(r)$ for all $r \in \mathcal{I}^+$ to the future of q. This is a consequence of the following fundamental proposition [19]:

"If p and r are joined by a causal curve which is not a null geodesic then they can also be joined by a timelike curve."

But $I^-(r)$ is open, so there is a neighbourhood of p which is contained in $I^-(r)$, hence contained in $J^-(\mathcal{I}^+) \cap M$. Therefore $J^-(\mathcal{I}^+) \cap M$ is open, and B is closed. In particular, the event horizon is contained in B so each null geodesic within the horizon has no future end points

on $\mathcal{I}^+$. Actually one has the stronger result that there are no future end points at all [19, 20]. This is one crucial property of the horizon (see also Hawking in [24] for a very readable account) and comes from the fact that H is a boundary.
The second property is that no points p, q in H can have timelike separation, for otherwise one could find $x \in B$ in a neighbourhood of p and $y \in M - B$ in a neighbourhood of q and a timelike curve from x to y. Then x could be seen at infinity contradicting the fact that it was in B.
From Def.(2) it follows that $J^+(\Sigma) \cap J^-(\mathcal{I}^+) \subset D^+(\Sigma)$. If in addition $J^+(\Sigma) \cap H \subset D^+(\Sigma)$, the space is said to be *strongly asymptotically predictable* from Σ. In other worlds, it is required that one can predict a neighbourhood of the event horizon. This is to avoid the possibility that a point in $J^+(\Sigma) \cap H$ which is not in $D^+(\Sigma)$ could be seen from infinity by the smallest perturbation. An example of a space which is not future asymptotically predictable is the Schwarzschild's solution with negative mass. It can be written

$$ds^2 = -\left(1 + \frac{2M}{r}\right) dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.11}$$

where $M > 0$ and $(v, u) \in \mathbb{R}^2$ are null coordinates. Setting $V = -\exp(-v/4M)$ and $U = \exp(u/4M)$ one has

$$ds^2 = -\frac{16M^2}{r}\exp(r/2M) dUdV + r^2(U,V)d\ell^2 \tag{2.12}$$

where r is implicitly determined from the equation

$$UV = -(r + 2M)\exp(-r/2M) \tag{2.13}$$

Space-time is the open region of the (V, U)-plane given by $0 > UV > -2M$, since $r > 0$, times the 2-sphere. The set $\{V = 0, U > 0\}$ is future null infinity $\mathcal{I}^+$ and the set $\{U = 0, V < 0\}$ is past null infinity. The point $V = U = 0$ represents spacelike infinity because it is at spacelike separation from all the other points. One may compress the (U, V) coordinates to a finite range setting $U = \sqrt{2M}\tan P$, $V = -\sqrt{2M}\cot Q$ with $0 < P, Q < \pi/2$ and $Q > P$. This gives the Penrose diagram shown in Fig. [1] from which one can see the space is weakly asymptotically simple but not future asymptotically predictable, the reason being that the singularity has a timelike character.

In a future strongly asymptotically predictable space one can construct a one parameter family, $\mathcal{C}$, of partial Cauchy surfaces, $\Sigma(t)$, in $D^+(\Sigma)$ such that [19]

(i) for $t_2 > t_1$, $\Sigma(t_2) \subset J^+(\Sigma(t_1))$.

(ii) each $\Sigma(t)$ intersects $\mathcal{I}^+$ in a 2-sphere $S(t)$.

(iii) for each $t > 0$, $\Sigma(t) \cup [\mathcal{I}^+ \cap J^-(S(t))]$ is a Cauchy surface for $D^+(\Sigma)$.

Although the detailed proof [19] of this statement is a little bit non trivial, one can see this remembering that there is a nowhere vanishing timelike vector field K in $(\tilde{M}, \tilde{g})$ which, being defined everywhere, is complete. One may deform K so as to yield a timelike vector field in M which is tangent to the boundary at $\mathcal{I}^+$. Thus a family of partial Cauchy surfaces $\Sigma'(t)$ can be defined by moving Σ a parameter distance t along the integral curves of K, and property (i) holds. Property (ii) holds if one takes the initial surface such that $\Sigma \cap \mathcal{I}^+$ is a 2-sphere. Then

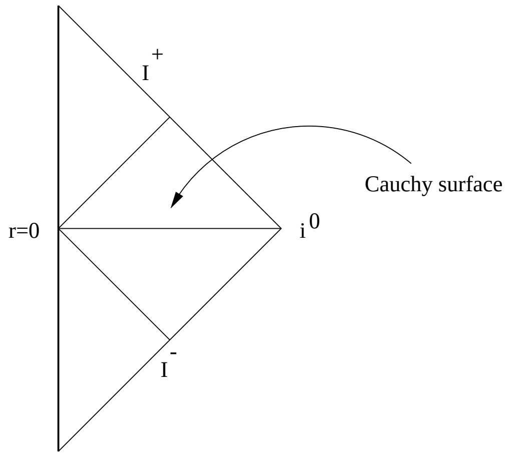

Figure 1: The Penrose's diagram of the negative mass Schwarzschild's solution showing the timelike character of the line $r = 0$ and a Cauchy's surface. Each point represents a 2-sphere.

the surfaces $\Sigma(t) = [\Sigma^{'}(t) \cap M] \cup [\mathcal{I}^{+} \cap J^{-}(S(t))]$ also have property (iii).
The position of the horizon in unknown to any observer at any finite time, because no one knows what will fall down the black hole. Nevertheless, one can say whether there will be a non trivial event horizon by seeking whether there are trapped surfaces inside spacetime.
If S is a spacelike orientable two manifold (this is a two-surface whose tangent vectors are spacelike) one can define on S two null vector fields l_a, n_a, which are orthogonal to S. We shall suppose that l_a is outgoing, n_a is ingoing and moreover that $g_{ab}l^a n^b = -1$. The induced metric in S can then be written as $\gamma_{ab} = g_{ab} + l_a n_b + n_a l_b$. The two null fundamental forms of S are the symmetric tensor fields on S defined by

$$\Theta^1_{ab} = \gamma^c_a \gamma^d_b \nabla_c l_d, \qquad \Theta^2_{ab} = \gamma^c_a \gamma^d_b \nabla_c n_d \tag{2.14}$$

where the indices are raised and lowered with g_{ab}. Then $\Theta^1_{ab} l^a = \Theta^2_{ab} l^a = \Theta^1_{ab} n^a = \Theta^2_{ab} n^a = 0$. The traces

$$\theta_1 = \nabla_a l^a \qquad \theta_2 = \nabla_a n^a \tag{2.15}$$

physically represent the rate of expansion of the two families of null geodesics orthogonal to S and tangent to l^a and n^a respectively. To see this, one computes the Lie derivative of the induced metric of S in the null direction l^a, with the result

$$\mathcal{L}_l(\gamma_{ab}) = \nabla_a l_b + \nabla_b l_a + \mathcal{L}_l(l_a n_b + l_b n_a) \tag{2.16}$$

The last two terms can be computed using the formula $\mathcal{L}_u v_a = u^c \nabla_c v_a + u_c \nabla_a v^c$, which is valid for any pair of vector fields u^a and v^a. Hence, letting γ to denote the determinant of γ_{ab}, one has

$$\mathcal{L}_l \sqrt{\gamma} = \frac{1}{2}\sqrt{\gamma}\gamma^{ab}\mathcal{L}_l \gamma_{ab} = \theta_1 \sqrt{\gamma} \tag{2.17}$$

as a simple calculation confirms. The area of a spacelike two section, $S(\lambda)$, of the outgoing null family which coincides with S at same value, λ_0, of the affine parameter λ, is $A(\lambda) = \int_{S(\lambda)} \sqrt{\gamma} d^2x$.

It follows that

$$\frac{dA(\lambda)}{d\lambda} = \int_{S(\lambda)} \theta_1(\lambda)\sqrt{\gamma}d^2x \tag{2.18}$$

This shows the relation between expansion and area increase (or decrease). At the value $\lambda = \lambda_0$, $\theta_1(\lambda_0)$ is equal to the trace of the second fundamental form Θ^1_{ab}. Therefore, if $\theta_1 < 0$ then the outgoing flash of light from S start converging. This motivates the definition

[Def.4] A closed trapped surface is a compact spacelike two manifold without boundary, such that $\theta_1 < 0$ and $\theta_2 < 0$.

In the following we shall use the following two results from the theory of geodesics, the proof of which can be found in HE, Wald or ON. The material in this section can also be found in Hawking's lecture in [24].

(a) Let S be a spacelike 2-manifold and $\gamma(\lambda)$ an outgoing null geodesic from $q \in S$, which is orthogonal to S. If $\theta_1(q) < 0$ and $R_{ab}K^aK^b \geq 0$ for all null vectors K, then there is a point conjugate to S along $\gamma(\lambda)$ to within an affine distance $\lambda_1 = 2/|\theta_1(q)|$.

(b) If there is a conjugate point to S along $\gamma(\lambda)$ between S and $p \in \gamma(\lambda)$, then there is a timelike curve from S to p.

Here, a point q is conjugate to S along γ if there is a solution of the equation of geodesic deviation

$$\frac{\nabla^2\xi^a}{d\lambda^2} + R^a_{bcd}v^b\xi^c v^d = 0, \qquad v^a = \frac{dx^a}{d\lambda} \tag{2.19}$$

which starts tangent to S and is zero at q. If this is the case, then the null geodesics starting from S in a neighbourhood of γ will converge toward q, although they don't need to intersect γ at q. A point like q, where future directed null geodesics intersect γ, even at infinitesimally small angles, represent a future end point for γ. It may also be remarked the significance of $R_{ab}K^aK^b \geq 0$ as an energy condition. This can be seen from Einstein equations since they imply $T_{ab}K^aK^b \geq 0$ for all null K^a if the condition holds. The validity of $T_{ab}\xi^a\xi^b \geq 0$ for all timelike ξ^a, and for all null ξ^a as well by continuity, is called the weak energy condition. It states that the rest energy density of matter, ρ, is non negative and that the pressures are less than ρ in absolute value[19].

The following property then shows that closed trapped surfaces cannot be seen at infinity in a black hole space-time.

[Prop.1] Let $\mathcal{F}$ be a closed trapped surface in $D^+(\Sigma)$ in a future asymptotically predictable space-time satisfying $R_{ab}K^aK^b \geq 0$ for all null vectors K. Then $\mathcal{F}$ cannot intersects $J^-(\mathcal{I}^+)$.

In fact, that $\mathcal{F}$ can be seen at infinity simply means that $J^+(\mathcal{F})$ intersects $\mathcal{I}^+$, and therefore there ought to be a null geodesic in the boundary, $\partial J^+(\mathcal{F})$, from $\mathcal{F}$ to $\mathcal{I}^+$ (actually this is the tricky part of the proof). Such a geodesic would intersect $\mathcal{F}$ orthogonally, because it is in $\partial J^+(\mathcal{F})$, and moreover it would have the properties

(i) it has infinite affine length, since it reaches $\mathcal{I}^+$.

(ii) it has a past end point on $\mathcal{F}$, because otherwise it would intersect Σ (a partial Cauchy surface), and pass to the past of Σ, which is impossible as $\mathcal{F} \subset D^+(\Sigma)$.

However, from (a) there is a conjugate point q along the geodesic within an affine distance $v_1 = 2/|\theta_1|$ from $\mathcal{F}$, and from (b) every point of the geodesic which is to the future of q is actually in the interior of $J^+(\mathcal{F})$, rather that in the boundary. This contradicts the assumption that the geodesic was in the boundary all the way out to $\mathcal{I}^+$.

Hence if a trapped surface exists and nature obey asymptotic predictability, there will be an event horizon somewhere in the future.
Recall that in a black hole space-time there is a family, $\mathcal{C}$, of partial Cauchy surfaces $\Sigma(t)$. These can be used to define the notion of a black hole at a particular instant of time.
[Def.5] A black hole at time t, $B(t)$, is a connected component of $B \cap \Sigma(t)$, where $\Sigma(t)$ is a partial Cauchy surface in the family $\mathcal{C}$, and B is the black hole region of spacetime. The boundary of a black hole, $\partial B(t)$, is a connected component of $H \cap \Sigma(t)$.
Black holes can collide and merge together like in ordinary collision processes. Strangely enough, one has the following result [19, 20],
[Prop.2] Black holes cannot bifurcate.

In fact, if a black hole $B(t)$ in a surface $\Sigma(t)$ bifurcates into two black holes $B_1(t^{'})$, $B_2(t^{'})$ in a later surface $\Sigma(t^{'})$, with $t^{'} > t$, then $B_1(t^{'}) \cap B_2(t^{'}) = \emptyset$ and both intersect $J^+(B(t)) \cap \Sigma(t^{'})$. Hence $B \subset I^-(B_1) \cup I^-(B_2)$ but no point, p, of $B(t)$ can be in $I^-(B_1) \cap I^-(B_2)$ since each inextendible, future directed curve from p would intersects $\Sigma(t^{'})$ either in B_1 or in B_2. Hence one could divide the future half cone at p in two disjoint open sets, according to whether a curve from p intersects B_1 or B_2, which is impossible since it is topologically connected. Therefore $I^-(B_1)$ and $I^-(B_2)$ are disjoint open sets whose union contains $B(t)$, which is again impossible since $B(t)$ is connected. It follows that $B_1(t^{'})$ and $B_2(t^{'})$ cannot be disjoint and therefore $B_1(t^{'}) = B_2(t^{'})$.
It may be noted that $J^+(B(t)) \cap \Sigma(t^{'}) \neq \emptyset$ because $\Sigma(t^{'})$ is a Cauchy surface lying to the future of $\Sigma(t)$. Hence black holes cannot even disappear. In contrast, black holes may merge and form new black holes. Apparently this seems to be the time inversion of the splitting process, which is forbidden as we have seen. The asymmetry is contained in the property of the event horizon that its null generators have no future end points. Thus if a black hole could split, there would be generators with a future end point, which is impossible. The argument does not work backward in time, since the generators may have past end points (where the horizon born). The time inverted situation refer to white holes. For them it is possible to bifurcate but impossible to merge.
Finally, one has the following weaker version of Hawking's area theorem. Let $\partial B(t) = H \cap \Sigma(t)$, where H is the event horizon of the black hole. Then $\partial B(t)$ is a spacelike two manifold which represents the boundary of the black hole region $B(t) = [M - J^-(\mathcal{I}^+)] \cap \Sigma(t)$ at time t. One has the
[Area Theorem] If (M, g) satisfies the null convergence condition, $R_{ab}K^aK^b \geq 0$ for all null K^a, then the area of $\partial B(t)$ is a non decreasing function of time.

This is because the null geodesic generators of the horizon which are orthogonal to $\partial B(t)$ have non negative expansion, $\theta(t) \geq 0$. For if $\theta(t) < 0$ at p, then by property (a) the null geodesic in H from p would have a conjugate point q within an affine distance $v_1 = 2/|\theta(t)|$. By property (b), all points beyond q along the geodesic would be at timelike separation from p, which cannot be the case since no points in H are timelike separated. Therefore they must belong to the interior of the black hole region. This means that the generator from p cannot remain in H all the way out to $\mathcal{I}^+$, or that it has a future end point at or before q. This is again impossible

since the geodesics generators of H have no future end points. Therefore $\theta(t)$ is non negative and Eq. (2.18) now gives the conclusion.
In practice, if θ were negative, the null geodesics in the horizon would start converging and fall inside the black hole. Thus they could not form the boundary of the exterior region.

2.4 Gravitational collapse

The relevance of the above (perhaps technical) definition of a black hole, roughly speaking a region of no escape, stems mainly from the fact that the gravitational collapse of a spherically symmetric star produces a black hole, rather than a naked singularity (a general reference for gravitational collapse is Ref. [25] and [26]. Many other references and some calculation can be found in Weinberg's book[27] and Ref. [28]).
The equations of hydrostatic equilibrium in general relativity imply that no regular, stable solutions exist for a star with mass M and radius R if $M/R > 4/9$ [27, 20]. Hence if a compact collapsing star will not manage to eject enough matter so as to fall below this inequality it will collapse to a singular state. By taking into account the degeneracy pressure of electrons (due to the Pauli exclusion principle), which is much stronger than the radiation pressure, one can see that no stable equilibrium configurations exist for cold stars whose mass is higher than a certain critical value of order $M_C \simeq (\hbar c)^{3/2} G^{-3/2} m_n^{-2} \simeq M_\odot$, where m_n is the nucleon mass (an informal discussion on this can be found in[19] and more detailed calculations are given in[27, 28]. See also [29, 30]). If the mass is lower than M_C then there is an equilibrium configuration, if not the star will collapse reducing the phase space available to electrons. Then it becomes favourable for protons to convert into neutrons by inverse beta decay after which the degeneracy pressure of cold degenerate neutrons can stop the collapse to another equilibrium configuration known as a neutron star. This happens for masses not greater than a certain critical value, after which a radial instability sets in and then the star is believed to collapse indefinitely, with a time scale of order $MG/c^3 \simeq 10^{-5}(M/M_\odot)$sec. Of course if the star reaches the Schwarzschild radius with nuclear density, the strongly interacting nuclei can stop the collapse[31]. However, the conditions at the Schwarzschild radius need not be so extreme (for example if 10^8 solar masses were to collapse, the density at the Schwarzschild radius would be only 10^2 times the density of water, which is more or less the central density of our sun. Nuclear densities can have small black holes of about 10^{13} grams).
Consider now a spherically symmetric star. By the Birkhoff theorem, the exterior geometry must be described by the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.20}$$

The radius of a static star must be greater than $r_g = 2M$, because there is no timelike Killing field for $r < r_g$ and the surface of a static star must correspond to the orbit of a timelike Killing field. Therefore, if the radius is less than r_g, then the star must either contract or expand, but the exterior solution is always given by the Schwarzschild metric. To show that it contains a black hole, one has to find the extension of the metric beyond the surface $r = 2M$, which corresponds to a coordinate singularity in the (t, r, θ, ϕ) local chart.

2.5 The Kruskal's manifold

One considers Eq.(2.20) and let $r^* = r + 2M\log(r/2M - 1)$, $v = t + r^*$ and $u = t - r^*$. In terms of (u, v) the metric takes the double null form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dvdu + r^2(v,u)d\ell^2 \tag{2.21}$$

which is still singular at $r = 2M$. The coordinate transformation $U = -\exp(-u/4M)$, $V = \exp(v/4M)$, gives the Kruskal form of the metric[32, 19]

$$ds^2 = -\frac{32M^3}{r}\exp(-r/2M)dVdU + r^2(V,U)d\ell^2 \tag{2.22}$$

where $r(V,U)$ is implicitly determined from the equation

$$VU = -\left(\frac{r}{2M} - 1\right)\exp(r/2M) < 1 \tag{2.23}$$

The left hand side of Eq.(2.23) is a monotonically increasing function of r in the full range $0 < r < \infty$, so the function $r(V,U)$ will be well defined in this range even if initially r was restricted to be greater than r_g. The original region of the Schwarzschild solution is given by $V > 0$ and $U < 0$, but the metric tensor is now defined and non singular in the larger open region $\mathcal{U}$ of the (V,U)-plane where $VU < 1$, corresponding to the condition $r > 0$. This is the maximally extended Schwarzschild solution, or the Kruskal manifold. It is the manifold $\mathcal{U} \times S^2$ with the Kruskal metric given above. The topology is clearly that of $\mathbb{R}^2 \times S^2$. The space is time orientable, since the vector field $K = \partial_V + \partial_U$ is everywhere timelike and non vanishing, it is connected, weakly asymptotically simple and strongly asymptotically predictable.
From Fig.[2], we see that it contains a black hole region, from which nothing can escape to infinity, as well as a white hole region, to which nothing can enter from past infinity. The boundary of these regions is the bifurcate null hypersurface $r = 2M$, which represents the event horizon of the black hole. There also exist another asymptotically flat region, the subset $V < 0$, $U > 0$, which is connected to the original static region by a throat (remember that each point in the diagram represents a 2-sphere).
The Penrose's diagram of the (V,U) section can be obtained by introducing the new coordinates

$$p = \arctan\left(\frac{V}{\sqrt{2M}}\right), \qquad q = \arctan\left(\frac{U}{\sqrt{2M}}\right) \tag{2.24}$$

which have finite range $-\pi/2 < p, q < \pi/2$, $-\pi/2 < p+q < \pi/2$. Then $\mathcal{I}^+$ is the open set $p = \pi/2$ or $q = \pi/2$, $\mathcal{I}^-$ is the open set $q = -\pi/2$ or $p = -\pi/2$. The points $i^0 = \{p = \pi/2, q = -\pi/2\}$ and $i^0 = \{p = -\pi/2, q = \pi/2\}$ (the two corners in Fig.[2]) represent spacelike infinity, because they are evidently at spacelike separation from all other points. The (U,V) part of the metric is conformally flat and has the form $\Omega(U,V)dUdV$ in the (U,V) coordinate system. Therefore light rays are represented by lines having the slope at $\pm 45^0$.

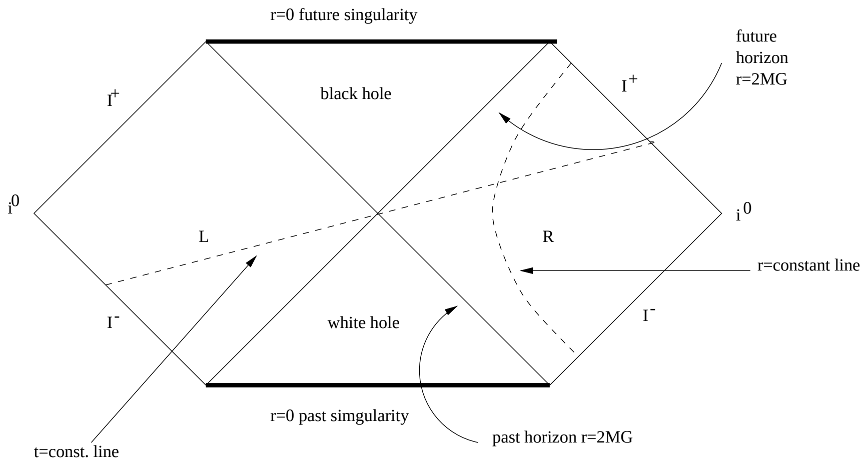

Fig.2 The Penrose's diagram of the Kruskal's manifold[1]. The upper region within the singularity and the horizons is the black hole, the lower one is the white hole, each point represents a 2-sphere.

For a collapsing star there is not a past event horizon since the past region is actually covered by the collapsing matter. The Penrose diagram representing the collapse situation is shown in Fig.[3]. The important point is that since the exterior solution is the Schwarzschild one there will be a black hole region to the future of the star, because the surface $r = 2M$ is an event horizon in the extended manifold. Moreover, the space is still future asymptotically predictable from a Cauchy surface, the reason being that the singularity is hidden inside the event horizon and is to the future of the exterior static region. Closely related is the fact that it has a spacelike character.

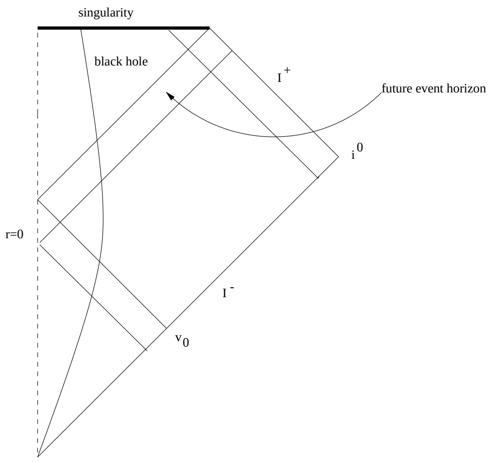

Fig.3 The Penrose's diagram of spherical collapse. The curved timelike line represents the history of the star surface, the straight lines are null geodesics and the one labelled by v_0 marks the birth of the event horizon.

These claims can be verified in the important case of incoherent collapsing matter, since an exact solution is then available[27, 33]. The interior metric of the spherical ball of incoherent

[1]The constant time lines are erroneously drown straigth, while they really terminate at spacelike infinity i^o

matter is simpler in a comoving system of coordinates, where it takes the form of a spatially homogeneous and isotropic metric

$$ds^2 = -dt^2 + R^2(t)[(1-kr^2)^{-1}dr^2 + r^2 d\ell^2] \tag{2.25}$$

If the energy momentum tensor is that of a perfect fluid with zero pressure, namely if $T_{ab} = \rho(t)u_a u_b$ and $u^2 = -1$, then the field equations give the curvature constant

$$k = \frac{8\pi}{3}\rho(0)$$

the conservation law

$$\rho(t)R(t)^3 = \rho(0)R(0)^3$$

and the evolution equation

$$\dot{R}^2(t) = k[R^{-1}(t) - 1]$$

to which one gives the initial condition $\dot{R}(0) = 0$. The scale of r is fixed if one also imposes that $R(0) = 1$. It can be easily shown that $R(t)$ vanishes when

$$t = t_c = \frac{\pi}{2\sqrt{k}} \tag{2.26}$$

so a ball with initial density $\rho(0)$ and initially at rest will collapse to a state of infinite density and curvature in the finite time t_c. The exterior solution is the Schwarzschild solution for some mass M. This solution fits continuously with the interior solution if and only if $M = 4\pi\rho(0)a^3/3$, where $r = a$ is the initial radius of the star. It it then easy to check the validity of the properties (1), (2) and (3) which were given in Sec.(2.2).
The study of linear perturbations in Schwarzschild space-time has shown that small departures from spherical symmetry do not produce a naked singularity but a future asymptotically predictable space-time. Numerical studies of linear perturbations of other black hole solutions also support this conclusion. Together with a number of unsuccessful theoretical attempts to prove the occurrence of naked singularities in gravitational collapse, this led to "the cosmic censorship conjecture" [34]: gravitational collapse always produces a black hole rather than a naked singularity (for a detailed discussion of cosmic censorship see Wald's [20]). In some situation, violation of cosmic censorship was shown to be equivalent to a violation of the second or third law of thermodynamics [35]. Therefore the notion of black hole that is commonly accepted is just that of a space-time containing a region of no escape but with the important requirement that one can still predict what happens at future infinity, namely that of a strongly, future asymptotically predictable space-time.

2.6 The final state of black holes

The likely outcome of gravitational collapse should be represented by a stationary solution outside the event horizon. This is because once matter falls inside the horizon, no matter nor gravitational radiation would escape to reach future infinity. Therefore it is important to study black hole solutions which are exactly stationary, which means that there is an isometry group $\psi_t : M \to M$ whose Killing vector field K^a is timelike at sufficiently large distances from the black hole. Moreover, one considers black holes for which it is possible for matter or radiation

to fall into, namely for which the boundary $\partial B(t)$ of the black hole is contained in $J^+(\mathcal{I}^-)$ for sufficiently large t. Now black holes that formed from the collapse of stellar masses which were initially dispersed, should be represented by spaces which have an asymptotically simple past. Hence they can be predicted from a Cauchy surface that was topologically $\mathbb{R}^3$, before the formation of the horizon. After the horizon formed, the external region of the surface should be topologically $\mathbb{R}^3$ minus a the compact region which contains the black holes. In the study of the possible final states, it is therefore convenient to forget about the remote past of the collapsed matter and look for spaces that can be predicted from a Cauchy surface which is topologically $\mathbb{R}^3$ in the external region of the black holes that possibly formed. The best reason for introducing such spaces, however, is that it is for them that one can prove the uniqueness theorems.
[Def.6] A space-time (M, g) is a regular predictable space if it is strongly future asymptotically predictable from a partial Cauchy surface Σ such that

(a) The intersection of Σ with the closure of $J^-(\mathcal{I}^+)$ in $\tilde{M}$ is $\mathbb{R}^3 - V$, where V is an open set with compact closure.

(b) Σ is simply connected.

(c) for sufficiently large t, $\partial B(t) \subset \bar{J}^+(\mathcal{I}^-)$, the bar over J^+ denoting the closure of J^+.

In such a space, all surfaces $\Sigma(t)$ have the properties (a) and (b) and moreover $\partial B(t)$ is compact and connected [19].
Next one considers a regular stationary predictable space-time which is also empty, i.e. $R_{ab} = 0$. If the corresponding Killing field is hypersurface orthogonal, $\varepsilon^{abcd} K_a \nabla_b K_c = 0$, the space-time is said to be static, in which case $K_a = \partial_a f$ for some function f. The expectation is then that the exterior region of a black hole will be isometric to a region of such a stationary or static regular predictable space. In a static regular predictable space-time, it can be proved that the Killing vector is actually timelike everywhere in the exterior region $J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)$ [19]. It follows then that time inversion is an isometry and therefore if there exist a future event horizon, then there must exist a past event horizon too. For stationary rotating black holes, it can be proved [19] that K_a must be spacelike in part of the exterior region $J^+(\mathcal{I}^-) \cap J^-(\mathcal{I}^+)$. This is essentially the Lense-Thirring dragging effect, for were K_a to be timelike and Killing everywhere, it would be possible for a particle to stay at rest relative to infinity, i.e. to move along an integral curve of K_a, instead to be forced to rotate around the black hole. The region where K_a is spacelike is called the ergosphere.
The first result toward the black hole uniqueness theorem of Carter [13] and Robinson [14], was given by Hawking who proved that a vacuum, stationary black hole solution must be static or axisymmetric [19]. This means that there exist another one parameter group of isometries, commuting with ψ_t, and whose orbits are closed and spacelike in the exterior region, in other words these are rotations around the symmetry axis. For such stationary and axisymmetric solutions one has then the theorem due to Carter [13, 24], according to which
Carter's Theorem All stationary, axisymmetric and regular predictable space-times such that

(1) $R_{ab} = 0$;

(2) the past event horizon $\dot{J}^+(\mathcal{I}^-)$ intersects the future event horizon $\dot{J}^-(\mathcal{I}^+)$ in a compact two manifold,

fall into disjoint families each depending on, at most, two parameters.

A priori, there could be many disjoint families, but Robinson demonstrated that, in fact, only one such family exists, namely the Kerr family with $|J| \leq M^2$ [14], M and J being the mass and the angular momentum of the solution. Therefore the only stationary axisymmetric black hole solutions must belong to this Kerr family. Hence, if cosmic censorship is correct and the final state is stationary, the exterior region of a rotating black hole must be described by the Kerr solution or its charged generalization, the Kerr-Newman metric. The theorem of carter contains as a special case the earlier theorem of Israel [36], according to which the Schwarzschild solution with positive mass is the unique static solution containing a black hole. One can admit the presence of the electromagnetic field, in which case there is a more general theorem according to which the final state will be represented by the Reissner-Nordström solution with $|Q| \leq M$. These conclusions are all the way striking. In fact, the gravitational field of an isolated boby always admits a multipole expansion with an infinite set of multipole moments [27]. Thus it would seem that a stationary black hole must also possess such an infinite set of multipole moments (hairs). But Carter theorem says that only two moments actually remain, these being the total mass and the angular momentum. Thus the dynamics of gravitational collapse is such that all the informations on the internal structure of the collapsing star are radiated away during the collapse. This is often called the Price theorem [37]:"anything which can be radiated gets radiated away completely". One can understand the loss of black hole hairs roughly as follows. The radius of a collapsing star approaches the event horizon at an exponentially fast rate and all gravitational multipole moments fall off at large t as $t^{-(2l+2)}$ or as $t^{-(2l+3)}$, depending on initial conditions [37]. The only hairs that remain are just the monopole and the dipole moment, because there in no monopole nor dipole gravitational radiation which can radiate them away. This also suggests that black holes might have a certain amount of entropy measuring how much information they contain, but that outside observers cannot probe since they can only measure the mass, the charge and the angular momentum of the black hole.

3 Black hole metrics

We begin now to describe briefly the unique stationary solution describing isolated black holes in general relativity. The most detailed account of their properties can be found in Chandrasekhar monograph [38] and also in [19], to which we mainly refer from now on. We shall also discuss some black hole's metric in Kaluza-Klein theory and solutions with a dilaton. As the Schwarzschild solution and its analytic extension has already been treated, we begin with the simplest charged black hole.

3.1 The Reissner-Nordström's metric

This solution, independently discovered by H. Reissner [39] and G. Nordström [40], represents the gravitational field outside a charged spherically symmetric body. The metric tensor is a two-parameter family labelled by r_+, r_-, and is

$$ds^2 = -r^{-2}(r - r_+)(r - r_-)dt^2 + r^2(r - r_+)^{-1}(r - r_-)-1dr^2 + r^2 d\ell^2 \tag{3.1}$$

where $d\ell^2$ is the standard metric of the unit 2-sphere. The electrostatic potential is $A_a = \delta^0_a r^{-1}\sqrt{r_+ r_-}$ and the field strength is $F_{ab} = \partial_a A_b - \partial_b A_a$. From this we deduce the charge, Q, and the mass, M, of the solution

$$M = \frac{r_+ + r_-}{2}, \qquad Q^2 = r_+ r_- \tag{3.2}$$

Obviously one has the Schwarzschild metric when $Q = 0$. If $M > |Q|$, the metric components presents two singularities at $r = r_\pm$, given in terms of mass and charge by

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{3.3}$$

and where the metric tensor blows up. These are singularities of the coordinate system, as can be seen by transforming to advanced, (v, r, θ, ϕ), or retarded, (u, r, θ, ϕ), Finkelstein null coordinates, where $v = t + r^*$, $u = t - r^*$ and

$$r^* = r + \frac{r_+^2}{r_+ - r_-} \log(r - r_+) - \frac{r_-^2}{r_+ - r_-} \log(r - r_-) \tag{3.4}$$

For example, using the parameters M, Q and the chart (v, r, θ, ϕ), one has

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dv^2 + 2dvdr + r^2 d\ell^2 \tag{3.5}$$

which is perfectly regular for $r > 0$, and in fact gives a future extension of the original chart (i.e. points with $r < r_+$ are to the future of those in the static region where (3.1) is valid). Similarly, the retarded chart (u, r, θ, ϕ) gives a past extension. The only real, irremovable singularity of the metric is in fact the locus where $r = 0$.
The 3-surface, $\mathcal{H}$, given by $r = r_+$ is null, that is the orthogonal four-vector $n_a = \partial_a r$ evaluated at $r = r_+$ is null (an hypersurface $f(x) = 0$ is null if $n_a = \partial_a f$ has zero norm at points where $f = 0$).
A null surface has half light cone from one side and the other half from the other side. Hence it acts as a one way membrane, letting photons or particles to cross only once (if strong causality holds).
Also, $\mathcal{H}$ being null, it is locally to the future or to the past of the asymptotically flat region where $r > r_+$. The portion $\mathcal{H}^+$ to the future is easily seen to be a future event horizon ($u \to \infty$ on $\mathcal{H}^+$), and the portion $\mathcal{H}_-$ to the past ($v \to -\infty$ there) is the past event horizon. Moreover, $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ and $\mathcal{H}^+$ intersects $\mathcal{H}^-$ in a two-sphere.
The area of any $t = constant$ cross section of the event horizon is

$$A = 4\pi r_+^2 = 4\pi(M + \sqrt{M^2 - Q^2})^2 \tag{3.6}$$

as it is obvious from the form of the metric tensor. This is the first important parameter of the horizon. The second is the acceleration, per unit coordinate time, given to a particle at rest near the event horizon. It is called the surface gravity and is denoted by k. So $k = -\lim_{r \to r_+} \sqrt{\alpha_a \alpha^a}$, where

$$\alpha^a = (u^0)^{-1} u^b \nabla_b u^a, \qquad u^a = (-g_{00})^{-1/2} \delta_0^a \tag{3.7}$$

is the four acceleration per unit coordinate time of a particle with four velocity u^a. One obtains easily

$$k = -\frac{1}{2}(\partial_r g_{00})_{|r=r_+} = \frac{r_+ - r_-}{2r_+^2} \tag{3.8}$$

which is evidently constant throughout the event horizon. For the Schwarzschild solution, which has $Q = 0$, one has $r_- = 0$ and $r_+ = 2M$ and therefore the surface gravity has the well known value $k = 1/4M$.
If now dA is calculated as a function of the two parameters M and Q one obtains the so called first law of black hole mechanics [41]

$$dM = \frac{k}{8\pi} dA + \Phi dQ \tag{3.9}$$

where $\Phi = Q r_+^{-1}$ is the electrostatic potential on the horizon.
The complete analytic extension of the metric (3.1) [42, 38, 19] is shown in Fig. []. In this larger manifold one can see the timelike character of the singularity, as well as the presence of Cauchy horizons.
However, if the horizon forms from the collapse of a charged spherical ball then the singularity is still hidden to the asymptotic observers by the event horizon. If $M < Q$ then the metric is inextendible and the manifold is simply the region $r > 0$ times a 2-sphere. The singularity in this case is naked. Finally, if $M = Q$ then we have the so called extreme solution, for which $k = 0$. If this solution results from the collapse of a spherical charged shell, then an horizon at $r = M$ forms but the shell bounces forming another asymptotically flat region [43], and singularities are not formed.

3.2 The Kerr's solution

This solution, discovered by R. P. Kerr [44], is believed to be the unique solution for the description of all rotating black holes that forms from the gravitational collapse of stellar masses. We write the metric in Boyer-Lindquist coordinates (t, r, θ, ϕ), which have the usual meaning at large distances from the body [45] and for which the physical meaning of the two parameters, a and M, characterizing the solution, is more evident. With the standard abbreviations [38]

$$\Delta = r^2 - 2Mr + a^2, \qquad \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \omega = \frac{a(r^2 + a^2 - \Delta)}{\Sigma^2}$$

the metric reads

$$ds^2 = -\rho^2 \frac{\Delta}{\Sigma^2} dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2}\left(d\phi - \frac{a(r^2 + a^2 - \Delta)}{\Sigma^2} dt\right)^2 \sin^2\theta \tag{3.10}$$

The coefficient of $-dt^2$ in (3.10, $N^2 = \Sigma^{-2}\rho^2\Delta$, is the lapse function of the foliation $t \to \Sigma_t$ of spacetime, determined by the stationary frame. The unit normal to a t =constant surface Σ_t is

$$u = N^{-1}(\partial_t + \omega \partial_\phi) \tag{3.11}$$

Examining the solution at large coordinate distances, one finds that M is the gravitational mass as measured from infinity and a is the angular momentum per unit mass as measured from infinity. Therefore $J = Ma$ is the angular momentum of the solution. If $M > a$ then the equation $\Delta = 0$ has two roots

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{3.12}$$

where the metric tensor blows up. These are coordinate singularities of the metric, since no component of the Riemann tensor diverges there in a local tetrad [38]. The surface $\mathcal{H}$ given by $r = r_+$ is null and therefore, as was already explained, has the property of an event horizon: either no particles can escape from beyond $\mathcal{H}$ or no particles can enter from the outer asymptotically flat region. Moreover, $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$, where $\mathcal{H}^+$ is to the future of the asymptotically flat region where $r > r_+$ and $\mathcal{H}^-$ is to the past. The area of a $t = constant$ section of the event horizon is easily seen to be

$$A = 4\pi(r_+^2 + a^2) = 8\pi M r_+ \tag{3.13}$$

The surface gravity of the horizon can be computed as for the Reissner-Nordström solution. It is the norm of the acceleration per unit coordinate time of a particle at rest relative to infinity, in the limit $r \to r_+$. With a little bit longer calculation one obtains

$$k = \frac{r_+ - r_-}{4Mr_+} \tag{3.14}$$

Another important region is where $\rho^2 = 0$. This is the only irremovable singularity in the metric [38, 19] and has the structure of a ring singularity in the equatorial plane $\theta = 0$. From Eq.(3.10) one can read off

$$g_{00} = -\left(1 - \frac{2Mr}{\rho^2}\right) \tag{3.15}$$

Therefore $g_{00} > 0$ if $R_- < r < R_+$, where $R_\pm$ are the roots of the equation $\rho^2 = 2Mr$, i.e.

$$R_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta} \tag{3.16}$$

The greater radius R_+ is the most significant. Since g_{00} is the length of the Killing, time translation symmetry generated by $K = \partial_t$, it follows that K is timelike only for $r > R_+$, it is null in the surface $r = R_+$ and spacelike in between R_- and R_+. The surface $r = R_+$ is thus the stationary limit surface of the Kerr geometry, in the sense that it is the inner boundary of the region where a particle can move along an integral curve of K (or that the world lines having r, θ and ϕ constants are timelike). The inner enclosed region is called the ergosphere. Note that the stationary limit surface is outside the horizon except at the poles $\theta = 0$ or $\theta = \pi$, where the two surfaces are tangent. In between the stationary limit surface and the horizon, a particle cannot stay at rest relative to infinity, i.e. to have constant values of r, θ and ϕ. The best we can have is constant values of r and θ, in which case the four-velocity is

$$g_{ab}u^a u^b = g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{\phi\phi}\dot{\phi}^2 < 0 \tag{3.17}$$

the overdot denoting proper-time derivatives. Since $g_{tt} > 0$ inside the ergosphere, $g_{t\phi} < 0$ everywhere and $\dot{t} > 0$ for a future directed curve, we see that $\dot{\phi} > 0$ is necessary for u^a to be timelike. Therefore a particle at a fixed coordinate distance from the body is forced to rotate relative to infinity.
However, since the coefficient of dt^2 in the metric (3.10) is negative in the ergosphere, the particle can have both r and θ constant, and then

$$\frac{d\phi}{dt} = \frac{2aMr}{\Sigma^2} \equiv \omega \tag{3.18}$$

gives a timelike line. This is the trajectory of a particle which is at rest relative to the surfaces of constant time. The value of ω on the horizon is referred to as the angular velocity of the horizon, say Ω, and is

$$\Omega = \frac{a}{2Mr_+} \tag{3.19}$$

If now dA is calculated as a function of M and J one obtains the first law for rotating black holes

$$dM = \frac{k}{8\pi}dA + \Omega dJ \tag{3.20}$$

where $J = Ma$ is the angular momentum of the Kerr solution. One can view the Kerr metric as an empty space solution of Einstein equations in which case there is a maximal extension to a larger manifold with a considerable complicated structure [45, 46, 47]. A detailed description is given in [38, 19]. However, the extended metric admits closed timelike curves near the ring singularity and therefore it is not a physically admissible solution (see also [30] for a discussion on this point).

All solutions so far discussed can be united into a single three parameter family which is known as the Kerr-Newman sequence [48]. The metric tensor is obtained by replacing $\Delta = r^2 - 2Mr + a^2$, in the Kerr solution, with $\Delta = r^2 - 2Mr + a^2 + Q^2$, and retaining the form of the metric tensor [38]. As before, M, a and Q have the physical meaning of mass, angular momentum per unit mass and electric charge. The first law takes the form

$$dM = \frac{k}{8\pi}dA + \Omega dJ + \Phi dQ \tag{3.21}$$

where $\Phi = Qr_+^{-1}$ is the electrostatic potential on the horizon.

3.3 A Kaluza-Klein black hole

If we start with the Einstein-Hilbert action in $5D$ and put the five-metric into the form (latin letters $a, b, c, ..$ are spacetime four dimensional indices)

$$ds^2 = e^{-4\phi/\sqrt{3}}(dx_5 + 2A_a dx^a)^2 + e^{2\phi/\sqrt{3}} g_{ab} dx^a dx^b \tag{3.22}$$

then the action becomes

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}[R - 2(\nabla\phi)^2 - e^{-2\phi/\sqrt{3}}F^2] \tag{3.23}$$

where $F^2 = F_{ab}F^{ab}$ with $F_{ab} = \partial_a A_b - \partial_b A_a$. It seems rather awkward to find an exact solution of the coupled equations arising from the action I. There is simple technique to generate solutions for I from simpler known solutions. We start with the product of $4D$ Schwarzschild spacetime with a line with metric $(dx_5)^2$. Thus we have the metric

$$ds^2 = -(1 - 2mr^{-1})dt^2 + (1 - 2mr^{-1})^{-1}dr^2 + r^2 d\ell^2 + (dx_5)^2 \tag{3.24}$$

which is obviously a solution of Kaluza-Klein theory with $\phi = A_a = 0$. Then we boost along the x_5-direction with boost parametr γ, compactify x_5 on a circle and compare with (3.22. In this way one obtains

$$ds^2 = -\Delta^{-1/2}(1 - 2mr^{-1})dt^2 + \Delta^{1/2}\left[(1 - 2mr^{-1})^{-1}dr^2 + r^2 d\ell^2\right] \tag{3.25}$$

where $\Delta = 1 + 2mr^{-1}\sinh^2\gamma$ and

$$e^{-4\phi/\sqrt{3}} = \Delta, \qquad A = \frac{m\sinh 2\gamma}{2r\Delta}\, dt \tag{3.26}$$

From this the electric charge is identified as

$$Q = \frac{m\sinh 2\gamma}{2} \tag{3.27}$$

and the physical mass (coming from the leading term in the asymptotic expansion of g_{tt} at large r) is

$$M = \frac{m}{2}(2 + \sinh^2\gamma) \tag{3.28}$$

The horizon is obviously at $r_+ = 2m$, so the surface gravity is

$$\kappa = -2^{-1}(g_{tt})'_{|r=r_+} = \frac{1}{4m\cosh\gamma} \tag{3.29}$$

and, finally, the horizon area, $A = 16\pi m^2 \cosh\gamma$. By direct verification, the first law take the form

$$dM = \frac{\kappa}{8\pi}\, dA + \Phi dQ \tag{3.30}$$

and there is also a Smarr-like formula for the total mass

$$M = \frac{\kappa A}{4\pi} + Q\Phi \tag{3.31}$$

We see that the description of charged black holes in Kaluza-Klein theory is very different from that provided by general relativity.

4 Quantum fields

Since black holes have so many remarkable properties it seems desirable to apply quantum ideas to them, one good motivation being that the classical theory predicts the occurrence of singularities after black hole formation. This is analogous to the collapse of the atom as was predicted by classical electrodynamics, and just as the resolution of this problem eventually led to the discovery of quantum theory so the resolution of the singular behaviour of gravitational collapse may lead to a quantum theory of gravity, which is still lacking at present. Since general relativity is a relativistic theory of gravity, one has to apply relativistic quantum mechanics, namely quantum field theory. Relatively poor results are known for interacting fields [49, 50], but since many deep aspects already appears when dealing with free fields one may restrict the discussion to a scalar free field theory. It is relatively straightforward to extend the results to fermion fields or to the electromagnetic field. Non abelian gauge fields are self interacting and therefore remain a challenge for future investigations.

4.1 Quantum theory of the scalar field

A real scalar field $\phi(x)$ in Minkowski space obeying the Poincaré invariant Klein-Gordon equation of motion, $[\Box - \mu^2]\phi(x) = 0$, can be quantized by the imposition of the following equal-time canonical commutation relations (CCR)

$$[\phi(t,X), P(t,Y)] = i\delta(X-Y), \qquad P(t,X) = \partial_t\phi(t,X) \tag{4.1}$$

all the other commutators vanishing (see any textbook in quantum field theory). Since in the CCR the time variable is singled out as a preferred coordinate, space-time was foliated with a family Σ_t of Cauchy surfaces t =constant, and when $x \in \Sigma_t$ we write $x = (t, X)$. Moreover one has $[\phi(x), \phi(y)] = iG(x,y)$, which defines the Pauli-Jordan function as the unique solution of the Klein-Gordon equation with the boundary conditions $G(x,y) = 0$ and $\partial_0 G(x,y) = \delta(X-Y)$ for $x, y \in \Sigma$. It vanishes for spacelike separation, i.e. when the square of the proper time $\sigma(x,y) = -g_{ab}(x-y)^a(x-y)^b < 0$, as one might expect from relativistic causality requirements. The quantization problem is to construct a representation of the CCR in a Hilbert space (hence the name quantum field) in which a unitary, completely reducible representation of the Poincaré group is defined (hence the name relativistic quantum field). The Fock representation is the most important up to now, and is defined informally as follows.

First, the Klein-Gordon equation has a conserved current $J_a = i\phi^* \overset{\leftrightarrow}{\partial_a} \psi$ which allows the definition of an invariant scalar product

$$(f,g) = i\int_\Sigma f^* \overset{\leftrightarrow}{\partial_a} g n^a d\Sigma \tag{4.2}$$

n^a being the future directed, timelike vector field normal to Σ. For functions of the form $\phi = (2E)^{-1/2}u(X)\exp(-iEt)$ it reduces to the usual $L^2(\mathbb{R}^3, d^3X)$-norm for $u(X)$. Then, one expands the field operator as

$$\phi(x) = \sum_i \frac{1}{\sqrt{2E_i}}[a_i u_i(X)e^{-iE_it} + a_i^\dagger \bar{u}_i(X)e^{iE_it}] \tag{4.3}$$

where $u_i(X)$ are the normalized eigenfunctions of the spatial part of the Klein-Gordon operator and the numbers E_i^2 are the corresponding eigenvalues, that is

$$[-\Delta + \mu^2]u_i = E_i^2 u_i \tag{4.4}$$

The sum over i is understood as a spectral sum, that is it may include an integral over the continuous part of the spectrum of the Laplace operator. The most common basis functions, tightly connected with the Poincaré symmetry, are of course the normalized plane waves

$$u_k(X) = (2\pi)^{-3/2}\exp(ik\cdot X), \qquad k \in \mathbb{R}^3 \tag{4.5}$$

for which $E^2 = k^2 + \mu^2$. The sum over i is to be replaced by the integral over k with the measure d^3k and the CCR translate into the following commutation relations

$$[a_k, a_q^\dagger] = \delta(k-q), \qquad [a_k, a_q] = [a_k^\dagger, a_q^\dagger] = 0 \tag{4.6}$$

One then introduces the vacuum, denoted by $|0>$ in this context, by the condition $a_k|0>= 0$. The one particle Hilbert space, H, is the space completion of the set of states of the form

$$|f>= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E_k}} f(k)a_k^\dagger|0> \tag{4.7}$$

for all $f(k)$ such that

$$\|f\|^2 =< f|f >= \int \frac{d^3k}{(2\pi)^3 2E_k} |f(k)|^2 < \infty \tag{4.8}$$

The Fourier transform of such an $f(k)$ is then a wave packet solution of the Klein-Gordon equation containing only positive frequencies and having positive norm with respect to the Klein-Gordon scalar product. The n-particle Hilbert space, H^n, is the symmetric tensor product of n copies of H and the Fock space is the direct sum $\mathcal{F} = \oplus_{n=0}^{\infty} H^n$, where $H^0 = \mathbb{C}|0>$ by definition. Since H is manifestly isomorphic to the space of square integrable function on the positive mass shell, it carries a unitary irreducible representation of the Poincaré group, which extends in a natural way on $\mathcal{F}$ as a unitary, completely reducible representation. One has then realized the Poincaré symmetry and the CCR very explicitly, and the physical picture underlying the Fock representation makes the scalar field to describe a system of non interacting, spin zero bosons. Charged, gauge and fermion fields can be treated in a similar way, as was explained in many excellent text books.
However, Eq. (4.3) can be taken to hold in the more general situation as when Σ is any Riemannian 3-manifold. Therefore one may consider space-times of the kind $\mathcal{M} = \mathbb{R} \times \Sigma$, where Σ is a Riemannian 3-manifold with metric γ_{ij}. The metric of $\mathcal{M}$ might be taken to be $g_{ab} = (-1, \gamma_{ij})$ (after Fulling, such universes are called ultrastatic space-times), or might be taken $g_{ab} = (g_{00}, \gamma_{ij})$ with g_{00} time independent (these are the static universes). There are many important examples of this more general situation. One is to take for Σ an open submanifold of Minkowski space-time, like the very important Rindler wedge, or take Σ to be a box with a given shape, as in the Casimir's configuration. Another one is $\Sigma = S^3$, in which case $\mathcal{M}$ is the Einstein static universe, for which many important calculations have been done. If Σ is geodesically complete then $\mathcal{M}$ is globally hyperbolic and Σ is a Cauchy surface [51]. Is Σ is multiply connected, then there is the possibility to consider twisted fields and inequivalent representations of the CCR. Denoting by Δ the Laplace-Beltrami operator in Σ one has $\Box - \mu^2 = -\partial_t^2 + \Delta - \mu^2$ in the ultrastatic case, and the CCR take the form

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \tag{4.9}$$

provided the eigenfunctions were normalized according to

$$\int_\Sigma \bar{u}_i(X) u_j(X) \sqrt{\gamma} d^3X = \delta_{ij} \tag{4.10}$$

The operator $[-\Delta + \mu^2]$ in a compact manifold with or without boundary is positive, each eigenvalue being finitely degenerate for well posed boundary conditions at the boundaries and infinity. In the non compact case there could be a continuous spectrum too.
For future reference, we end this section with a concise digression on the Wightman functions. From Eq. (4.3) and the definition of the vacuum it follows immediately that $< 0|\phi(x)|0 >= 0$. However, higher order monomials will not vanish in general, the meaning being that the vacuum contains non trivial correlations among the fields at different space-time points. The simplest correlations are described by the Wightman functions, or two-point correlation functions

$$W^+(t, X, Y) =< 0|\phi(t, X)\phi(0, Y)|0 > \tag{4.11}$$

$$W^-(t, X, Y) =< 0|\phi(0, Y)\phi(t, X)|0 >= (W^+(t, X, Y))^* \tag{4.12}$$

It is then easy to find the following representations

$$W^{\pm}(t,X,Y)=\sum_i \frac{1}{2E_i}u_i^*(X)u_i(Y)e^{\mp iE_it} \tag{4.13}$$

which can be derived directly from the definitions. From these expressions, it follows that W^+ is analytic in the lower half complex t-plane and W^- is analytic in the upper half complex plane. Now $W^+(t,X,Y)-W^-(t,X,Y)=G(x,y)$ is the commutator function. It follows that $W^+=W^-$ whenever $\sigma(x,y)<0$ (remember that $\sigma(x,y)$ is the square of the proper time between x and y so this is the condition for spacelike separation). It follows from this that an analytic function $\mathcal{G}(z,X,Y)$ exists such that

$$\mathcal{G}(z,X,Y)=\begin{cases} W^+(z,X,Y) & if \quad \mathrm{Im}\, z<0 \\ W^-(z,X,Y) & if \quad \mathrm{Im}\, z>0 \end{cases} \tag{4.14}$$

where $z=t+is$. The Wightman functions determine the Green functions zoo of quantum field theory. We have seen that $\mathrm{Im}\, W^+(t,X,Y)$ is the commutator. The real part is known as the Hadamard elementary function, denoted by $G^{(1)}(x,y)$. Thus

$$G^{(1)}(x,y)=<0|\phi(x)\phi(y)+\phi(y)\phi(x)|0> \tag{4.15}$$

is the anti-commutator function. An elementary calculation gives the following expression in Minkowski space

$$G^{(1)}(x,y)=\frac{\mu}{4\pi\sqrt{\sigma}}\theta(\sigma)N_1\left(\mu\sqrt{\sigma}\right)+\frac{\mu}{2\pi^2\sqrt{-\sigma}}\theta(-\sigma)K_1\left(\mu\sqrt{-\sigma}\right) \tag{4.16}$$

where $N_1(z)$ is the Neumann function of order one and $K_1(z)$ is the McDonald function of order one [52]. Both have a branch point at $z=0$ and are commonly defined in the complex plane cut along the negative real axis. One can see that $G^{(1)}(x,y)$ has locally the Hadamard singularity structure for $\sigma(x,y)$ small

$$G^{(1)}(x,y)=\frac{u(x,y)}{\sigma}+v(x,y)\log\sigma+w(x,y) \tag{4.17}$$

where u, v and w are smooth functions expandable in powers of σ with smooth coefficients. This is a very important property, since when it holds in a curved background one can define a renormalized stress tensor T_{ab} that satisfies the physical requirements expressed by the Wald's axioms [53]. The Wald-Narcowich-Fulling's theorem [54, 55] states that that the Hadamard function has the Hadamard singularity structure in any globally hyperbolic ultrastatic spacetime.
The positive frequency Wightman function in Minkowski space is

$$W^+(x,x')=\frac{\mu}{4\pi^2}\frac{1}{\sqrt{-\sigma+2i(t-t')\varepsilon}}K_1\left(\mu\sqrt{-\sigma+2i(t-t')\varepsilon}\right) \tag{4.18}$$

and the zero mass limit is easily seen to be

$$W^+(x,x')=-\frac{1}{4\pi^2}\frac{1}{\sigma(x,x')-2i(t-t')\varepsilon} \tag{4.19}$$

The analytically continued Wightman function $\mathcal{G}(z,X,Y)$ can be restricted to the imaginary axis. The restriction $S(s,X,Y)=\mathcal{G}(is,X,Y)$ is known as the Schwinger function. It is the

unique Green function of the elliptic operator $L = -\partial_s^2 - \Delta + \mu^2$ which decays (exponentially) at euclidean infinity in all directions. It can be obtained from Eq. (4.18) putting $t - t' = is$ and $\varepsilon = 0$. Conversely, one may start from the Schwinger function and analytically continuing back arrive to the Wightman functions. This continuation will depend on how one reaches the cuts on the real time axis, and this is how starting from a unique Euclidean Green function one obtains the various Green functions in Minkowski space.
Another important class of Wightman functions is obtained by defining field correlations in a thermal state. The existence of the thermal partition function normally requires systems in a finite volume and the positivity of the quantum hamiltonian. Then one may take Σ to be a compact manifold (for example a box in flat space). The thermal Wightman functions are then defined by thermal averaging the field operators

$$W_\beta^+(t, X, Y) = Z^{-1}\,\mathrm{Tr}[e^{-\beta\mathcal{H}}\phi(t_2, X)\phi(t_1, Y)] = Z^{-1}\,\mathrm{Tr}[e^{-\beta\mathcal{H}}\phi(t, X)\phi(0, Y)] \tag{4.20}$$

$$W_\beta^-(t, X, Y) = Z^{-1}\,\mathrm{Tr}[e^{-\beta\mathcal{H}}\phi(t_1, Y)\phi(t_2, X)] = Z^{-1}\,\mathrm{Tr}[e^{-\beta\mathcal{H}}\phi(0, Y)\phi(t, X)] \tag{4.21}$$

where $t = (t_2 - t_1)$ and $Z = \mathrm{Tr}\exp(-\beta\mathcal{H})$ is the partition function of the Fock hamiltonian $\mathcal{H}$ at inverse temperature β . Since $\mathcal{H}$ is positive definite, which can always be achieved by normal ordering the field operators, one can write

$$W_\beta^+(z, X, Y) = Z^{-1}\,\mathrm{Tr}[e^{i(z+i\beta)\mathcal{H}}\phi(0, X)e^{-i\mathcal{H}z}\phi(0, Y)], \qquad -\beta < \mathrm{Im}\, z < 0 \tag{4.22}$$

$$W_\beta^-(z, X, Y) = Z^{-1}\,\mathrm{Tr}[\phi(0, Y)e^{i\mathcal{H}z}\phi(0, X)e^{-i(z-i\beta)\mathcal{H}}], \qquad 0 < \mathrm{Im}\, z < \beta \tag{4.23}$$

where $z = t + is$, and moreover these functions will be analytic in the indicated strips of the complex plane. Once more

$$W_\beta^+(t, X, Y) - W_\beta^-(t, X, Y) = [\phi(t, X), \phi(0, Y)] \tag{4.24}$$

so that $W_\beta^+ = W_\beta^-$ if $\sigma(x, y) < 0$. Therefore an analytic function $\mathcal{G}_\beta(z, X, Y)$ must exist in the strip $-\beta < \mathrm{Im}\, z < \beta$, such that

$$\mathcal{G}_\beta(z, X, Y) = \begin{cases} W_\beta^+(z, X, Y) & if \quad -\beta < \mathrm{Im}\, z < 0 \\ W_\beta^-(z, X, Y) & if \quad \beta > \mathrm{Im}\, z > 0 \end{cases} \tag{4.25}$$

From the cyclic property of the traces it follows the important property

$$W_\beta^+(z - i\beta, X, Y) = W_\beta^-(z, X, Y) \tag{4.26}$$

which is known as the KMS condition [56, 57, 58]. Hence one can extend the function $\mathcal{G}_\beta$ to the complex plane as an analytic periodic function

$$\mathcal{G}_\beta(z + in\beta, X, Y) = \mathcal{G}_\beta(z, X, Y), \qquad n = \pm 1, \pm 2, \ldots \tag{4.27}$$

except for the half lines $\mid \mathrm{Re}\, z \mid > d(X, Y)$, $\mathrm{Im}\, z = \pm n\beta$, where $d(X, Y)$ denotes the geodesic distance in the manifold Σ. The thermal Schwinger function

$$S_\beta(s, X, Y) = \mathcal{G}_\beta(is, X, Y) \tag{4.28}$$

will then be periodic with period β, i.e., $S_\beta(s + \beta, X, Y) = S_\beta(s, X, Y)$. It will also be the periodic Green function of the elliptic operator $\mathcal{L} = -\partial_s^2 - \Delta + \mu^2$, namely the Green function of $\mathcal{L}$ on the cylinder $S^1 \times \Sigma$, whose circumference is equal to β. A straightforward computation gives the formula, for $0 < s < \beta$

$$S_\beta(s, X, Y) = \sum_i \frac{u_i(X)\bar{u_i}(Y)}{2E_i}[1 - \exp(-\beta E_i)]^{-1}[\exp(-E_i s) + \exp(E_i(s - \beta))] \tag{4.29}$$

which can be extended to other values of s by periodicity. Notice that the results so far obtained are essentially a consequence of the CCR, and in particular of the causal structure of the commutator function.

If Σ is non compact, then the partition function will (possibly) not exist and the thermal averages will not be defined by equations like Eq. (4.20) or Eq. (4.21). However, the periodic solutions of the elliptic operator $\mathcal{L} = -\partial_s^2 - \Delta + \mu^2$ may still exist. One simple method to construct them would be to start from the Green function S_∞ of $\mathcal{L}$ on $\mathbb{R} \times \Sigma$ with the euclidean metric $g_{ab} = (1, \gamma_{ab})$ and then forming the sum over images

$$S_\beta(s, X, Y) = \sum_{n=-\infty}^{\infty} S_\infty(s + n\beta, X, Y) \tag{4.30}$$

This will be a periodic solution of $\mathcal{L}S_\beta = 1$ (in operator notation) by construction. However, one has to keep in mind that for gauge theories in covariant gauges the method of images may well not works, due to the presence of unphysical states in the Hilbert space of the theory [59]. In such cases S_β must be obtained directly from the defining equations and boundary conditions. Then, by analytic continuation back to real time, one should find periodic distributions which solve the Klein-Gordon equation and obey the KMS condition, even if the partition function does not exist. Hence they are interpreted as describing a system in thermal equilibrium. Indeed, for free fields this condition uniquely determine the state [56] (this can be inferred from the observation that the Schwinger function does not depend on the choice of a state. Hence the analytic continuation is unique, apart from the usual ambiguities which result from the branching points). For interacting systems at low temperatures it may not, since physically there may be more than one thermodynamic phase at that temperatures.

4.2 Quantum fields in spaces with an asymptotically simple past

The extension of the above procedure to a general curved space-time is plagued with several well known problems [60, 3, 50, 61]. However, the space-times that result from the gravitational collapse of isolated bodies all have an asymptotically simple past, in the sense of being predictable from a Cauchy surface whose causal past is isometric to the causal past of a similar Cauchy surface in an asymptotically simple space. This is because the singularities in gravitational collapse always are to the future of the matter configuration that possibly collapsed. In a space-time which has an asymptotically flat and stationary past and future infinity, the interaction with the gravitational field is negligible in the asymptotic regions, and therefore one may hope to define the Fock representation in each one of the two asymptotically flat regions [3, 50, 49].

Let us consider a scalar field propagating in the space-time geometry of a space asymptotically stationary with an asymptotically simple past. Then there exists an asymptotic time translation symmetry, i.e. a vector field ξ^a, which is timelike and which generates a one parameter isometry group of the asymptotic metric near infinity. The asymptotic symmetry of space-time may indeed be much larger [20, 62, 63], and in all situations that we shall consider it contains in fact a nearly exact Poincaré symmetry. But generally, the minimum requirement is just the existence of a time translation symmetry at infinity, in order to be able to define asymptotic vacua. Next one assumes that the surface $\mathcal{I}^-$ is a Cauchy surface for massless fields (we mainly consider massless fields, but for massive fields one can take as a Cauchy surface a slice arbitrarily close to $\mathcal{I}^-$ and everywhere spacelike). The rationale for this is that one has in mind the collapse of some matter configuration which was nearly stationary in the remote past, in particular that there was no past event horizon, as in Fig. [3]. In this situation $\mathcal{I}^-$ is in fact a Cauchy surface for massless fields. Then one chooses a complete orthonormal set $\{P_i(x)\}$ of ingoing solutions of the Klein-Gordon equation, which are positive frequency relative to the time translation symmetry at past infinity. Therefore one can write

$$\phi(x) = \sum_i [A_i P_i(x) + A_i^\dagger P_i^*(x)], \qquad [A_i, A_j^\dagger] = \delta_{ij} \tag{4.31}$$

The $|in>$ vacuum is then defined by demanding $A_i|in>= 0$, from which one can build the Fock space of incoming states.

At future infinity, matter is different depending on whether the collapsing matter formed or not an event horizon. If not, then $\mathcal{I}^+$ is a Cauchy surface and one can write a similar decomposition for $\phi(x)$ as above, using a complete orthonormal set $\{F_i(x)\}$ of solutions which are positive frequency at future infinity

$$\phi(x) = \sum_i [B_i F_i(x) + B_i^\dagger F_i^*(x)], \qquad [B_i, B_j^\dagger] = \delta_{ij} \tag{4.32}$$

and the $|out>$ vacuum will then be defined by demanding that $B_i|out>= 0$. If then the field was in the $|in>$ vacuum, one can compute the expectation value of any asymptotic observable at future infinity. This will be the matrix element $<in|\mathcal{O}(B_i, B_j^\dagger)|in>$, where $\mathcal{O}(B_i, B_j^\dagger)$ is some self-adjoint operator (typically a polynomial) acting in the out Fock space. All of this is as in conventional S-matrix theory, the only departure from the conventional being that the $|in>$ vacuum is in general a many particle out-state, even for free fields. The concrete calculation of the expectations values is the only non trivial part of the game, and can be reduced to the purely classical problem of finding how the two set $\{P_i(x)\}$ and $\{F_j(x)\}$ are connected. In general, since both sets are complete, one may write either

$$F_i(x) = \sum_j [\alpha_{ij} P_j(x) + \beta_{ij} P_j^*(x)] \tag{4.33}$$

or

$$P_i(x) = \sum_j [\alpha_{ji}^* F_j(x) - \beta_{ji} F_j^*(x)] \tag{4.34}$$

the two equations being just the inverse of each other. The complex numbers α_{ij} and β_{ij} are called the Bogoljubov coefficients, after Bogoljubov technique to improve the ground state of superconducting matter. Indeed, Eq.s (4.33), (4.34) induce a corresponding mixing of operators

$$A_j = \sum_i [\alpha_{ij} B_i + \beta_{ij}^* B_i^\dagger] \tag{4.35}$$

$$B_i = \sum_j [\alpha_{ij}^* A_j - \beta_{ij}^* A_j^\dagger] \tag{4.36}$$

This shows how the $|in>$ vacuum is in general a many-particle out-state, provided only the coefficients β_{ij} be non vanishing. In fact, the expected number of out-particles in the wave packet $F_i(x)$ when the field is in the $|in>$ vacuum, is

$$N_i = < in|B_i^\dagger B_i|in> = \sum_k |\beta_{ik}|^2 \tag{4.37}$$

The second possibility is the formation of a future event horizon, $\mathcal{H}^+$. Then $\mathcal{I}^+$ is not a Cauchy surface since it fails to register the part of the field that crossed the horizon entering the black hole region, i.e. that part with no outgoing components. However, if we content ourselves to analyse the quantum field at future infinity in the asymptotically flat region exterior to the black hole, then one may replace the $F_i(x)$'s with a complete orthonormal set of purely outgoing solutions $Q_j(x)$, which have zero Cauchy data on the event horizon. Therefore one can expand the field in the form

$$\phi(x) = \sum_i [C_i Q_i(x) + C_i^\dagger Q_i^*(x)] + \phi_H(x), \qquad [C_i, C_j^\dagger] = \delta_{ij} \tag{4.38}$$

where $\phi_H(x)$ is that part with no outgoing components and therefore has zero Cauchy data on $\mathcal{I}^+$ and is orthogonal to the set $\{Q_i(x)\}$. One can also expand $\phi_H(x)$ into a basis of solutions with zero Cauchy data at infinity, say $H_i(x)$. However, as there is no natural separation of this set into positive and negative frequency solutions we leave it undetermined. The fact that one has no natural definition of a particle state in a generic, curved space-time is a well known piece of knowledge. Nevertheless one can still use a particular set $H_i(x)$ to build a Fock space of horizon states [64, 2] (which will not be observed at infinity) and use them to calculate the behaviour of the observables near the horizon (prominent is certainly the quantum stress tensor), rather than ill defined or ambiguous particle states.
Since the set $\{P_i(x)\}$ is complete, one has again the expansion

$$Q_i(x) = \sum_j [\alpha_{ij} P_j(x) + \beta_{ij} P_j^*(x)], \qquad \hat{P}_i(x) = \sum_j [\alpha_{ji}^* Q_j(x) - \beta_{ji} Q_j^*(x)] \tag{4.39}$$

where $\hat{P}_i(x)$ is the projection of $P_i(x)$ onto the subspace spanned by the set $\{Q_i(x)\}$. From this one derives the mixing between operators

$$C_i = \sum_j [\alpha_{ij}^* A_j - \beta_{ij}^* A_j^\dagger] \tag{4.40}$$

and the non trivial problem is still the purely classical problem of computing the Bogoljubov coefficients. Charged fields require some minor modification of the formalism. The Klein-Gordon equation is

$$g^{ab}(\nabla_a + ieA_a)(\nabla_b + ieA_b)\phi - \mu^2\phi = 0 \tag{4.41}$$

and a mass parameter μ has been introduced. The field can be expanded either in the form

$$\phi(x) = \sum_i [A_i P_{i,e}(x) + B_i^\dagger P_{i,-e}^*(x)] \tag{4.42}$$

with $[A_i, B_j^\dagger] = \delta_{ij}$, or in the form

$$\phi(x) = \sum_i [C_i Q_{i,e}(x) + D_i^\dagger Q_{i,-e}^*(x)] + \phi_H(x) \tag{4.43}$$

with $[C_i, D_j^\dagger] = \delta_{ij}$. The functions $P_{i,e}$ form a complete orthonormal set of solutions of Eq.(4.41) which are positive frequency at past infinity. Similarly, the set $\{Q_{i,e}\}$ is positive frequency at future infinity and has zero Cauchy data on the event horizon. The Bogoljubov relations get modified according to

$$Q_{i,e}(x) = \sum_j [\alpha_{ij,e} P_{j,e}(x) + \beta_{ij,e} P^*_{j,e}(x)] \tag{4.44}$$

This is not invertible since the $Q_{i,e}$'s do not form a complete set. However, the projection of $P_{i,e}$ onto the subspace spanned by the set $\{Q_{i,e}\}$ can be computed. Denoting the projection by $\hat{P}_{i,e}$, one has

$$\hat{P}_{i,e}(x) = \sum_j [\alpha^*_{ji,e} Q_{j,e}(x) - \beta_{ji,-e} Q^*_{j,e}(x)] \tag{4.45}$$

From this one obtains the operator mixing

$$C_i = \sum_j [\alpha^*_{ij,e} A_j - \beta^*_{ij,e} B_j^\dagger] \tag{4.46}$$

$$D_i = \sum_j [\alpha^*_{ij,-e} B_j - \beta^*_{ij,-e} A_j^\dagger] \tag{4.47}$$

which is totally independent on the horizon states, since only the Fock space of outgoing states is involved in these equations.

4.3 Rindler's quantization

The simplest example of a space containing an event horizon, is the region of Minkowski space which is accessible to a uniformly accelerated observer. Quantum field theory in accelerated systems played a very important role to better understand the quantum properties of black holes. However, the intrinsic relevance of the subject largely justifies the great amount of investigations which have been done (two general references here are [65] and [66]). Indeed, the predictions to be made may well be the only ones that can be proved or disproved by experiments in the near future [67].

The Rindler's wedge W_R [68] is the open submanifold of Minkowski's space defined as the set of points whose coordinates satisfy the restrictions $x^2 - t^2 > 0$, $x > 0$. A left Rindler region W_L can then be obtained by reflecting $(t, x, x_t) \to (-t, -x, x_t)$, $x_t = (y, z)$ standing for the coordinates of the transverse 2-plane. They are not Lorentz invariant submanifolds, since they are defined relative to a reference frame, but they are invariant under boosts in the x-direction. A global coordinate system for W_R is given by (τ, ξ, x_t), where $x = \xi \cosh\tau$, $t = \xi \sinh\tau$, $\xi > 0$ and $x_t = (y, z)$ are the two transverse coordinates. Thus any line $\xi = \xi_0$, $y = y_0$, $z = z_0$ represents the trajectory of a uniformly accelerated particle with proper acceleration $g = \xi_0^{-1}$ and proper time $s = g\tau$. The Minkowski metric in the (τ, ξ, x_t) coordinate system takes the form

$$ds^2 = -\xi^2 d\tau^2 + d\xi^2 + dx_t^2, \qquad dx_t = (dy, dz) \tag{4.48}$$

In a slightly different form (by replacing τ with $g\tau$), this is also the metric describing the space-time observations of an observer standing at rest in a uniformly accelerating tetrad, Fermi-transported along the world line of the origin with proper acceleration g. By the equivalence principle, the metric also describes the gravitational field of a large body (like our earth) in the region where it can be considered uniform. The causal structure is shown in Fig.[4]. The region R is W_R, which is separated from the future region (F) and from the past region (P) by an event horizon, the lightlike plane $t = \pm x$.

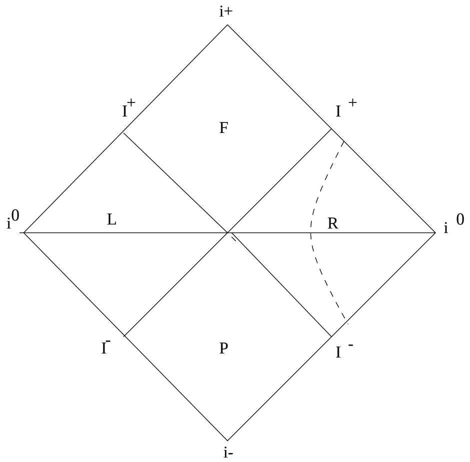

Fig.4 The (t, x) section of Minkowski's space, where each point represents the transverse (y, z)-plane. The dotted line is the world history of a uniformly accelerated point particle.

As the Rindler's region is a static, globally hyperbolic space-time one may pose the question: what is like to be the quantum theory relative to the given metric? In fact, this is a non trivial question, since it gives rise to an inequivalent quantum theory from the Minkowski based quantization. At the same time, this will tell us how an accelerating observer would describe its quantum observations, which seems in itself a very interesting question.
The canonical quantization of a massless scalar field in W_R can be done concretely [60, 69, 70, 71, 5, 72]. The field operator is

$$\phi = \int \frac{d^2k}{2\pi} \int_0^\infty \frac{d\omega}{\pi} [\sinh(\pi\omega)]^{1/2} K_{i\omega}(k\xi) \left[A_{k\omega}^{(R)} e^{i(k_t x_t - \omega\tau)} + A_{k\omega}^{(R)\dagger} e^{-i(k_t x_t - \omega\tau)} \right] \tag{4.49}$$

where $k_t \in \mathbb{R}^2$ is the transverse momentum and $k = \sqrt{k_t^2}$. The suffix (R) is to remind that one is in W_R. A similar expansion can be written in the left wedge W_L, with corresponding operators $A_{k\omega}^{(L)}$, $A_{k\omega}^{(L)\dagger}$. $K_{i\omega}(k\xi)$ is a Mc Donald function and the modes $\phi_{\omega k} = \exp(ik_t x_t \pm i\omega\tau) K_{i\omega}(k\xi)$ are the unique solution of the wave equation which are bounded at infinity. The strange factor $\sqrt{\sinh(\pi\omega)}$ was inserted to insure the validity of the canonical commutation relations

$$\left[A_{k\omega}^{(R)}, A_{k'\omega'}^{(R)\dagger} \right] = \delta^{(2)}(k_t - k_t') \delta(\omega - \omega') \tag{4.50}$$

The state such that $A_{k\omega}^{(R)} |F >= A_{k\omega}^{(L)} |F >= 0$ is called the Fulling vacuum. Notice that ω and k_t are decoupled (even for massive fields), which is a rather unfamiliar case. A plausible physical explanation is offered in [73] and an interacting field theory where this decoupling may be absent is discussed in [74].
The Wightman functions for this state are easily founded. From the definition one obtains the integral representation ($p = (\tau, \xi, x_t)$)

$$W^+(p, p') = \int_0^\infty \frac{d\omega}{\pi^2} \int \frac{d^2k_t}{4\pi^2} e^{ik_t(x_t - x_t')} \sinh \pi\omega e^{-i\omega(\tau - \tau' - i\varepsilon)} K_{i\omega}(k\xi) K_{i\omega}(k\xi') \tag{4.51}$$

and $W^-(p,p') = W^{+*}(p,p')$. Remarkably, the integral can be expressed in terms of elementary functions, giving

$$W^+(p,p') = -\frac{1}{4\pi^2}\frac{\alpha}{\xi\xi'\sinh\alpha}\frac{1}{(\tau-\tau'-i\varepsilon)^2-\alpha^2} \tag{4.52}$$

where we defined α through the equation

$$\cosh\alpha = \frac{\xi^2+\xi'^2+|x_t-x_t'|^2}{2\xi\xi'} \tag{4.53}$$

This result was first discovered by Dowker [75, 76]. To obtain the finite temperature Wightman functions one has to perform the sum over the images in imaginary time. This gives

$$W_\beta^+(p,p') = -\frac{1}{4\pi^2}\frac{\alpha}{\xi\xi'\sinh\alpha}\sum_{n=-\infty}^{\infty}\frac{1}{(\tau-\tau'-i\varepsilon-in\beta)^2-\alpha^2} \tag{4.54}$$

The infinite sum is a standard one and gives the result [76]

$$W_\beta^+(p,p') = \frac{1}{4\pi\beta\xi\xi'\sinh\alpha}\left[\frac{\sinh\frac{2\pi}{\beta}\alpha}{\cosh\frac{2\pi}{\beta}\alpha-\cosh\frac{2\pi}{\beta}(\tau-\tau'-i\varepsilon)}\right] \tag{4.55}$$

Eq.(4.55) shows that the Fulling thermal states are not Hadamard states for a general value of β since the singularity structure of the Wightman function has not the Hadamard form. In a sense therefore, they are not physically realizable states (for a through discussion of Hadamard, quasi-free states and their properties see Ref. [77]). Indeed, all such states have infinite proper energy relative to the Minkowski vacuum but for the special value $\beta = 2\pi$. The physical significance of this choice is well known, for it leads to

$$\begin{aligned} W_{2\pi}^+ &= \frac{1}{4\pi^2}\frac{1}{\xi^2+\xi'^2+|x-x'|^2-2\xi\xi'\cosh(\tau-\tau'-i\varepsilon)} \\ &= -\frac{1}{4\pi^2}\frac{1}{(t-t'-i\varepsilon)^2-(x-x')^2-|x-x'|^2} \end{aligned} \tag{4.56}$$

This is the Wightman's function which characterizes the Minkowski's vacuum, which is therefore a KMS state [77, 57] under the translation group $\tau \to \tau + is$. Thus any observer which is at rest in the Rindler coordinate system will perceive the Minkowski's vacuum state as a thermal bath. Similar results can be proved for free fields with higher spin as well, and the conclusion will also hold for interacting quantum theories at the perturbative level [78]. This is easy to understand on a formal level, since the perturbative expansion only involves the Green functions of the free theory.

4.4 The accelerated detector

In order to strengthen the above interpretation, one may calculate the response to the vacuum of an idealized monopole detector linearly coupled to the quantum field, and in a state of uniformly accelerated motion [5, 50, 3, 65]. The interaction hamiltonian is taken to be

$$H = \lambda D(\tau)\phi(x(\tau)), \qquad D(\tau) = e^{iH_0\tau}D_0e^{-iH_0\tau} \tag{4.57}$$

where H_0 is the monopole hamiltonian and the field is evaluated on the monopole trajectory, given by $x = g^{-1}\cosh g\tau$, $t = g^{-1}\sinh g\tau$. Assuming the coupling $\lambda \ll 1$, the total transition probability per unit proper time from the ground state $|E_0, M>=|E_0>|M>$, where $|M>$ stands for the Minkowski's vacuum, to the excited states (the total absorption probability) will be given by standard perturbation theory to first order in λ by

$$R_- = \lambda^2 \sum_E |<E|D_0|E_0>|^2 \int_{-\infty}^{\infty} e^{-i(E-E_0)s} W^+(x(s), x(0)) ds \tag{4.58}$$

The total emission probability can be computed in the same way as

$$R_+ = \lambda^2 \sum_E |<E|D_0|E_0>|^2 \int_{-\infty}^{\infty} e^{+i(E-E_0)s} W^+(x(s), x(0)) ds \tag{4.59}$$

The functions

$$F_\pm(E) = \int_{-\infty}^{\infty} e^{\pm iEs} W^+(x(s), x(0)) ds \tag{4.60}$$

are called the detector's response functions. The scalar field Wightman's function in the integral is obtained from Eq. (4.56) by putting $\xi = \xi^{'} = g^{-1}$ and $\tau - \tau^{'} = gs$. Then

$$W^+(x(s), x(0)) = -\frac{g^2}{16\pi^2 \sinh^2[g(s - i\varepsilon)/2]} \tag{4.61}$$

This function has infinitely many double poles in the complex s-plane, at the points

$$s_j = \frac{2\pi i}{g} j \qquad j \in \mathbb{Z} \tag{4.62}$$

Since $E - E_0 > 0$ the first integral can be calculated closing the contour in the lower half complex plane and summing over the residues. In this way one obtains

$$R_- = \frac{\lambda^2}{2\pi} \sum_E \frac{(E - E_0)|<E|D_0|E_0>|^2}{\exp[2\pi(E - E_0)/g] - 1} \tag{4.63}$$

Similarly, the total transition probability from the excited states to the ground state is

$$R_+ = \frac{\lambda^2}{2\pi} \sum_E \frac{(E - E_0)|<E|D_0|E_0>|^2 \exp[2\pi(E - E_0)/g]}{\exp[2\pi(E - E_0)/g] - 1} \tag{4.64}$$

If the detector were at rest and immersed in a thermal bath at $\beta = 2\pi g^{-1}$, the transition probabilities would have been the same. Clearly then, the detector experiences a thermal bath of scalar radiation with a distinguished temperature

$$T = \frac{\hbar g}{2\pi c} \tag{4.65}$$

This is a temperature within the statistical meaning of the concept, as a closer analysis of the emission-absorption process shows [79, 65, 18].

4.5 Bogoljubov's mixing

To see the particle content of Rindler states, one can expand the plane wave into Rindler modes using the formula [52]

$$\begin{aligned} u(k,k_t|x) &= \frac{1}{\pi}\int_0^\infty d\omega \left(\frac{E+k}{E-k}\right)^{-i\omega/2} e^{\pi\omega/2}K_{i\omega}(k_t\xi)e^{i\omega\tau-ik_tx_t} \\ &+ \frac{1}{\pi}\int_0^\infty d\omega \left(\frac{E+k}{E-k}\right)^{i\omega/2} e^{-\pi\omega/2}K_{i\omega}(k_t\xi)e^{-i\omega\tau-ik_tx_t} \end{aligned} \tag{4.66}$$

where $u(k,k_t|x) = \exp i(Et-kx-k_tx_t)$ with $E^2 = k^2+k_t^2$. This equation gives the restriction of the plane wave to W_R. The restriction of $u(k,k_t|x)$ to the left wedge W_L can be obtained taking the complex conjugate of the right hand side and performing the space inversion, $x_t \to -x_t$, afterward. The quantum field can be expanded in the usual fashion into plane waves

$$\phi(x) = \int \frac{dkd^2k_t}{(2\pi)^{3/2}\sqrt{2E}}\left[A(k,k_t)\bar{u}(k,k_t|x) + A^\dagger(k,k_t)u(k,k_t|x)\right] \tag{4.67}$$

By writing the restriction of the quantum field to W_R and comparing it with Eq. (4.49) by using Eq. (4.66), it follows

$$A^{(R)}_{\omega k_t} = \int \frac{dk}{\sqrt{4\pi E}}\left(\frac{E+k}{E-k}\right)^{i\omega/2}\left[e^{\pi\omega/2}A(k,k_t) + e^{-\pi\omega/2}A^\dagger(k,-k_t)\right][\sqrt{\sinh\pi\omega}]^{-1} \tag{4.68}$$

where $A(k,k_t)$ is the Minkowski annihilation operator. Of course $A^{(R)\dagger}_{\omega k_t}$ can be obtained by taking the adjoint of this equation. This is a concrete examples of Bogoljubov transformations. They are not invertibles, and this is why the Minkowski vacuum appears to be described by a density matrix in W_R. To find a sufficient number of equations, one restricts the field to the left wedge W_L so as to find also

$$A^{(L)}_{\omega k_t} = \int \frac{dk}{\sqrt{4\pi E}}\left(\frac{E+k}{E-k}\right)^{-i\omega/2}\left[e^{\pi\omega/2}A(k,-k_t) + e^{-\pi\omega/2}A^\dagger(k,k_t)\right][\sqrt{\sinh\pi\omega}]^{-1} \tag{4.69}$$

whose adjoint will give also $A^{(L)\dagger}_{\omega k_t}$. If $|M>$ denotes the Minkowski vacuum, it follows from these equations that

$$A^{(R)\dagger}_{\omega k_t}A^{(R)}_{\omega k_t}|M> = A^{(L)\dagger}_{\omega k_t}A^{(L)}_{\omega k_t}|M> \tag{4.70}$$

Hence the Minkowski vacuum may be said to contain an equal number of left and right Rindler quanta (rindlerons) for any given energy and momentum. However, the stress tensor of the thermal Fulling state with $\beta = 2\pi$ vanishes identically, since this state is just the Minkowski vacuum. To obtain a flux of particle one has to artificially impose suitable boundary conditions on the horizon bounding the Rindler region. This is just what has been done (see Sec.(5.1) below) in the Kruskal manifold, whose causal structure is akin to that of Minkowski space and its division into Rindler regions.
It also appears that the Minkowski's vacuum is a coherent superposition of left-right rindlerons with equal energy and momentum. This means that the formal solution of the above equations is [66]

$$|M> = \prod_j N_j \sum_{n_j=0}^{\infty} e^{-2\pi\omega n_j}|n_j,L>\otimes|n_j,R> \tag{4.71}$$

with $j = (\omega, k_t)$ and $N_j = (1 - \exp(2\pi\omega))^{1/2}$ is the normalization factor for each internal sum. Notice the strong correlation between the left-right sectors. An observer in one region detecting the presence of n_j rindlerons with energy ω fixes the state of the other region to be precisely a state with n_j rindlerons with the same energy. This is a kind of Einstein-Podolski-Rosen correlation, and as such it cannot be used to transmit information from one region to the other [79].

5 Particle emission from black holes

Black holes are likely to produce strong gravitational fields and strong gravitational fields are expected to create particle out from the vacuum. Indeed, neglecting the back reaction of the matter fields on the metric, Hawking was successful in the task of computing the particle spectrum created by the gravitational collapse of a compact star [1]. Soon after, Wald [2] computed precisely the final quantum state.

We shall treat with a certain detail the Schwarzschild black hole first. As the main arguments works well also for more complicated black hole solutions, even for the extreme ones indeed, we shall have not to repeat them. Certainly it is the motion of the collapsing matter which is responsible for particle creation. However, the asymptotic steady flux first discovered by Hawking contains some more subtle elements that caused a bit of surprise. First, the particles created in strong external fields by a Schwinger type emission process are normally emitted in pure states and do not show additional randomness beyond the quantum one. However, the Hawking emission is perfectly thermal being described by a Planck distribution with a distinguished temperature! Second, it involves certain universal features of the collapse when the black hole is close to form, in particular the fact that the radius of the collapsing star approaches the event horizon at an exponentially fast rate which is controlled by the surface gravity of the black hole. Therefore, even if the quantum fields were in the vacuum state long before the collapse took place, the field modes reaching infinity from the surface of the star underwent a very large an time increasing red shift. How it could be that an exponentially time-increasing red shift corresponds to thermal particle emission can actually be seen using a simplified $2D$-dimensional model [5], which is also of independent interest.

5.1 The 2D Unruh's vacuum

One begins replacing the collapse problem with a simpler field theory in the Kruskal's manifold. Forgetting about the two angular variables we are left with the manifold $\mathbb{R}^2$ with metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dudv = -\frac{32M^3}{r} \exp(r/2M) dUdV \tag{5.1}$$

where we remind that $v = t + r^*$ and $u = t - r^*$ are advanced and retarded times respectively, and $r^* = r + 2M \log[(r/2M) - 1]$. The Kruskal's coordinates are here related to u and v by

$$U = -\exp(-u/4M), \qquad V = \exp(v/4M) \tag{5.2}$$

The coordinates U, V cover the entire manifold, the line $U = 0$ is the future event horizon and the line $V = 0$ is the past event horizon. We also recall that there are two asymptotically flat and static regions which are separated from the black and white hole regions by the bifurcate event horizon. We shall concentrate our attention to the static region R, sitting to the right of

the horizon (see Fig. [2]). The left region L, is causally separated from R.
Since in the real collapse problem there is no white hole region, one may impose the restriction that no particles come out from the white hole region. This condition defines the so called Unruh state. More precisely, since the metric is conformally flat, the massless Klein-Gordon equation is just the same as in Minkowski space, namely

$$\frac{\partial^2\phi}{\partial U\partial V}=0, \qquad \frac{\partial^2\phi}{\partial u\partial v}=0 \tag{5.3}$$

depending one uses Kruskal, (U,V), or retarted, (u,v), coordinates. Three states can then be defined according to the above coordinates.

(i) The Unruh vacuum [5]: the union $\Sigma^-=H^-\cup i^-\cup\mathcal{I}^-$ is a Cauchy surface for the massless Klein-Gordon equation in R, so one can expand the field in the form

$$\phi(x)=\int_0^\infty\frac{d\omega}{\sqrt{4\pi\omega}}[a_\omega e^{-i\omega U}+a^\dagger_\omega e^{i\omega U}+b_\omega e^{-i\omega v}+b^\dagger_\omega e^{i\omega v}] \tag{5.4}$$

The modes $U_{in}=\exp(-i\omega v)$ are incoming from past infinity $\mathcal{I}^-$ while the modes $\hat{U}_{in}=\exp(-i\omega U)$ are outgoing from the white hole region. The state $|U>$ such that

$$a_\omega|U>=b_\omega|U>=0 \tag{5.5}$$

is called the Unruh vacuum. Evidently it does not contain Minkowski particles incoming from infinity or Kruskal particles coming out from the white hole. One uses this terminology because at infinity a positive frequency wave in the Finkelstein time v is also positive frequency in Minkowski time, since $v=t+r^*$. On the other hand, no such correspondence holds for the $\hat{U}_{in}$ modes.

(ii) The Hartle-Hawking vacuum [80]: the field is expanded in the form

$$\phi(x)=\int_0^\infty\frac{d\omega}{\sqrt{4\pi\omega}}[A_\omega e^{-i\omega U}+A^\dagger_\omega e^{i\omega U}+B_\omega e^{-i\omega V}+B^\dagger_\omega e^{i\omega V}] \tag{5.6}$$

where this time V replaces v in the advanced part of the field. The Hartle-Hawking state $|H>$ is defined by

$$A_\omega|H>=B_\omega|H>=0 \tag{5.7}$$

Evidently it does not contains Kruskal particles falling down the black hole by crossing the future event horizon H^+.

(iii) The Boulware vacuum [71]: the field is expanded in the form

$$\phi(x)=\int_0^\infty\frac{d\omega}{\sqrt{4\pi\omega}}[C_\omega e^{-i\omega u}+C^\dagger_\omega e^{i\omega u}+D_\omega e^{-i\omega v}+D^\dagger_\omega e^{i\omega v}] \tag{5.8}$$

The Boulware state $|B>$ is defined by

$$C_\omega|B>=D_\omega|B>=0 \tag{5.9}$$

It does not contain incoming particles from past infinity or outgoing particles to future infinity.

A crucial observation is that the Boulware outgoing modes are positive frequency relative to the observer clock at future infinity. Wave packets formed by superposition of them will have zero Cauchy data on the future event horizon, because $u \to \infty$ near the horizon. On the other hand, wave packets formed by superposition of ingoing modes will have zero Cauchy data at $\mathcal{I}^+$. Therefore one decomposes $|B>=|out> \otimes |hor>$, where $|hor>$ is the vacuum for states describing particles crossing the horizon and $|out>$ can be taken as the vacuum for states describing particles escaping to infinity.
Let us take now the field in the Unruh vacuum. Given an outgoing wave packet of the kind

$$F(u) = \int_0^\infty \exp(-i\omega u) f(\omega) d\omega$$

one defines the smeared operator

$$C(F) = \int_0^\infty C_\omega f(\omega) d\omega$$

Then the operator counting the number of particles at infinity in the outgoing state $F(u)$ is $N(F) = C^\dagger(F)C(F)$. The simplest observable that one can compute is the average number of particles at future infinity in the state $F(u)$. This is given by

$$N_F =< U|N(F)|U >= \int_0^\infty \int_0^\infty d\omega d\omega' f^*(\omega) < U|C^\dagger_\omega C_{\omega'}|U > f(\omega') \tag{5.10}$$

The matrix element in the integral can be computed by finding the Bogoljubov transformation relating the Unruh modes to the out modes, $\exp(-i\omega u)$. However, it is only the restriction of the Unruh modes on the portion $U < 0$ of the past horizon that can be observed at infinity. Therefore the Bogoljubov transformation takes the form

$$\frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u} = \int_0^\infty \frac{1}{\sqrt{4\pi\sigma}} [\alpha_{\omega\sigma} e^{-i\sigma U} + \beta_{\omega\sigma} e^{i\sigma U}] \theta(-U) d\sigma \tag{5.11}$$

The red shift effect enters here because

$$U = -e^{-u/4M} \tag{5.12}$$

and so the frequency at infinity ω_∞ is related to the frequency ω at the past horizon by the exponential red shift $\omega_\infty = \omega \exp(-u/4M)$. Using Eq. (5.12) to invert Eq. (5.11), one may rotate the contour in the complex U-plane due to the restriction $U < 0$. With a simple calculation one finds

$$\alpha_{\omega\sigma} = -\frac{i}{2\pi} \frac{\sigma^{-i4M\omega}}{\sqrt{\omega\sigma}} \Gamma(1 + i4M\omega) e^{2\pi M\omega} \tag{5.13}$$

$$\beta_{\omega\sigma} = \frac{i}{2\pi} \frac{\sigma^{-i4M\omega}}{\sqrt{\omega\sigma}} \Gamma(1 + i4M\omega) e^{-2\pi M\omega} \tag{5.14}$$

where $\Gamma(z)$ is the Euler's gamma function. The Bogoljugov coefficients satisfy the completeness relation

$$\int_0^\infty [\alpha^*_{\omega\sigma}\alpha_{\omega'\sigma} - \beta^*_{\omega\sigma}\beta_{\omega'\sigma}] d\sigma = \delta(\omega - \omega') \tag{5.15}$$

Using Eq. (5.11) and Eq. (5.8) one finds the relation between C_ω, $C^\dagger_\omega$ and the Unruh operators

$$C_{\omega'} = \int_0^\infty d\omega[\alpha^*_{\omega'\omega} a_\omega - \beta^*_{\omega'\omega} a^\dagger_\omega], \qquad C^\dagger_{\omega'} = \int_0^\infty d\omega[\alpha_{\omega'\omega} a^\dagger_\omega - \beta_{\omega'\omega} a_\omega] \tag{5.16}$$

One can now compute the matrix element $< U|C^\dagger_\omega C_{\omega'}|U >$. From Eq. (5.16) it follows

$$< U|C^\dagger_\omega C_{\omega'}|U >= \int_0^\infty \beta_{\omega\sigma}\beta^*_{\omega'\sigma} d\sigma = [\exp(8\pi M\omega) - 1]^{-1}\delta(\omega - \omega') \tag{5.17}$$

and therefore

$$N_F = \int_0^\infty \frac{|f(\omega)|^2}{\exp(8\pi M\omega) - 1} d\omega \tag{5.18}$$

which has the form of a Planckian spectrum at the inverse temperature $\beta = 8\pi M$. It was clearly the exponential form of the red shift that caused the thermal spectrum. Exactly the same occurs for an accelerated mirror in hyperbolic motion, which emits a thermal spectrum of radiation with temperature $T = g/2\pi$, g being the acceleration of the mirror [50, 81, 3, 18]. This is no coincidence as the Hawking temperature is just $T = k/2\pi$, $k = 1/4M$ being the surface gravity of the Schwarzschild black hole.

One can define the Unruh vacuum for fermions as well. Very briefly, the Weyl equation for a two-component Weyl spinor splits into a pair of decoupled equations for the positive/negative chirality components, say $\phi_\pm$. From the conformal invariance of the Weyl equation, one finds $\phi_\pm = N^{-1/4}\chi_\pm$ where $N = (1 - 2M/r)$ and $\chi_\pm$ obey the trivial equations

$$\partial_v \chi_+ = 0, \qquad \partial_u \chi_- = 0 \tag{5.19}$$

The normalized modes which define the Unruh vacuum are those which are positive frequency relative to the Kruskal coordinate U if they are outgoing from the past horizon, and to the advanced Finkelstein time v if they are ingoing from past infinity. The formers are easily seen to be

$$\phi_{\omega+} = N^{-1/4}(-kU)^{1/2}e^{-i\omega U}\theta(-U) \tag{5.20}$$

where $k = 1/4M$ is the surface gravity. One can expand this mode into outgoing solutions which define the $|out >$ vacuum. One then determines the relevant Bogoljugov coefficient to be

$$\beta_{\omega\omega'} = e^{i\pi/4}\frac{1}{2\pi\sqrt{k\omega'}} e^{-\pi\omega/2k}(\omega')^{-i\omega/k}\Gamma(1/2 + i\omega/k) \tag{5.21}$$

from which one obtains the Fermi-Dirac distribution

$$\int_0^\infty \beta^*_{\omega\sigma}\beta_{\omega'\sigma} d\sigma = [\exp(8\pi M\omega) + 1]^{-1}\delta(\omega - \omega') \tag{5.22}$$

One can do similar calculations for the Hartle-Hawking or Boulware states as well. Then one finds the Boulware state is empty at future infinity (as it was at past infinity), and the Hartle-Hawking state corresponds to a thermal equilibrium state with the Hawking temperature. However, the only state for which the stress tensor is regular on the event horizon is the Hartle-Hawking equilibrium state, since the Unruh state gives a divergence on the past horizon and the Boulware state a divergent result on both horizons [82]

5.2 The 4D Unruh's vacuum

The state of affairs described above is essentially the same in the four dimensional eternal Schwarzschild black hole. Here again one can define the Unruh, Hartle-Hawking and Boulware states, all having the same interpretation as in the $2D$-dimensional case [66]. The only changes are that a part of the incoming modes gets back scattered to infinity and a part gets transmitted into the hole by the potential barrier, Eq. (5.27). This implies the luminosity formula (Eq. (5.49) in the next section), if the field were in the Unruh state. The Hartle-Hawking state corresponds to a black hole in thermal equilibrium with a massless radiation gas at the Hawking temperature and the Boulware state corresponds to our intuitive notion of empty space. Since the stress tensor diverges at the horizon in the Boulware state, we may say that such a state is appropriate to describe the empty space outside a static, spherically symmetric star. A detailed account of these claims can be found in Candelas paper [82], from which we quote the following results: for $r \to \infty$ the stress tensor in the Boulware state vanishes. In the Hartle-Hawking state one has

$$< H|T_{ab}|H >\sim \frac{1}{2\pi^2}\int_0^\infty \frac{\omega^3 d\omega}{\exp(8\pi M\omega)-1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \tag{5.23}$$

corresponding to a black body equilibrium radiation. In the Unruh state one has

$$< U|T_{ab}|U >\sim \frac{L}{4\pi r^2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.24}$$

where L is the luminosity of the black hole. It corresponds to a flux of radiation at infinity. Near the horizons the results are more complicated but for the Boulware vacuum, for which the stress tensor diverges as fast as $(r-2M)^{-2}$. We refer the readers to Candelas paper for further clarifications (he also computed the response of a particle detector in the three vacua) or to Ref. [66], and we turn instead the attention to the collapse situation.

5.3 The spherically symmetric collapsing star

For a real collapsing star there will not be a past horizon and consequently there is no such a thing as the Unruh vacuum. Instead, the space-time will be asymptotically simple in the past as it is shown in the following space-time diagram of the collapse

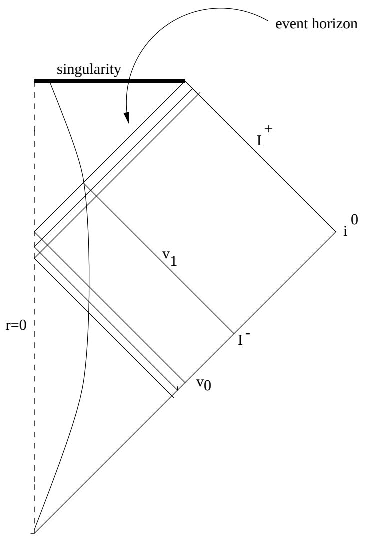

Fig.5 The Penrose's diagram for a black hole that is formed through spherical gravitational collapse. The value v_0 of the advanced Finkelstein time marks the latest ray which just fails to reach infinity and v_1 marks the birth of the black hole.

Since the collapse is spherically symmetric, the positive frequency incoming modes from past infinity have the form

$$P_{j\omega} = (4\pi\omega)^{-1/2} r^{-1} e^{-i\omega t} \overset{\leftarrow}{R}_{l\omega}(r) Y_{lm}(\theta, \phi) \qquad j = (l, m) \tag{5.25}$$

where $Y_j(\theta, \phi)$ are the normalized spherical harmonics. $P_{j\omega}$ will be a solution of the Klein-Gordon equation in the Schwarzschild geometry of the collapsing star if the radial function satisfies the ordinary differential equation

$$\left[\frac{d^2}{dr^{*2}} + \omega^2 - V_l(r)\right] \overset{\leftarrow}{R}_{l\omega}(r) = 0, \qquad r^* = r + 2M \log\left(\frac{r}{2M} - 1\right) \tag{5.26}$$

The meaning of r^* is that constant values of retarded time $u = t - r^*$, or of advanced time $v = t + r^*$, give the world line of radially outgoing or ingoing light rays, namely they specify the radial null geodesics. The potential function is

$$V_l(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right] \tag{5.27}$$

The shape of the potential barrier is shown in the figure

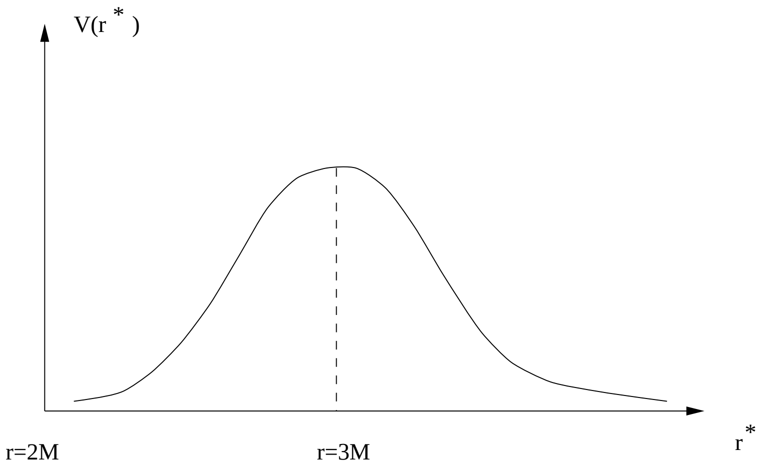

Fig.6 A sketch of the potential barrier around a Schwarzschild's black hole as a function of r^*. The coordinate r is measured in units of Schwarzschild's radii.

As a function of r^* the potential has a bell-like shape and decreases exponentially fast to zero for $r \to 2M$. Therefore one takes the radial function $\overleftarrow{R}_{l\omega}(r)$ obeying the asymptotic condition appropriate for incoming modes

$$\overleftarrow{R}_{j\omega} \simeq \begin{cases} e^{-i\omega r^*} + A_j(\omega) e^{i\omega r^*} & r^* \to \infty \\ B_j(\omega) e^{-i\omega r^*} & r^* \to -\infty \end{cases} \tag{5.28}$$

which defines the reflection and transmission coefficients from the potential barrier. As in the usual scattering theory one has then the unitarity equation

$$|A_j(\omega)|^2 + |B_j(\omega)|^2 = 1 \tag{5.29}$$

The normalization factor has been chosen so as to insure that $< P_{j\omega}, P_{i\omega'} >= \delta_{ji}\delta(\omega - \omega')$, where the brackets denote the Klein-Gordon scalar product. Since $\mathcal{I}^-$ is a Cauchy surface for massless fields, the field can be expanded everywhere in the form

$$\phi(x) = \sum_j \int_0^\infty [A_{j\omega} P_{j\omega} + A^\dagger_{j\omega} P^*_{j\omega}] d\omega \tag{5.30}$$

where

$$[A_{i\omega}, A^\dagger_{j\omega'}] = \delta_{ij}\delta(\omega - \omega'), \qquad [A^\dagger_{i\omega}, A^\dagger_{j\omega'}] = [A_{i\omega}, A_{j\omega'}] = 0$$

The $|in>$ vacuum state is then defined by the usual condition $A_{j\omega}|in>= 0$.
In the extended Schwarzschild geometry there also exist outgoing solutions from the past horizon behaving asymptotically as

$$\overrightarrow{R}_{j\omega} \simeq \begin{cases} e^{i\omega r^*} + A_j(\omega) e^{-i\omega r^*} & r^* \to -\infty \\ B_j(\omega) e^{i\omega r^*} & r^* \to \infty \end{cases} \tag{5.31}$$

that will be later convenient to have at hand. In the stationary Schwarzschild geometry there also exist solutions $Q_{j\omega}$ with the asymptotic form

$$Q_{j\omega} \simeq (4\pi\omega)^{-1/2} r^{-1} Y_j(\theta,\phi) \begin{cases} e^{-i\omega u} + A^*_j(\omega) e^{-i\omega v} & r^* \to \infty \\ B^*_j(\omega) e^{-i\omega u} & r^* \to -\infty \end{cases} \tag{5.32}$$

Wave packets formed by superposition of the $Q_{j\omega}$ are purely outgoing, i.e. they have zero Cauchy data on the event horizon, and therefore they can be taken to define the $|out>$ vacuum state long after the completion of the collapse. Because of the horizon, $\mathcal{I}^+$ is not a Cauchy surface for massless fields, so the field operator can also be expanded in the form

$$\phi(x) = \sum_j \int_0^\infty [C_{j\omega} Q_{j\omega} + C^\dagger_{j\omega} Q^*_{j\omega}] d\omega + \phi_H(x) \tag{5.33}$$

where $\phi_H(x)$ is that part of the field which at late times is transmitted by the potential barrier down the black hole, i.e. that part which has no outgoing components. Fortunately one doesn't have to worry about if all that one wants is the radiation at infinity. Rather, it is needed to discuss the negative energy flow into the hole [64] and also to compute the density matrix representing the final quantum state [2]. The Q-modes are normalized to a δ-function and therefore

$$[C_{i\omega}, C^\dagger_{j\omega'}] = \delta_{ij}\delta(\omega - \omega') \tag{5.34}$$

The $|out>$ vacuum state at $\mathcal{I}^+$ is now defined by the condition $C_{j\omega}|out>=0$. It is important to stress that $|out>$ is the physical (or may be we should say "more familiar") vacuum state at very late times because the Schwarzschild time coordinate becomes the observer proper time at infinity, thus allowing an unambiguous definition of particle states. Moreover, there exist an asymptotic, nearly exact Poincaré symmetry which permits the definition of mass and spin. In other words, this is the state whose excitations are the observed elementary particles like photons, electrons or neutrinos (after all, we observe these particles in the very weak gravitational field of the earth). Similarly $|in>$ is the physical, or familiar, vacuum state at very early times. By following the general scheme of Sec.(4.2) and using spherical symmetry one has the Bogoljubov transformation

$$Q_{j\omega} = \int_0^\infty [\alpha_{j\omega\omega'} P_{j\omega'} + \beta_{j\omega\omega'} P^*_{j\omega'}] d\omega' \tag{5.35}$$

because the in-modes form a complete set. This leads to a corresponding Bogoljubov mixing between operators. For example

$$C_{j\omega'} = \int_0^\infty d\omega [\alpha^*_{j\omega'\omega} A_{j\omega} - \beta^*_{j\omega'\omega} A^\dagger_{j\omega}] \tag{5.36}$$

from which one can immediately obtain the adjoint $C^\dagger_{j\omega}$.
One important issue now is to determine the evolution of an ingoing mode $P_{j\omega}$ in the geometry of the collapsing matter. A fraction of $P_{j\omega}$ will be clearly back scattered to infinity with unchanged frequency, and a fraction will cross the star matter after which it is converted into an outgoing mode. Because the height of the potential barrier is of order M^{-2}, this fraction will be significant for frequencies larger than M^{-1}. The expression of this outgoing part on the surface of the star, when this is about to form the event horizon, will be the interesting thing. The answer is

$$P_{j\omega} \simeq (4\pi\omega)^{-1/2} e^{-i\omega v_1} (r)^{-1} \exp\left[i\omega C e^{-ku}\right] Y_{lm}(\theta,\phi)\theta(-U) \tag{5.37}$$

where $U = -\exp(-ku)$, $k = (4M)^{-1}$ is the surface gravity and C and v_1 are positive constant, the only relics of the collapse process. The argument has been given by Hawking backward in time [1], by asking instead: what would have been the form of an ingoing mode that evolved

into some outgoing positive frequency mode $Q_{j\omega}$? We choose to give the argument by following the evolution forward in time. The ray tracing argument of Hawking will also be replaced by the continuity condition of the mode on the star surface.
A posteriori (see Fig. [5]), only rays very close to the horizon contribute significantly to the steady Hawking flux. In this case, a mode at infinity with finite frequency must have started from the star surface with a huge frequency because of the very large red shift. Because the effective frequency on the star was very large one can use geometric optics to find the evolution of the modes through the interior of the star. Let v_0 be the value of advanced time that marks the formation of the event horizon. From Fig. [5], we see that a high frequency *in*-mode must have started at past infinity at advanced time v very close to v_0, in order for it to arrive at future infinity near the event horizon. After having crossed the star, this mode then reached the star surface at an advanced time $v^{'}$ which must have been very close to the value v_1 which marks the birth of the black hole. The phase of the mode will be the crucial quantity: it is a function $\Theta(u)$ of the retarded time, because now the wave has been converted into an outgoing mode. One can describe the motion of the star surface by giving the advanced time as a function, $v_s(U)$, of the Kruskal retarded coordinate $U = -\exp(-ku)$. Then $v^{'} = v_s(U)$. We choose to do this because v and U are well defined and finite at the horizon ($u = \infty$ there). At $v^{'}$, the positive frequency ingoing mode enters the collapsing star essentially with unit amplitude, because $\omega > M^{-1}$, and phase $\Theta^{'}(v^{'}) = -\omega v^{'}$. This is because the phase was $-\omega v^{'}$ near past infinity and according to geometric optics the phase is a constant along the ray. The continuity of the field on the star surface now gives the phase $\Theta(u) = \Theta^{'}(v^{'}) = -\omega v^{'} = -\omega v_s(U)$ and the amplitude will be unity. It remains to find the asymptotic form of the function $v_s(U)$. As has been noted by Price [37], the motion of the star surface acquires a universal behaviour near the formation of the horizon, independently from the details of the collapse. In fact, for $v_0 - v$ small and positive, U must be very small and negative. Hence

$$v_s(U) \simeq v_1 + CU + O(U^2) = v_1 - Ce^{-u/4M} + O(U^2) \tag{5.38}$$

where C is a positive constant characteristic of the collapse process. This gives immediately Eq.(5.37), the step function $\theta(-U)$ just encoding the condition that $v_0 - v$ was small and positive. As it is apparent from Eq.(5.37), the modes are no longer monochromatic at late times due to the time increasing time dilation which is imparted to them by the motion of the collapsing star. Notice also that $P_{j\omega}$ in (5.37) is positive frequency with respect to retarded Kruskal time U, which was the boundary condition on the past horizon used to define the Unruh state in the Kruskal manifold.
To calculate the Bogoljubov coefficients one takes the Fourier transform of $\exp(-i\omega U)$, leading to (see also Ref. [11, 12])

$$e^{-i\omega U} = \int_0^\infty \frac{d\omega^{'}}{2\pi k} \left\{ e^{\pi\omega^{'}/2k}\omega^{i\omega^{'}/k}\Gamma(-i\omega'/k)e^{-i\omega^{'}u} + e^{-\pi\omega^{'}/2k}\omega^{-i\omega^{'}/k}\Gamma(i\omega'/k)e^{i\omega^{'}u} \right\}$$

This can be proved by making use of the method of residues, summing over the poles of the Euler's Γ-function in the upper half complex plane. Therefore, from Eq.(5.37) it follows that, as $r \to 2M$

$$P_{j\omega} \simeq \int_0^\infty \frac{d\omega^{'}}{\sqrt{4\pi\omega^{'}}} r^{-1} r^*_{\omega^{'},\omega} e^{-i\omega^{'}u} Y_j(\theta,\phi) - \int_0^\infty \frac{d\omega^{'}}{\sqrt{4\pi\omega^{'}}} r^{-1} s_{\omega^{'},\omega} e^{i\omega^{'}u} Y_j(\theta,\phi) \tag{5.39}$$

where we defined, for real x and $x^{'}$

$$r_{xx'} = -\frac{i}{2\pi}\frac{(x^{'})^{-ix/k}}{\sqrt{xx^{'}}}\Gamma[1+ik^{-1}x]e^{\pi x/2k} \tag{5.40}$$

$$s_{xx'} = \frac{i}{2\pi}\frac{(x^{'})^{-ix/k}}{\sqrt{xx^{'}}}\Gamma[1+ik^{-1}x]e^{-\pi x/2k} \tag{5.41}$$

and the constant C has been absorbed into a redefinition of the scale of U, or equivalently into a redefinition of the origin of u. From Eq.(5.31) it follows that a wave at the horizon with phase $\Theta = -\omega^{'}u$ and unit amplitude arrives at infinity with phase $-\omega^{'}u$ and amplitude $B_j(\omega^{'})$. In the language of quantum theory, the particles associated to each monochromatic component in the wave packet (5.39) will tunnel through the potential barrier surrounding the black hole with probability $|B_j(\omega^{'})|^2$. Therefore the mode at future infinity will be

$$P_{j\omega} \simeq \int_0^\infty \frac{d\omega^{'}}{\sqrt{4\pi\omega^{'}}}r^{-1}\left[\alpha^*_{l\omega^{'}\omega}e^{-i\omega^{'}u} - \beta_{l\omega^{'}\omega}e^{i\omega^{'}u}\right]Y_j(\theta,\phi) \tag{5.42}$$

where

$$\alpha_{l\omega\omega'} = -\frac{i}{2\pi}e^{-i\omega^{'}v_1}B^*_l(\omega)\frac{(\omega^{'})^{-i4M\omega}}{\sqrt{\omega^{'}\omega}}\Gamma(1+i4M\omega)e^{2\pi M\omega} \tag{5.43}$$

$$\beta_{l\omega\omega'} = \frac{i}{2\pi}e^{i\omega^{'}v_1}B^*_l(\omega)\frac{(\omega^{'})^{-i4M\omega}}{\sqrt{\omega^{'}\omega}}\Gamma(1+i4M\omega)e^{-2\pi M\omega} \tag{5.44}$$

The Bogoljugov β-coefficients satisfy the relation

$$\int_0^\infty \beta^*_{l\omega\sigma}\beta_{l\omega^{'}\sigma}d\sigma = |B_l(\omega)|^2[\exp(8\pi M\omega)-1]^{-1}\delta(\omega-\omega^{'}) \tag{5.45}$$

If now the quantum field were in the $|in>$ ground state, the expected number of particles at infinity with energy ω would be just given by (5.45) with $\omega = \omega^{'}$, and therefore it would be infinite, as it is proportional to $\delta(0)$. This reflects the fact that particles are being emitted at a steady rate for an infinite time, as judged from infinity. As in the scattering theory, the way out is to write, for very large T

$$\delta(\omega-\omega^{'}) = \frac{1}{2\pi}\int_{-T/2}^{T/2} e^{i(\omega-\omega^{'})t}dt \tag{5.46}$$

so that $\delta(0) = T/2\pi$. Writing Eq.(5.45) for $\omega = \omega^{'}$ and dividing by T gives then the expected number of emitted particles arriving at infinity per unit time in the frequency range $d\omega$ around ω and angular momentum (l,m). This is the celebrated Planckian spectrum first discovered by Hawking

$$dN_{l\omega} = |B_l(\omega)|^2[\exp(8\pi M\omega)-1]^{-1}\frac{d\omega}{2\pi} \tag{5.47}$$

leading to a temperature $T = (8\pi M)^{-1}$, or

$$T = \frac{\hbar c^3}{8\pi MG} \tag{5.48}$$

One has the same result in the collapse of a spherical thin shell, where the knowledge that the interior of the shell is flat Minkowski space allows more detailed calculations [83]. Because $dN_{l\omega}$ is independent on m and since for each l there are $(2l+1)$ available states, the luminosity of the black hole will be

$$L = \sum_{l=0}^{\infty}(2l+1)\int_0^{\infty} \omega dN_{l\omega} = \sum_{l=0}^{\infty}(2l+1)\int_0^{\infty} |B_l(\omega)|^2 \frac{\omega}{\exp(8\pi M\omega)-1}\frac{d\omega}{2\pi} \tag{5.49}$$

Few remarks are in order. First notice that $T = k/2\pi$ where $k = 1/4M$ is the surface gravity of the black hole. This is the same relationship between acceleration and temperature that was discovered in Rindler space, with no black holes at all. The thermal character of the emission also suggests that the final state at $\mathcal{I}^+$ be a mixed state as indeed a closer examination confirms [2]. Hence even if classically it is possible to predict future infinity from knowledge of data at past infinity, it is not possible to do so if quantum effects are taken into account. Indeed, since the final state at $\mathcal{I}^+$ is a density matrix one can only measure the probabilities for the particles to be emitted but not the amplitudes. This breakdown of predictability was discussed at length in [64]. Another classical result that is violated by quantum effects is the area theorem. Clearly the emission decreases the mass so that $\dot{A} = 32\pi M\dot{M} < 0$, the dot denoting time derivative. We see also that the frequencies that predominantly contribute to the Hawking emission are centered around $\omega \simeq (M)^{-1}$. Hence a mode that started from the star at retarded time u must have had a huge frequency of the order $\omega\exp(ku)$. This exceedes the Planck mass after a time of order $u \simeq M\log M$. Hence the black hole acts as a magnifying lens, making the consequences of high energy physics visible to an asymptotic observer. As it stands, this could be a severe problem since one has no knowledge of the physics at the Planck scale. However, the energy density of the quantum fields near the horizon is of order $\rho \simeq M^{-4}$ [82, 84, 85, 86, 11], and therefore

$$\rho \simeq \frac{\hbar c^9}{M^4 G^4} \simeq 10^{-36}(M_\odot/M)^4 erg\cdot cm^{-3} \tag{5.50}$$

which is quite small for large black holes, although the red shifted energy still approach the Planck scale.

5.4 Probability distributions and fluctuations

The particles being emitted randomly, there will be fluctuations around the expected particle numbers. It has been shown that not only is the expected number of particles thermal but also the probability distribution of the particles in the out-states at infinity are identical in all aspects to that of black body radiation at temperature $k/2\pi$ [2, 4, 64, 10]. By choosing a suitable set of horizon states it is not hard to see that particles are created in pairs (see also Ref. [80] for a path integral derivation making clear what we are saying), one crossing the horizon with negative energy and the other escaping to infinity with probability $\Gamma_{l\omega} = |B_l(\omega)|^2$.
To compute the fluctuations in the number of particles emitted in a normalized mode $q_j(\omega)$, one needs to calculate the expected square of the number operator for that mode, namely

$$< N_j^2 >=< in|C_j^\dagger C_j C_j^\dagger C_j|in >=< in|C_j^\dagger C_j|in > + < in|C_j^\dagger C_j^\dagger C_j C_j|in > \tag{5.51}$$

where C_j is the smeared operator

$$C_j = \int_0^{\infty} C_{j\omega} q_j(\omega)d\omega, \qquad \|q_j\|^2 = \int_0^{\infty} |q_j(\omega)|^2 d\omega = 1 \tag{5.52}$$

We further take $q_j(\omega)$ sharply peaked around some frequency and put

$$< \omega >= \int_0^\infty \omega |q_j(\omega)|^2 d\omega \tag{5.53}$$

The matrix element $< C^\dagger C^\dagger C C >$ is easy to compute using the Bogoljubov transformation (5.36), and the matrix element $< C^\dagger C >$ is known from our previous calculations. The mode being sharply peaked one obtains

$$< in|C_j^\dagger C_j|in >= \frac{\Gamma_j e^{-2\pi k^{-1} <\omega>}}{1 - e^{-2\pi k^{-1} <\omega>}} \tag{5.54}$$

where

$$\Gamma_j = \int_0^\infty |q_j(\omega)|^2 \Gamma_{l\omega} d\omega \tag{5.55}$$

is the transmission probability for a particle in mode $q_j(\omega)$ to escape the potential barrier. With the some understanding the result for the mean square fluctuation is

$$\sigma_j^2 =< N_j^2 > - < N_j >^2 = \frac{\Gamma_j e^{-2\pi k^{-1} <\omega>} \left[1 + (\Gamma_j - 1) e^{-2\pi k^{-1} <\omega>} \right]}{\left(1 - e^{-2\pi k^{-1} <\omega>} \right)^2} \tag{5.56}$$

In a similar way it is possible to compute the higher moments $< N_j^k >$. Using the information that particle numbers in different modes are mutually independent random variables, we now show that all moments are derivable from the following probability distribution

$$P_j(N) = \frac{\left(\Gamma_j e^{-2\pi k^{-1} \omega} \right)^N [1 - \exp(-2\pi k^{-1} \omega)]}{[1 - (1 - \Gamma_j) \exp(-2\pi k^{-1} \omega)]^{N+1}} \tag{5.57}$$

for N particles to arrive at infinity in a mode carrying angular momentum (l, m) and whose energy is sharply peaked around ω [64, 10]. In fact, in order that N particles arrive at infinity in a given mode it is necessary that any $M \geq N$ particles were in the given mode at the horizon, out of which N escaped to infinity with probability Γ_j^N (since Γ_j is the probability for one particle to escape) and $M - N$ went across the horizon with probability $(1 - \Gamma_j)^{M-N}$. Putting $x = \exp(-2\pi k^{-1} \omega)$, the probability of this random event, i.e. to have $M \geq N$ particles at the horizon out of which any N among them escape to infinity, is

$$P_{M,N} = C(M, N)(1 - x) x^M \Gamma_j^N (1 - \Gamma_j)^{M-N} \tag{5.58}$$

where $C(M, N)$ is the binomial coefficient taking care that the emitted bosons are indistinguishable, and $(1 - x)x^M$ is the usual thermal probability to have M particles in the given mode. Therefore the total probability to have N particles at infinity will be

$$P_j(N) = \sum_{M \geq N} P_{M,N} \tag{5.59}$$

which is just Eq.(5.57), as can be seen by summing the elementary series. It is then a simple matter to verify that the mean square deviation is derivable from Eq.(5.57) by the usual rules of probability calculus. Similarly, the joint probability distribution for N_1 particles in mode (ω_1, l_1), $\cdots$, N_k particles in mode (ω_k, l_k) is

$$P(N_1, ..., N_k) = \prod_i P(N_i) \tag{5.60}$$

by the statistical independence of the emission process. For fermions one can have either zero or one particle in a given mode. The thermal probability being $x^k/(1+x)$, $k=0,1$, the probability for one or zero particles at infinity will be

$$P_j(1) = \frac{x\Gamma_j}{1+x}, \qquad P_j(0) = 1 - P_j(1) = \frac{1+x-x\Gamma_j}{1+x} \tag{5.61}$$

5.5 The Reissner-Nordström's black hole

We consider now the charged and rotating black hole solutions. First, they are far from be a trivial generalization of the previous case. For example, (i) there is no analogue of the Birkhoff theorem for Kerr black holes, (ii) the Kerr geometry is much more complicated than Schwarzschild, (iii) the new phenomenon of super radiance comes into play for both rotating and charged solutions, (iv) the potential barrier is frequency, and charge, dependent. Indeed, although the physical ideas behind the quantum emission are sound, there is still no sufficiently rigorous treatment of all points involved. Second, this provides a useful check that the basic ideas leading to the quantum emission process are not limited to spherical symmetry but are instead of much greater validity.
As was shown in Sec.(3.1), the event horizon in the R-N solution is the null hypersurface located at $r = r_+ = M + \sqrt{M^2 - Q^2}$. The electrostatic potential of the event horizon is $\Phi = Q/r_+$ and the surface gravity is

$$k = \frac{r_+ - r_-}{2r_+^2} \tag{5.62}$$

The tortoise coordinate

$$r^* = r + \frac{r_+^2}{r_+ - r_-}\log(r - r_+) - \frac{r_-^2}{r_+ - r_-}\log(r - r_-) \tag{5.63}$$

is necessary to define the retarded and advanced times, $u = t - r^*$, $v = t + r^*$ respectively. Constant values of u and v specify the radial null geodesics of the R-N geometry. The Klein-Gordon equation for a charged scalar field is

$$g^{ab}(\nabla_a + ieA_a)(\nabla_b + ieA_b)\phi - \mu^2\phi = \Box\phi + g^{00}[2ieA_0\partial_0 - e^2A_0^2]\phi - \mu^2\phi = 0 \tag{5.64}$$

Besides the charge one has now the additional complication due to the mass μ. This can be handled by introducing the radial momentum p such that $\omega^2 = p^2 + \mu^2$, ω being the frequency. Eq.(5.64) has a complete set of positive frequency solutions of the kind ($j = (l, m)$)

$$\phi_{j\omega,e}(x) = (4\pi p)^{-1/2} r^{-1} R_{j\omega,e}(r) Y_j(\theta,\phi) e^{-i\omega t}, \qquad \omega \geq \mu \tag{5.65}$$

which have been normalized to a δ-function of the frequency, rather than p. The radial function obeys the ordinary differential equation

$$\frac{d^2 R_{j\omega,e}}{dr^{*2}} + V_{l\omega}(r) R_{j\omega,e} = 0 \tag{5.66}$$

The potential function is now frequency dependent, being given by

$$\begin{aligned} V_{l\omega}(r) \;=\; & \left(\omega - \frac{eQ}{r}\right)^2 \\ & - \left[\frac{l(l+1)}{r^4} + \frac{2(r-M)}{r^5} - \frac{2(r-r_+)(r-r_-)}{r^6} + \frac{\mu^2}{r^2}\right](r-r_+)(r-r_-) \end{aligned} \tag{5.67}$$

and converges to $\omega^2 - \mu^2$ as $r \to \infty$ and to $(\omega - e\Phi)^2$ as $r \to r_+$. The modes may then be distinguished by the asymptotic behaviour they take at infinity and the horizon. Up to a logarithmic phase due to the Coulomb field, the ingoing modes from past infinity describing the scattering states behave asymptotically

$$\overleftarrow{R}_{j\omega,e}(r) \simeq \begin{cases} e^{-ipr^*} + \overleftarrow{A}_{j,e}(\omega)e^{ipr^*} & r^* \to \infty \\ \overleftarrow{B}_{j,e}(\omega)e^{-i(\omega - e\Phi)r^*} & r^* \to -\infty \end{cases} \tag{5.68}$$

where $\omega \geq \mu$ and $p > 0$. With the same understanding, the outgoing modes from the past horizon behave

$$\overrightarrow{R}_{j\omega,e}(r) \simeq \begin{cases} e^{i(\omega - e\Phi)r^*} + \overrightarrow{A}_{j,e}(\omega)e^{-i(\omega - e\Phi)r^*} & r^* \to -\infty \\ \overrightarrow{B}_{j,e}(\omega)e^{ipr^*} & r^* \to \infty \end{cases} \tag{5.69}$$

Taken together, they form a complete orthonormal set in the static exterior region of the charged black hole. The constancy of the Wronskian of two radial solutions implies the unitarity equation [87]

$$p\overrightarrow{B}_{j,e} = (\omega - e\Phi)\overleftarrow{B}_{j,e} \tag{5.70}$$

which relates the ingoing/outgoing transmission factors, and

$$(\omega - e\Phi)|\overleftarrow{B}_{j,e}(\omega)|^2 = p[1 - |\overleftarrow{A}_{j,e}(\omega)|^2] \tag{5.71}$$

Therefore those modes for which $\omega - e\Phi < 0$ are scattered with greater amplitude than they had initially, a phenomenon known as super radiance. It was first discovered in [88, 89] by analysing diffusion of electromagnetic waves from a rotating cylinder and from the Kerr black hole.
Now if the black hole formed from the collapse of a spherical charged ball, the space-time will be asymptotically simple in the past. As in Sec.(4.2), one introduces a complete set of solutions $P_{j\omega,e}$ which are positive frequency and ingoing at past infinity, and expands the quantum field in the form

$$\phi = \sum_j \int_\mu^\infty [A_{j\omega}P_{j\omega,e} + B_{j\omega}^\dagger P_{j\omega,-e}^*]d\omega \tag{5.72}$$

This is because the complex conjugate of a solution of the charged Klein-Gordon equation is a solution of the Klein-Gordon equation with reversed charge. The $|in>$ vacuum is then defined by $A_{j\omega}|in>= B_{j\omega}|in>= 0$. Similarly, one introduces a complete set $\{Q_{j\omega,e}\}$, of positive frequency solutions which are outgoing to infinity, so that

$$\phi = \sum_j \int_\mu^\infty d\omega[C_{j\omega}Q_{j\omega,e} + D_{j\omega}^\dagger Q_{j\omega,-e}^*] + \phi_H \tag{5.73}$$

where ϕ_H is that part which has no outgoing components, and the $|out>$ vacuum is defined by $C_{j\omega}|out>= D_{j\omega}|out>= 0$. Then one has the Bogoljubov transformations

$$C_{j\omega'} = \int_\mu^\infty d\omega[\alpha_{j\omega'\omega,e}^* A_{j\omega} - \beta_{j\omega'\omega,e}^* B_{j\omega}^\dagger] \tag{5.74}$$

$$D_{j\omega'}^\dagger = \int_\mu^\infty d\omega[\alpha_{j\omega'\omega,-e}B_{j\omega}^\dagger - \beta_{j\omega'\omega,-e}A_{j\omega}] \tag{5.75}$$

The coefficients can be computed in the ultra relativistic limit $\omega \gg \mu$. To see this, one needs the following result: let v_0 be the advanced time that marks the birth of the event horizon. Then the outgoing component of the ultra relativistic incoming modes $P_{j\omega,e}$, with $\omega \gg \mu$, labelled by v such that $v - v_0$ was small and negative, is

$$P_{j\omega,e} \simeq (4\pi p)^{-1/2} r^{-1} Y_j(\theta,\phi) \exp(iCpe^{-ku}) e^{-ie\Phi t - \omega v_1} \theta(-U) \tag{5.76}$$

where v_1 is the value of advanced time that marks the birth of the black hole, i.e. the value at which matter disappears from sight by crossing the event horizon, and $U = -\exp(-ku)$.
To show this, let $\Theta(r^*, t)$ be the phase of $P_{j\omega,e}$. Since $v - v_0$ is small and negative, the mode emerges from the star at a value v' which must be very close to v_1. From Eq.(5.68), the ingoing mode which enters the star from infinity at v' arrives on the horizon with the phase $\Theta' = -(\omega - e\Phi)r^* - \omega t = -\omega v' - e\Phi t + \alpha$, where α is an unessential constant. This is because both $v' = t + r^*$ as well as the phase are constant along the rays in geometric optics. With the same approximation, either there is complete transmission or there is complete reflection. The reflected waves arrive at infinity with the same frequency as the initial ones and have nothing to do with the creation of particles. Therefore the frequency must be so high that $\overleftarrow{B}_{j,e}(\omega)$ is nearly one. Since the field is continuous, one has then $\Theta(r^*, t) = \Theta'$ and so

$$P_{j\omega,e} \simeq (4\pi p)^{-1/2} r^{-1} Y_j(\theta,\phi) e^{-ipv' - ie\Phi t} \theta(v_0 - v) \tag{5.77}$$

where $\omega \simeq p$ has been used due to the ultra relativistic condition $\omega \gg \mu$. Now events labelled by v_1 have $U = 0$ and since v' is very close to v_1 one can expand

$$v' = v_1 + CU + O(U^2) \tag{5.78}$$

where C is a positive constant (since $U < 0$) depending on the collapse process (it is related to the radial velocity of the ball surface on the event horizon). Substituting this into $P_{j\omega,e}$ gives the result, the step function $\theta(-U)$ just encoding the fact that $v - v_0$ was small and negative.
To analyse the particle content at future infinity one has to find how the mode $P_{j\omega,e}$ will propagate to infinity in the Reissner-Nordström geometry, given that its form near the horizon is Eq.(5.76). To this aim, consider the identity (the easiest way to prove it is by the method of residues)

$$e^{-i\omega U} = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi k} e^{\pi\omega'/2k} \omega^{i\omega'/k} \Gamma(-i\omega'/k) e^{-i\omega' u} \tag{5.79}$$

The field at future infinity can be found using the identity and making the shift $\omega' \to \omega' - e\Phi$. In this way, at $r \to r_+$ and for large $\omega \simeq p$, one obtains

$$\begin{aligned} P_{j\omega,e} \simeq & \int_0^{\infty} \frac{d\omega'}{\sqrt{4\pi(\omega' - e\Phi)}} r^{-1} r^*_{\omega' - e\Phi, p} e^{-i(\omega' - e\Phi)u - ie\Phi t} Y_j(\theta,\phi) \\ & - \int_o^{\infty} \frac{d\omega'}{\sqrt{4\pi(\omega' + e\Phi)}} r^{-1} s_{\omega' + e\Phi, p} e^{i(\omega' + e\Phi)u - ie\Phi t} Y_j(\theta,\phi) \end{aligned} \tag{5.80}$$

where $r_{xx'}$ and $s_{xx'}$ were defined in Eq.(5.40), (5.41). From Eq.(5.69) one sees that a wave at the horizon with phase $\Theta = -(\omega' - e\Phi)u - e\Phi t$ and unit amplitude arrives at infinity with phase $\omega' t - p' r^*$ and amplitude $\overrightarrow{B}_{j,e}(\omega')$, if $\omega' \geq \mu$. This is because the solutions outside of this range are exponentially damped at infinity, as it follows from the radial differential equation (5.66). The solution then tunnels through the potential barrier arriving at future infinity in the form

$$P_{j\omega,e} \simeq \int_\mu^\infty \frac{d\omega'}{\sqrt{4\pi p'}} r^{-1} \left[\alpha^*_{j\omega'\omega,e} e^{-i\omega' t + ip' r^*} - \beta_{j\omega'\omega,-e} e^{i\omega' t - ip' r^*} \right] Y_j(\theta,\phi) \tag{5.81}$$

One can read off from this the Bogoljubov coefficients

$$\alpha_{j\omega'\omega,e} = -\frac{i}{2\pi} \overset{\leftarrow *}{B}_{j,e}(\omega') \frac{(p)^{-i(\omega'-e\Phi)/k}}{\sqrt{pp'}} \Gamma[1 + ik^{-1}(\omega' - e\Phi)] e^{\pi(\omega'-e\Phi)/2k} \tag{5.82}$$

$$\beta_{j\omega'\omega,e} = \frac{i}{2\pi} \overset{\leftarrow *}{B}_{j,e}(\omega') \frac{(p)^{-i(\omega'-e\Phi)/k}}{\sqrt{pp'}} \Gamma[1 + ik^{-1}(\omega' - e\Phi)] e^{-\pi(\omega'-e\Phi)/2k} \tag{5.83}$$

where use of Eq.(5.70) has been done. As in Sec.(5.3), this gives the average number of particles with charge e arriving at infinity per unit time, in the frequency interval $d\omega$ around ω and with angular momentum j, in the form

$$dN_{j\omega,e} = |\overset{\leftarrow}{B}_{j,e}(\omega)|^2 \frac{\omega - e\Phi}{p} \frac{1}{\exp[2\pi k^{-1}(\omega - e\Phi)] - 1} \frac{d\omega}{2\pi} \tag{5.84}$$

From Eq. (5.71) we may write, alternatively

$$dN_{j\omega,e} = [1 - |\overset{\leftarrow}{A}_{j,e}(\omega)|^2] \frac{1}{\exp[2\pi k^{-1}(\omega - e\Phi)] - 1} \frac{d\omega}{2\pi} \tag{5.85}$$

The superradiance condition, $\omega - e\Phi < 0$, ensures the positivity of $dN_{j\omega,e}$ for all frequencies and the temperature of the black hole is $T = k/2\pi$. Since the emitted particles carry energy and charge, the black hole luminosity, L, and charge loss are

$$L = \sum_j \int_\mu^\infty \omega[dN_{j\omega,e} + dN_{j\omega,-e}] \tag{5.86}$$

$$\frac{dQ}{dt} = -e \sum_j \int_\mu^\infty [dN_{j\omega,e} - dN_{j\omega,-e}] \tag{5.87}$$

Because $e/\mu \gg 1$ for all known elementary particles, one expects a rapid discharge of the black hole [87]. In the zero temperature limit, namely for large mass black holes, one obtains

$$dN_{j\omega,e} = -[1 - |\overset{\leftarrow}{A}_{j,e}(\omega)|^2]\theta(\omega - e\Phi) \frac{d\omega}{2\pi} \tag{5.88}$$

where $\theta(x)$ is the step function. This result represents the spontaneous emission corresponding to the classical superradiant effect. It is a Schwinger type process of pair creation [87], one partner escaping to infinity with positive energy ω and the other falling down the hole with negative energy $\omega - e\Phi$ as measured at infinity but with positive energy as measured locally near the horizon. This is because the local energy of the superradiant modes is positive [3]. Hence positive energy is flowing out as well down the black hole. This helps to explain the fact that the spontaneous emission respects the area theorem, unlike the thermal Hawking process. Indeed, the energy ω of the emitted particles is much lower than the black hole mass and therefore the black hole parameters M, Q and Φ would evolve adiabatically with $\dot{M} \ll \omega M$, the dot denoting time derivatives. From the first law of black hole mechanics one obtains

$$\frac{k}{8\pi}\dot{A} = (\dot{M} - \Phi\dot{Q}) = -\sum_j \int_\mu^{e\Phi} (\omega - e\Phi) dN_{j\omega,e} \tag{5.89}$$

where $A = 4\pi r_+^2 = 4\pi(M+\sqrt{M^2 - Q^2})^2$. Therefore $\dot{A} > 0$, which implies that the hole discharge more rapidly than it looses mass, as was noted above. At the end Q will vanish, A will reach the limit value $8\pi M^2$ and the spontaneous emission stops.

5.6 The Kerr's black hole

It was established in Sec.(3.2) that the hypersurface $r = r_+ = M+\sqrt{M^2 - a^2}$ is an event horizon in the Kerr geometry, having the surface gravity

$$k = \frac{r_+ - r_-}{4Mr_+} \tag{5.90}$$

and the angular velocity

$$\Omega = \frac{a}{2Mr_+} \tag{5.91}$$

relative to infinity. As usual, one introduces the tortoise coordinate

$$r^* = r + \frac{2Mr_+}{r_+ - r_-}\log(r - r_+) - \frac{2Mr_-}{r_+ - r_-}\log(r - r_-) \tag{5.92}$$

and defines retarded and advanced coordinates by $u = t-r^*$, $v = t+r^*$. Constant values of u and v give the radial null geodesics in each θ =constant section. In the stationary Kerr geometry the Klein-Gordon equation is separable. Following de Witt notation [3], we shall denote the outgoing modes from the past horizon (in the exact Kerr geometry) by $\overrightarrow{\phi}_{j\omega}$, where

$$\overrightarrow{\phi}_{j\omega} = (4\pi\omega)^{-1/2}(r^2 + a^2)^{-1/2} S_j(\theta)\, \overrightarrow{R}_{j\omega}\, e^{im\phi}e^{-i\omega t} \tag{5.93}$$

where $\omega > 0$ and $S_j(\theta)$ are the real spheroidal harmonics for scalar fields[3]. The index j stands for (l, m) with $l \geq 0$ and $|m| \leq l$ and similarly j' stands for (l', m') and so on, unless otherwise stated. The radial function then obey the differential equation

$$\frac{d^2 R_{j\omega}}{dr^{*2}} + V_{j\omega}(r)R_{j\omega} = 0 \tag{5.94}$$

where the potential converges rapidly to $(\omega - m\Omega)^2$ as $r \to r_+$ and to ω^2 as $r \to \infty$ [3]. Then the appropriate asymptotic form for outgoing modes will be

$$\overrightarrow{R}_j \simeq \begin{cases} e^{i(\omega-m\Omega)r^*} + \overrightarrow{A}_j(\omega)e^{-i(\omega-m\Omega)r^*} & r^* \to -\infty \\ \overrightarrow{B}_j(\omega)e^{i\omega r^*} & r^* \to \infty \end{cases} \tag{5.95}$$

The asymptotic form of ingoing modes from past infinity will be

$$\overleftarrow{R}_j \simeq \begin{cases} e^{-i\omega r^*} + \overleftarrow{A}_j(\omega)e^{i\omega r^*} & r^* \to \infty \\ \overleftarrow{B}_j(\omega)e^{-i(\omega-m\Omega)r^*)} & r^* \to -\infty \end{cases} \tag{5.96}$$

The reflection/transmission coefficients for both kind of modes can be related by noting that the Wronskian of two solutions of the radial equation is a constant. One finds easily the unitarity relations [90, 3]

$$\omega \overrightarrow{B}_j = (\omega - m\Omega) \overleftarrow{B}_j \tag{5.97}$$

which relates the ingoing/outgoing transmission factors, and

$$(\omega - m\Omega)| \overleftarrow{B}_j (\omega)|^2 = \omega[1 - | \overleftarrow{A}_j (\omega)|^2] \tag{5.98}$$

From this we see that for $\omega > 0$ one has superradiance whenever $\omega - m\Omega < 0$ [88, 89, 91]. This is the wave counterpart of a well known process of energy extraction from the Kerr black hole, first discussed by Penrose [34] and Penrose-Floyd [92]. An extensive discussion and more detailed considerations relative to the process can be found in [38, 93, 94, 95].
Now one considers a black hole that formed from the collapse of some initial matter configuration. Then space-time has an asymptotically simple past but the event horizon will eventually form. One has to apply then the general scheme of Sec.(4.2). One introduces a complete orthonormal set of solutions $P_{j\omega} \equiv P_{l,m,\omega}$ of the Klein-Gordon equation (in the collapse geometry) which are ingoing and positive frequency at past infinity, long before the onset of collapse. The quantum field can be expanded in the form

$$\phi = \sum_j \int_0^\infty [A_{l,m,\omega} P_{l,m,\omega} + A^\dagger_{l,-m,\omega} P^*_{l,-m,\omega}] d\omega \tag{5.99}$$

This decomposition then defines the $|in>$ (initial) vacuum state. In the final stationary Kerr geometry there also exist solutions with the asymptotic form

$$Q_{j\omega} \simeq (4\pi\omega)^{-1/2}(r^2+a^2)^{-1/2} S_j(\theta) \begin{cases} e^{-i\omega u + im\phi} + \overleftarrow{A}_j^* (\omega) e^{-i\omega v + im\phi} & r^* \to \infty \\ \overleftarrow{B}_j^* (\omega) e^{-i(\omega - m\Omega)u + im(\phi - \Omega t)} & r^* \to -\infty \end{cases} \tag{5.100}$$

Wave packets formed by superposition of the $Q_{j\omega}$ are purely outgoing and have zero Cauchy data on the event horizon, so they can be taken to define the $|out>$ vacuum state long after the completion of the collapse. Thus, one can also write

$$\phi = \sum_j \int_0^\infty [C_{l,m,\omega} Q_{l,m,\omega} + C^\dagger_{l,-m,\omega} Q^*_{l,-m,\omega}] d\omega + \phi_H \tag{5.101}$$

where ϕ_H is the part with no outgoing components and $C_{l,m,\omega}|out>= 0$. This induces the mixing between operators

$$C_{lm'\omega'} = \sum_{|m|\le l} \int_0^\infty d\omega [\alpha^*_{\omega'\omega, m'm} A_{l,m,\omega} - \beta^*_{\omega'\omega, m'm} A^\dagger_{l,-m,\omega}] \tag{5.102}$$

It can now be argued that the outgoing part of the ingoing modes emerge from the collapsing star when this is about to form the horizon with the asymptotic form (where C is a positive constant and $U = -\exp(-ku)$)

$$P_{j\omega} \simeq (4\pi\omega)^{-1/2}(r^2+a^2)^{-1/2} \sum_{j'} \exp(i\omega C e^{-ku}) e^{im(\phi - \Omega t)} \Lambda_{jj'} S_{j'}(\theta)\theta(-U) \tag{5.103}$$

provided that (i) the geometry rapidly settles down to the Kerr geometry, (ii) ω be sufficiently large so as to make possible the use of geometric optics, and (iii) the field be continuos everywhere. $\Lambda_{jj'}$ is a unitary matrix which takes into account the fact that the θ-dependence of $P_{j\omega}$ is not given in general by the function $S_j(\theta)$. However, a posteriori the emission process is independent from $\Lambda_{jj'}$ [11, 12], because to get the expected number of emitted particles one has to square and sum over j' the Bogoljubov coefficient, which are proportional to $\Lambda_{jj'}$. Since this is unitarity, this has no effect on the final formula and therefore one may safety set $\Lambda_{jj'} = \delta_{jj'}$. The argument for (5.103) is just the same as for the spherical black holes, the only relevant difference being that in the present case there is no analogue of the Birkhoff theorem. The exterior geometry is then given by the Kerr metric only asymptotically at late time. However the advanced time coordinate can still be defined but will be given by $t+r^*$ only asymptotically. Let v_0 the value of v which marks the birth of the event horizon: then the ingoing wave front labelled by v_0 just fails to reach infinity. Eq.(5.103) gives the asymptotic form of those ingoing modes which were labelled by v-values for which $v - v_0$ was small and negative. Let also v_1 the value of the advanced time that marks the birth of the black hole. Then the ingoing mode that started with v very close to v_0 will emerge from the star as an outgoing mode at some advanced time v' which must be very close to v_1. In the case of spherical collapse these quantities were illustrated in Fig. [5], where since the exterior geometry is given by the Schwarzschild metric all the time the v coordinate is always $t + r^*$.
From conditions (i), (ii) and Eq.(5.96) we see that the phase of the ingoing mode labelled by v' is $\Theta(v', \phi, t) = -\omega v' + m(\phi - \Omega t)$, and by (iii) this is also the phase of the outgoing mode we are presently interested. For ω higher than the potential barrier around the star we may also put $\overleftarrow{B}_j(\omega) = 1$, since there is complete transmission in geometric optics under the given conditions. Also the factor $\phi - \Omega t$ is finite on the horizon since in each surface $\theta =$constant, the ingoing null geodesics obey the equations [38]

$$\frac{dt}{d\lambda} = \frac{r^2 + a^2}{\Delta}\omega \qquad \frac{dr}{d\lambda} = -\omega \qquad \frac{d\phi}{d\lambda} = -\frac{a}{\Delta}\omega \tag{5.104}$$

where $\Delta = r^2 - 2Mr + a^2$. As a simple consequence of the equations, the quantity $\phi - \Omega t + \Omega r$ is a constant (i.e. it is independent from λ) which shows, incidentally, that the azimuthal angle diverges while approaching the horizon along null geodesics, i.e. the geodesic spirals around the horizon infinitely many times as seen from infinity. Now all points with $v = v_1$ lie on the event horizon of the Kerr geometry and therefore have vanishing Kruskal retarded U-coordinate, where $U = -\exp(-ku)$ and k is the surface gravity of the Kerr black hole. Points with $v = v'$ very close to v_1 have small and negative values of U and therefore one can expand

$$v' = v_1 + CU + O(U^2) \tag{5.105}$$

where $C > 0$. Hence the phase of the outgoing part of $P_{j\omega}$ must be

$$\Theta = -\omega v_1 + \omega C e^{-ku} + m(\phi - \Omega t) + O(U^2) \tag{5.106}$$

and this gives Eq.(5.103), the step function $\theta(-U)$ just encoding the condition that $v - v_0$ was small and negative.
To analyse the particle content at future infinity, one has to find how the mode $P_{j\omega}$ propagates to infinity in the Kerr geometry, given that its form near the horizon was Eq.(5.103). This can

be done using the identity (5.39) given in Sec.(5.5) and making the shift $\omega' \to \omega' - m\Omega$. One obtains in this way the asymptotic formula at future infinity

$$
\begin{aligned}
P_{j\omega} &\simeq \sum_{m'} \int_0^\infty \frac{d\omega'}{\sqrt{4\pi(\omega' - m'\Omega)}} (r^2+a^2)^{-1/2} \left[r^*_{\omega' - m'\Omega, \omega} \delta_{m',m} \vec{B}_{lm'}(\omega') e^{-i(\omega' u - im'\phi)} \right. \\
&- \left. s_{\omega' - m'\Omega, \omega} \delta_{m',-m} \vec{B}_{lm'}(\omega')^* e^{i(\omega' u - im'\phi)} \right] S_j(\theta) \qquad (5.107)
\end{aligned}
$$

By introducing the Bogoljubov coefficients

$$
\alpha_{\omega\omega',mm'} = -\frac{i}{2\pi} \overleftarrow{B}_j(\omega)^* \frac{(\omega')^{-i(\omega - m\Omega)/k}}{\sqrt{\omega\omega'}} \Gamma[1 + ik^{-1}(\omega - m\Omega)] e^{\pi(\omega - m\Omega)/2k} \delta_{m,m'} \qquad (5.108)
$$

$$
\beta_{\omega\omega',mm'} = \frac{i}{2\pi} \overleftarrow{B}_j(\omega)^* \frac{(\omega')^{-i(\omega - m\Omega)/k}}{\sqrt{\omega\omega'}} \Gamma[1 + ik^{-1}(\omega - m\Omega)] e^{-\pi(\omega - m\Omega)/2k} \delta_{m,-m'} \qquad (5.109)
$$

the asymptotic formula can be written in the form

$$
P_{j\omega} \simeq \sum_{m'} \int_0^\infty \frac{d\omega'}{\sqrt{4\pi\omega'}} \frac{1}{\sqrt{r^2+a^2}} \left[\alpha^*_{\omega'\omega,mm'} e^{-i(\omega' u - im'\phi)} - \beta_{\omega'\omega,mm'} e^{i(\omega' u - im'\phi)} \right] S_j(\theta)
$$

showing the outgoing part of the quantum field at future infinity as a superposition of those outgoing solutions which defined the $|out>$ vacuum state.

From Eq.(5.102), one obtains the expected number of particles arriving at infinity per unit time with quantum numbers $j = (l, m)$ in the frequency range $d\omega$ around ω, as

$$
dN_{j\omega} = |\overleftarrow{B}_j(\omega)|^2 \frac{\omega - m\Omega}{\omega} \frac{1}{\exp[2\pi k^{-1}(\omega - m\Omega)] - 1} \frac{d\omega}{2\pi} \qquad (5.110)
$$

Therefore the black hole behaves as a hot black boby radiator with the temperature $T = k/2\pi$. From Eq.(5.98), one may also write

$$
dN_{j\omega} = [1 - |\overleftarrow{A}_j(\omega)|^2] \frac{1}{\exp[2\pi k^{-1}(\omega - m\Omega)] - 1} \frac{d\omega}{2\pi} \qquad (5.111)
$$

and the luminosity of the hole will be

$$
L = \sum_j \int_0^\infty \omega dN_{j\omega} \qquad (5.112)
$$

The black hole also looses angular momentum at the rate

$$
\frac{dJ}{dt} = -\sum_j \int_0^\infty m dN_{j\omega} \qquad (5.113)
$$

where m is the azimuthal quantum number. When the mass of the black hole is very large only the superradiant modes contribute and the formula reduces to the Unruh-Starobinsky emission, which was obtained by ignoring the gravitational collapse [90, 88, 89, 91]. The spontaneous Unruh-Starobinsky emission is not very efficient to extract energy and angular momentum from the hole (except for gravitational waves) and, unlike the thermal emission, it respects the Hawking's area theorem [3]. The reason is analogous to that causing superradiance in the charged black hole case. A direct computation yields

$$
\frac{k}{8\pi} \dot{A} = -\sum_{j, m\Omega > 0} \int_0^{m\Omega} (\omega - m\Omega) dN_{lm\omega} \qquad (5.114)
$$

which is positive since $\omega - m\Omega < 0$ in the superradiant regime.

5.7 The luminosity of black holes

The efficiency of particle emission from black holes is clearly an important issue of the evaporation phenomenon. As the temperature of a black hole is inversely proportional to the black hole mass and the area A is proportional to M^2, the total power emitted is proportional to AT^4 or M^{-2}. From this it follows that the lifetime is proportional to M^3. We shall see below that black hole formed by stellar collapse, thus having $M \geq M_\odot$, have a lifetime of order 10^{66} yr. Therefore the thermal emission is insignificant for such black holes. However, it is important for primordial black holes, for which M is less than 10^{15} g, and the lifetime is less than the age of the universe.

Another problem is how rapidly a charged rotating black hole discharge and spins down. The main difference in the rapidity of the two processes can be seen as follows. The two parameters

$$a_* = \frac{J}{M^2}, \qquad Q_* = \frac{Q}{M} \tag{5.115}$$

are constrained by the inequality

$$a_*^2 + Q_*^2 \leq 1 \tag{5.116}$$

This is because the solutions with $a_*^2 + Q_*^2 > 1$ do not describe black holes and contains naked singularities. A charged emitted particle carries off the charge $\Delta Q = e = 0.085$ and an angular momentum $-\Delta J = m$, both of order unity. Hence due to the constraint the number of charged particles needed to neutralize the hole is Q/e which is at most of order M, and the number of particles needed to spin down the hole is J/m which is at most of order M^2. Thus the hole can discharge quickly [96, 87, 97] but the loss of angular momentum requires the same number of particles as the loss of mass. The question of the evolution of a rotating black hole was analysed by Page [98] in great detail. By considering all the known particles with masses less than 20 Mev, the temperature of a black hole with mass of order 10^{16} g, he found that the emission of angular momentum increases greatly with a_*. Moreover, more than one half of the energy is emitted after a_* reaches a small value of the order of 0.06. From this point the power is within 1% its Schwarzschild value and therefore the earlier assumption that decaying black holes have negligible rotation is valid.

In what follows we shall consider therefore the case of non rotating uncharged black holes. Although we calculated the emission rate only for spinless particles, it is clear that the black hole will emit several species of massless and massive particles. In this case the total luminosity of the black hole can be computed by summing over all particle species.

Let $\Gamma_{slm\omega\lambda}$ be the absorption probability for a particles of the s-th specie and elicity λ, emitted in a mode with energy ω and angular momentum (l,m). Hence $\Gamma_{slm\omega\lambda}$ is the square modulus of the transmission factor of the relevant potential barrier of each particle specie, times the degeneracy of the polarization states. The total luminosity of the hole is given by [3, 7, 99]

$$L = \frac{1}{2\pi} \sum_{s,l,m,\lambda} \int_0^\infty \Gamma_{slm\omega\lambda} \frac{\omega d\omega}{\exp(2\pi k^{-1}\omega) - (-1)^{2\lambda}} \tag{5.117}$$

where the sum over (l,m) is such that $l-\lambda$ and $l-|m|$ are non negative integers. The temperature of the spectrum of the Schwarzschild black hole is then

$$T = \frac{k}{2\pi} = \frac{1}{8\pi M} \simeq 6 \cdot 10^{-8} \frac{M_\odot}{M} K^o \tag{5.118}$$

where $M_\odot$ is the solar mass. Hence the emission is totally irrelevant for astrophysical black holes. As for the luminosity, one needs to compute the transmission factor from the potential barrier around the hole. This is clearly the non trivial part of the luminosity calculation. The absorption probability can be calculated analytically only in the low frequency limit $M\omega \ll 1$, or in geometrical optics limit $M\omega \gg 1$. The expression of $\Gamma_{slm\omega\lambda}$ for boson fields in the low frequency limit has been derived by Starobinsky and Churilov [100] and extended to fermion fields by Page [99]. However, the thermal factor in the luminosity formula suggest a peak for $k^{-1}\omega \simeq M\omega$ of order one, so both limits are like to produce inaccurate results.
One can see the geometric optics limit explicitly without too much effort in the case $\lambda = 0$, i.e. massless, spin zero particles. Then the barrier (see Sec.(5.3)) has a sharp maximum at about $r = 3M$ and for s-waves the maximum is at $r = 8M/3$. The maximum value is about

$$\max V_l = \frac{l(l+1)}{27M^2} \tag{5.119}$$

and for $l = 0$

$$\max V_0 = \frac{27}{1024M^2} \tag{5.120}$$

In the geometric optics approximation, there is no transmission when $\omega^2 \leq \max V_l$, that is when $l \geq \sqrt{27}M\omega$ (since then Eq. (5.26) has no oscillating solutions). Hence one can take

$$B_l(\omega) \simeq \theta(\sqrt{27}M\omega - l) \tag{5.121}$$

where $\theta(x)$ is the step function. Using

$$\sum_{l=0}^{\infty}(2l+1)\theta(\sqrt{27}M\omega - l) \simeq 27M^2\omega^2 \tag{5.122}$$

the geometric optics limit of the luminosity is

$$L = -\frac{dM}{dt} = \frac{9}{10\pi 2^{12}}(M)^{-2} \tag{5.123}$$

Therefore the lifetime of a black hole with initial mass M is

$$\tau \simeq \frac{10\pi 2^{12}}{27}\frac{M^3}{M_P^3}T_P, \qquad M_P^2 = \frac{\hbar c}{G}, \qquad T_P = \frac{\hbar}{M_p c^2} \tag{5.124}$$

where $\hbar$ and c have been restored, M_P is the Planck mass and T_P is the Planck time. This is about the age of the universe for a black hole with a mass of 10^{15} grams.
We now give a more accurate formula by quoting the numerical result obtained by Page. The temperature of the black hole is lower than the electron mass when

$$\frac{\hbar c}{8\pi MG} \leq m_e \tag{5.125}$$

Therefore the black hole must have a mass $M > 10^{17}$g, in order for it not to emit known massive particles. For such a black hole, the total power emitted taking into account four kinds of neutrinos, the photon and the graviton is

$$L = 2.011 \times 10^{-4}\hbar c^6 G^{-2} M^{-2} = 2.28 \times 10^{-54} L_\odot (M_\odot/M)^2 \tag{5.126}$$

where 81.4% is in the four kinds of neutrinos, 16.7% is in photons and 1.9% is in gravitons. Here $M_\odot = 1.99 \times 10^{33} g$ is the solar mass and $L_\odot = 3.9 \times 10^{33}$erg sec^{-1} is the solar luminosity.

For $5 \times 10^{14} < M < 10^{17}$ the black hole emits ultrarelativistic $e^\pm$ which may be treated as massless fermions, and the power is

$$L = 3.6 \times 10^{-4} \hbar c^6 G^{-2} M^{-2} = 4.07 \times 10^{-54} L_\odot (M_\odot/M)^2 \tag{5.127}$$

of which 45% is in electrons and positrons, 45% is in neutrinos, 9% is in photons and 1% is in gravitons.

The lifetime of a black hole can be obtained from the rate of mass loss, namely the negative of the luminosity

$$\frac{dM}{dt} = -\frac{\hbar c^4}{G^2}\frac{\alpha}{M^2} \tag{5.128}$$

where α is a numerical coefficient that depends on which particle species is emitted significantly, and thus it depends on the black hole mass. Taking α to be its value at the initial mass M_0, which might not be true for $M < 10^{14} g$, one has

$$\tau \simeq \frac{G^2}{\hbar c^4}\frac{M_0^3}{3\alpha} \tag{5.129}$$

Then for $M \gg 10^{17}$ one has $\alpha = 2.011 \times 10^{-4}$ and

$$\tau = 2.16 \times 10^{66} (M/M_\odot)^3 yr \tag{5.130}$$

and for $5 \times 10^{14} < M < 10^{17}$ one has $\alpha = 3.6 \times 10^{-4}$ and

$$\tau = 1.25 \times 10^{66} (M/M_\odot)^3 yr \tag{5.131}$$

The above results were obtained by Page by calculating the absorption probabilities and then combining them with the Hawking radiation formula. These absorption probability $\Gamma_{slm\omega\lambda}$, were calculated by the method of Ref. [101], by numerically integrating out to a sufficiently large radius a purely ingoing solution chosen on the horizon. Then $\Gamma_{slm\omega\lambda}$ was calculated as the ratio of the energy going down the hole to the energy of the ingoing wave at infinity.

5.8 Extreme black holes

We shall study now the extreme limit of the Reissner-Nordström solution, when $M = |Q|$ and the Coulomb repulsion of like charges might stop the collapse and turn it into a bounce. That this is indeed the case was explained in [43, 102] (see also Ref. [103, 104]), where the equations of motion of a collapsing charged spherical shell were analysed in great details. The equations can be deduced by relating the discontinuity of the extrinsic curvature across the shell with the surface energy density in the shell, the Israel-Kuchar equations [105, 106].

Following Boulware [43], let ρ be the surface energy density of the shell in the comoving frame and $\mathcal{M} = 4\pi R^2 \rho$ the total proper mass, R being the shell radius. For an extreme shell, let us introduce the parameter

$$a = \frac{\mathcal{M}^2 - M^2}{2\mathcal{M}} \tag{5.132}$$

where M is the total gravitational mass of the shell. The exterior metric is the extreme Reissner-Nordström metric

$$ds^2 = -\frac{(r-M)^2}{r^2}dt^2 + \frac{r^2}{(r-M)^2}dr^2 + r^2 d\ell^2 \tag{5.133}$$

and the interior is flat Minkowski space. The shell position is specified by $r = R(\tau)$, $t = T(\tau)$, where τ is the proper time along the shell history. The equations of motion are then

$$\dot{R} = -\sqrt{\left[\frac{M}{\mathcal{M}} + \frac{a}{R}\right]^2 - 1}, \qquad \dot{T} = \frac{R}{R-M}\sqrt{1 + \frac{R^2\dot{R}^2}{(R-M)^2}} \tag{5.134}$$

A simple, but important, consequence of the equations is that if $M > \mathcal{M}$, then the shell radius has a turning point at $R_T = (M + \mathcal{M})/2 < M$ and therefore it reaches the horizon value $R = M$, after which the shell disappears from sight by forming a future event horizon. However it subsequently bounces and expands by forming another asymptotically flat region with a past horizon and singularities do not form. The important region will be the one containing the future event horizon. Although the equations of motion are exactly integrables, one needs only the asymptotic expansion of the functions $R(\tau)$ and $T(\tau)$ near the horizon. More precisely, one needs the advanced time v of the shell as a function of the retarded time u, namely the relation $v(u)$ near the horizon to leading non trivial order. One finds

$$\dot{R} = a_0 + a_1\varepsilon + a_2\varepsilon^2 + O(\varepsilon^3), \qquad \varepsilon = R - M \tag{5.135}$$

$$\dot{T} = b_{-2}\varepsilon^{-2} + b_{-1}\varepsilon^{-1} + b_0 + O(\varepsilon) \tag{5.136}$$

where the exact form of the coefficients is left apart, since the only important fact is that both a_0 and b_{-2} are non vanishing. The advanced/retarded coordinate times are respectively $t \pm r^*$, where

$$r^* = r + 2M\log\left(\frac{r}{M} - 1\right) - \frac{M^2}{r-M} \tag{5.137}$$

shows the characteristic first order pole of extreme black holes. From the equations it now follows that $v(u)$ remains finite on the horizon while u diverges to infinity. Therefore one finds

$$v(u) = v_1 - Cu^{-1} + O(u^{-2}\log u), \qquad u \to \infty \tag{5.138}$$

where v_1 is the value of advanced time that marks the birth of the black hole and C is a positive constant depending on M and $\mathcal{M}$. This remarkable behaviour is to be compared with the non extreme black hole, where $v(u)$ approaches v_1 exponentially fast in retarded time. This can be traced back to the first order pole singularity that controls the behaviour of r^* near the horizon in contrast with the logarithmic singularity of the non extreme case.

5.9 Quantum theory

The asymptotic behaviour just discussed has the consequence that the phase of a positive frequency ingoing mode from past infinity will be red shifted at a much lower rate than for non extreme black holes. Indeed, it is pretty straightforward to deduce the asymptotic form of such a mode near the event horizon, when the black hole is about to form. Precisely one has: let $P_{j\omega,e}$

be a positive frequency solution of the charged Klein-Gordon equation in the collapse geometry of the infalling shell which is ingoing from past infinity and therefore (j standing for (l,m) from now on)

$$g^{ab}(\nabla_a + ieA_a)(\nabla_b + ieA_b)P_{j\omega,e} - \mu^2 P_{j\omega,e} = 0 \tag{5.139}$$

$$P_{j\omega,e} = (4\pi p)^{-1/2} r^{-1} Y_j(\theta,\phi) R_{j\omega}(r) e^{-i\omega t}, \qquad \omega = \sqrt{p^2+\mu^2} \tag{5.140}$$

where p is the radial momentum and the modes are normalized to a δ-function of ω. Then, asymptotically for large u, for $\omega \gg \mu$ and for $r \to M$, the outgoing part of $P_{j\omega,e}$ after reflection through the shell is

$$P_{j\omega,e} \simeq (4\pi p)^{-1/2} r^{-1} Y_j(\theta,\phi) \exp(iCpu^{-1}) e^{-i\sigma e t - ipv_1} \qquad \sigma = \frac{Q}{|Q|} \tag{5.141}$$

where e is the charge of the field. Then, as in Sec.(5.5), one may neglect the mass μ from now on, its only role being to set the lower bound for integration over energy. The quantum field

$$\phi = \sum_j \int_\mu^\infty d\omega [A_{j\omega} P_{j\omega,e} + B^\dagger_{j\omega} P^*_{j\omega,-e}] \tag{5.142}$$

is assumed to be in the $|in>$ vacuum state relative to the above decomposition. In order to discuss the particle content at future infinity, one should expand the in-mode into positive and negative frequency waves of the kind $\exp(\pm i\omega u)$, namely waves which are outgoing to infinity. This is a rather delicate matter. Considering Eq.(5.141) as an holomorphic function of u in the lower half complex u-plane, one argues that the Fourier expansion of $P_{j\omega,e}$ can only contains the factors $\exp(-i\omega u)$, which just share with $P_{j\omega,e}$ the same holomorphic property. One immediately concludes that there is no neutral scalar radiation at infinity from an extreme black hole (no negative frequency components). In two dimensions this was already known after Davies analysis [35]. However, for charged fields things are different due to the superradiant phenomenon. Indeed, in the Reissner-Nordström geometry the radial outgoing waves from the past horizon have the asymptotic behaviour [87]

$$\vec{R}_{j\omega} \simeq \begin{cases} e^{i(\omega-\sigma e)r^*} + \vec{A}_{j,e}(\omega) e^{-i(\omega-\sigma e)r^*} & r^* \to -\infty \\ \vec{B}_{j,e}(\omega) e^{ipr^*} & r^* \to \infty \end{cases} \tag{5.143}$$

which defines the reflection and transmission coefficient of the potential barrier surrounding the black hole. One defines similar reflection-transmission coefficients for ingoing waves from past infinity. One has then the unitarity relations [87]

$$p\, \vec{B}_{j,e}(\omega) = (\omega - \sigma e)\, \overleftarrow{B}_{j,e}(\omega) \tag{5.144}$$

$$(\omega - \sigma e) |\overleftarrow{B}_{j,e}(\omega)|^2 = p[1 - |\overleftarrow{A}_{j,e}(\omega)|^2] \tag{5.145}$$

From Eq.(5.143), a positive frequency wave packet emerging from the horizon and containing the factor $\exp(-i\omega u)$ arrives at infinity with the time dependence $\exp[-i(\omega + \sigma e)u]$. This is holomorphic in the lower half complex u-plane if $\omega + \sigma e > 0$, but is holomorphic in the upper half complex u-plane if $\omega + \sigma e < 0$, which is just one superradiance condition. Therefore it

may appear in the expansion of $P_{j\omega,e}$ giving a negative frequency component at infinity which corresponds to particle emission in the quantum theory.
We now show that this is actually the case. Let $\nu \in \mathbb{C}$ such that $-1 < \mathrm{Re}\,\nu < 1/2$ and $J_\nu(z)$ the regular Bessel function of order ν. Then

$$\int_0^\infty e^{-i\omega' u}(\omega')^{\nu/2} J_\nu(2\sqrt{\omega\omega'})d\omega' = \frac{\omega^{\nu/2}e^{-i\pi\nu/2}}{iu^{\nu+1}}\exp\left(\frac{i\omega}{u}\right)\theta(u) \tag{5.146}$$

By forming a wave packet one can easily show that the limit $\nu \to -1$ can be safety taken. This means that in the distributional sense one has

$$\int_{-\infty}^\infty e^{-i\omega' u}(\omega')^{-1/2} J_{-1}(2\sqrt{\omega\omega'})\theta(\omega')d\omega' = \omega^{-1/2}\exp\left(\frac{i\omega}{u}\right)\theta(u) \tag{5.147}$$

Making the shift $\omega' \to \omega' - \sigma e$ and using Eq.(5.141) one obtains, asymptotically for $r \to M$

$$\begin{aligned} P_{j\omega,e} &\simeq \frac{\sqrt{C}}{\sqrt{4\pi}}\int_0^\infty e^{-i(\omega'-\sigma e)u - i\sigma e t} J_{-1}\left(2\sqrt{Cp(\omega'-\sigma e)}\right)\frac{\theta(\omega'-\sigma e)}{\sqrt{\omega'-\sigma e}}d\omega' \\ &+ \frac{\sqrt{C}}{\sqrt{4\pi}}\int_0^\infty e^{i(\omega'+\sigma e)u + i\sigma e t} J_{-1}\left(2\sqrt{Cp(-\sigma e-\omega')}\right)\frac{\theta(-\sigma e-\omega')}{\sqrt{-\sigma e-\omega'}}d\omega' \end{aligned}$$

This solution tunnels through the potential barrier surrounding the black hole, each monochromatic component with $\omega' \geq \mu$, carrying the amplitude $\overrightarrow{B}_{j,e}(\omega')$ to infinity (see Eq.(5.143)). Therefore

$$\begin{aligned} P_{j\omega,e} &\simeq \frac{\sqrt{C}}{\sqrt{4\pi}}\int_\mu^\infty e^{-i\omega' t + ip' r^*} J_{-1}\left(2\sqrt{Cp(\omega'-\sigma e)}\right)\overrightarrow{B}_{j,e}(\omega')\frac{\theta(\omega'-\sigma e)}{\sqrt{\omega'-\sigma e}}d\omega' \\ &+ \frac{\sqrt{C}}{\sqrt{4\pi}}\int_\mu^\infty e^{i\omega' t - ip' r^*} J_{-1}\left(2\sqrt{Cp(-\sigma e-\omega')}\right)\overrightarrow{B}^*_{j,-e}(\omega')\frac{\theta(-\sigma e-\omega')}{\sqrt{-\sigma e-\omega'}}d\omega' \end{aligned} \tag{5.148}$$

and we can see the presence of a negative frequency component if $\sigma e < 0$. Similarly, a negative frequency ingoing mode will have a positive frequency component at future infinity if $\sigma e > 0$. By following the usual procedure, the Bogoljubov coefficient for particle creation can be found easily from Eq.(5.148). It is

$$\begin{aligned} \beta_{j\omega'\omega,e} &= -\frac{\sqrt{Cp'}}{\sqrt{\sigma e-\omega'}}J_{-1}\left(2\sqrt{Cp(\sigma e-\omega')}\right)\overrightarrow{B}^*_{j,e}(\omega')\theta(\sigma e-\omega') \\ &= \frac{\sqrt{C(\sigma e-\omega')}}{\sqrt{p'}}J_{-1}\left(2\sqrt{Cp(\sigma e-\omega')}\right)\overleftarrow{B}^*_{j,e}(\omega')\theta(\sigma e-\omega') \end{aligned} \tag{5.149}$$

by using Eq.(5.144). Therefore one has

$$\int_0^\infty \beta^*_{j\omega'\omega,e}\beta_{j\omega''\omega,e}dp = -[1-|\overleftarrow{A}_{j,e}(\omega')|^2]\theta(\sigma e-\omega')\delta(\omega'-\omega'') \tag{5.150}$$

by using Eq.(5.145) and the completeness relation for Bessel functions. Hence the luminosity of the black hole is

$$L = -\frac{1}{2\pi}\sum_j\int_\mu^{\sigma e}[1-|\overleftarrow{A}_{j,e}(\omega)|^2]\omega d\omega = -\frac{1}{2\pi}\sum_{l\geq 0}(2l+1)\int_\mu^{\sigma e}[1-|\overleftarrow{A}_{l,e}(\omega)|^2]\omega d\omega \tag{5.151}$$

as $\overleftarrow{A}_{j,e}(\omega)$ does not depend on the azimuth quantum number m. Similarly, one obtains the rate of charge loss

$$\dot{Q} = \frac{e}{2\pi}\sum_{l\geq 0}(2l+1)\int_{\mu}^{\sigma e}[1 - |\overleftarrow{A}_{l,e}(\omega)|^2]d\omega \tag{5.152}$$

As σ is the sign of Q, on the average the black hole will emit particles having the same sign as Q and the black hole tends to discharge. We point out that these results are the zero temperature limit of those pertinent to non extreme black holes. In this sense, therefore, the Hawking temperature of an extreme black hole vanishes. Another observation is that the luminosity and charge loss both vanish for particles with charge to mass ratio is unity. So the only possibility for an extreme black hole to be stable under quantum emission seems to be the non existence of particles with $e/\mu \neq 1$. This is not the case and the black hole will evolve toward the non extreme sequence. Finally, the lack of neutral scalar emission is just what is needed to avoid naked singularities, since neutral particles would decrease the mass but not the charge of the black hole. Likewise, a calculation with fermions shows that neutrinos are not emitted too, which may be regarded as cosmic censorship at work.
A similar analysis can be done for an extreme, rotating shell. This time, however, we do not have at disposal exact equations since the exterior geometry is not given by the Kerr metric all the time (a solution which is valid to first order in the angular velocity of the shell but to all orders for the mass and radial velocity has been worked out in [107]). Hence one assumes that the Kerr geometry settles asymptotically, when the event horizon is about to form. Then asymptotically $\dot{\phi} \simeq \Omega\dot{T}$, which is also quite a well established result in the theory of gravitational collapse (this can be seen explicitly in [107]). From the timelike condition on the four-velocity, $g_{ab}u^a u^b = -1$, it then follows that

$$\dot{T} = \frac{2M^2}{(R-M)^2}\sqrt{\frac{(R-M)^2}{M^2(1+\cos^2\theta)} + \dot{R}^2} \tag{5.153}$$

Therefore, since $\dot{R}$ converges to a finite value on the horizon one gets

$$\dot{R} = a_0 + a_1\varepsilon + O(\varepsilon^2), \qquad \dot{T} = b_{-2}\varepsilon^{-2} + b_{-1}\varepsilon^{-1} + b_0 + O(\varepsilon) \tag{5.154}$$

with θ-independent leading coefficients, and $\varepsilon = R - M$. This is the behaviour which occurred in the Reissner-Nordström case, and therefore it leads to a quantum emission which is the zero surface gravity limit of Eq.(5.112). In this sense the temperature of the extreme Kerr black hole vanishes.

6 Black hole entropy

The thermal character of the Hawking emission implies a flux of entropy from the black hole. That a black hole could behave as an entropy store, was anticipated in the classical theory on the ground of the uniqueness theorems about the final state of gravitational collapse. It was also anticipated that this entropy should be proportional to the quasi stationary value of the area of the event horizon, on the ground of the first law of black hole mechanics, with the surface gravity playing the role of temperature. We can adress this question now with increased strength, since we know that black holes have an effective temperature as a result of the quantum emission process, which is just proportional to the surface gravity.

6.1 The thermodynamic argument

This is based on the idea that since the collapsing matter and the quantum field together form an isolated system, the total entropy cannot decreases.
For a Kerr-Newman black hole, one can write the rate of spinless particle production as

$$dN_{lm\omega,e} = \frac{\Gamma_{lm\omega,e}}{\exp[2\pi k^{-1}(\omega - m\Omega - e\Phi)] - 1}\frac{d\omega}{2\pi} \tag{6.1}$$

The black hole rate of mass loss, the rate of charge loss and the rate of angular momentum loss are then, respectively, given by

$$\dot{M} = -\sum_{lm}\int_{\mu}^{\infty}\omega[dN_{lm\omega,e} + dN_{lm\omega,-e}] \tag{6.2}$$

$$\dot{Q} = -e\sum_{lm}\int_{\mu}^{\infty}[dN_{lm\omega,e} - dN_{lm\omega,-e}] \tag{6.3}$$

and

$$\dot{J} = -\sum_{lm}\int_{\mu}^{\infty}m[dN_{lm\omega,e} + dN_{lm\omega,-e}] \tag{6.4}$$

Before defining the entropy emission, one needs to have certain geometrical quantities of the Kerr-Newman sequence. From Sec.(3.2), one determines the position of the horizon at

$$r_+ = M + \sqrt{M^2 - a^2 - Q^2} \tag{6.5}$$

and $r_- = 2M - r_+$. A direct calculation from the definition (see Ref. [41]) gives the surface gravity

$$k = \frac{r_+ - r_-}{2(r_+^2 + a^2)} \tag{6.6}$$

the area of the event horizon is

$$A = 4\pi(r_+^2 + a^2) \tag{6.7}$$

The electrostatic potential on the horizon, Φ, and its angular velocity, Ω, are respectively

$$\Phi = \frac{Qr_+}{r_+^2 + a^2}, \qquad \Omega = \frac{a}{r_+^2 + a^2} \tag{6.8}$$

Finally, the angular momentum of the black hole is $J = Ma$. If now dA is calculated as a function of the black hole parameters M, Q and J, with a simple calculation one obtains the stationary limit of the first law of black hole mechanics [41]

$$dM = \frac{k}{8\pi}dA + \Omega dJ + \Phi dQ \tag{6.9}$$

Since a particle arriving at infinity with energy ω started from the star near the horizon with the energy $\omega - m\Omega \pm e\Phi$, the rate of entropy emission is

$$\frac{dS}{dt} = T^{-1}\sum_{lm}\int_{0}^{\infty}[(\omega - m\Omega - e\Phi)dN_{lm\omega,e} + (\omega - m\Omega + e\Phi)dN_{lm\omega,-e}] \tag{6.10}$$

where $T = k/2\pi$ is the temperature of the black hole. In other words, the radiated entropy is T^{-1} times the radiated energy minus the electrostatic and rotational energy carried away by the radiation. It may be shown that this radiated entropy is consistent with the statistical formula $S = -\sum_i P_i \log P_i$, where P_i is the probability of occurrence of a given radiation state at infinity [95, 10]. Each radiation state is uniquely specified by a set $\{N_k\}$ of occupation numbers and P_i is then given by Eq.(5.60).
From Eq.(6.2), (6.3) and (6.4) one can also write the entropy rate as

$$\dot{S} = -\frac{2\pi}{k}\left(\dot{M} - \Omega\dot{J} - \Phi\dot{Q}\right) \tag{6.11}$$

If during the evaporation process the black hole remains in the Kerr-Newman sequence, then using the first law one obtains

$$\dot{S} = -\frac{1}{4}\dot{A} \tag{6.12}$$

The total radiated entropy $\dot{S}_m$, will be greater than this since the star emitted also electromagnetic and gravitational radiation before collapsing to a black hole state. Therefore $\dot{S}_m \geq \dot{S}$, and one obtains the generalized second law

$$\frac{d}{dt}\left(S_m + \frac{A}{4}\right) \geq 0 \tag{6.13}$$

Thus one arrives at a very important conclusion. One has to assign to a black hole the total entropy [9, 108]

$$S_H = \frac{A}{4} = \pi\left(2M^2 - Q^2 + 2(M^4 - M^2Q^2 - J^2)^{1/2}\right) \tag{6.14}$$

For a spherically symmetric hole this is (in ordinary units)

$$S_H = \frac{A}{4L_P^2} = \frac{4\pi G M^2}{\hbar c} \simeq 11.2 \times 10^{76}(M/M_\odot)^2 \tag{6.15}$$

$L_P \simeq 10^{-33}$cm being the Planck length. This is consistent with the thermodynamic relation $dM = TdS$, which was actually used by Hawking to deduce the expression of S_H. This quickly led to a reformulation of the second law into a "generalized second law": $S_H + S_m$ never decreases with time, where S_m is the entropy of matter and radiation outside the black hole [9, 10, 108]. Without the black hole contribution the second law can be violated, e.g. by sending matter down the hole. Note that for an extreme black hole the above thermodynamics argument breaks down, since the surface gravity and the temperature are zero for such a black hole, but not the horizon area (see, however, Ref. [35] for a different point of view).
The existence of S_H suggests that a black hole has a finite number, albeit a large one, of internal quantum configurations. Indeed, the Compton wavelength $\lambda = m^{-1}$, of the particles from which the black hole formed might have to be restricted to be less that the radius of the black hole and that therefore the number of internal configurations might be finite. So if $\lambda = m^{-1} \leq M$ then $m \geq M^{-1}$ and this number is of order $M/m \simeq M^2$. Thus we expect indeed that $S_H = KM^2$, where K is some constant possibly of order one. Jacob Bekenstein [9] was the first to push forward this idea and also to estimate $K = \ln 2/8\pi$.

6.2 Thermodynamical considerations

The black hole entropy

$$S_H = \frac{A}{4} = \pi \left(2M^2 - Q^2 + 2(M^4 - M^2Q^2 - J^2)^{1/2}\right) \tag{6.16}$$

and temperature

$$T = \frac{k}{2\pi} = \frac{(M^4 - M^2Q^2 - J^2)^{1/2}}{2\pi M[2M^2 - Q^2 + 2(M^4 - M^2Q^2 - J^2)^{1/2}]} \tag{6.17}$$

together with the first law $dM = TdS + \Omega dJ + \Phi dQ$, and the generalized second law allow one to laid down the foundations for a black hole thermodynamics [35, 109, 78, 108, 110] (a review is in [81]). As an application, one can derive the integral mass formula first noted by Smarr [111] and derived classically in full generality in [41]. In fact, from S_H one can read the scaling law

$$S_H(\lambda M, \lambda^2 J, \lambda Q) = \lambda^2 S_H(M, J, Q) \tag{6.18}$$

Differentiating this with respect to λ at $\lambda = 1$ and using the first law one gets immediately the black hole analogue of the Gibbs-Duhem relation

$$M = 2TS_H + Q\Phi + 2\Omega J = \frac{k}{4\pi}A + Q\Phi + 2\Omega J \tag{6.19}$$

which is just the integral mass formula. This equation may be regarded as one of the fundamental laws of black hole thermodynamics.
As another application let us consider thermodynamic fluctuations. According to this thermodynamic theory the probability for a small fluctuation in M, Q or J occurring is proportional to $\exp(\Delta S_T)$, where ΔS_T is the variation in the total entropy caused by the fluctuations [29]. Therefore one should enclose the black hole inside a large box and let the hole to come in stable equilibrium with a radiation gas at the black hole temperature. It is known that in such circumstances the equilibrium is stable if the volume satisfies the bound $V_s < V \leq V_0$ for a given total energy E [108, 81, 78], where $V_s \simeq E^3$ is a Schwarzschild volume (to avoid the collapse of the box),

$$V_0 = \frac{3\pi^2 2^{20} E^5}{125(n_b + 7n_f/8)} \tag{6.20}$$

n_b is the number of zero mass boson fields and n_f is the number of fermion fields. However, for small fluctuations in a quasi-stationary Schwarzschild black hole it is sufficient to consider the variations of S_H considering the black hole as a small subsystem of its environment. The thermodynamic fluctuations dominate over the quantum fluctuations if a certain condition holds [29]. This is that the characteristic time, τ, for the black hole parameters to change appreciably be much larger than $1/T$. Taking τ to be of order of the black hole lifetime gives the condition $\tau \simeq M^3 \gg M$, namely M should be much larger than the Planck mass $M_P \simeq 10^{-5}$g. Given this, the maximum value of the entropy is clearly the Schwarzschild value at $Q = J = 0$. Hence for black holes it is entropically favourable to spin down and discharge, in accord with the luminosity calculations of Page. Expanding the entropy around $J = Q = 0$ up to quadratic orders one gets

$$S_H = 4\pi M^2 - 2\pi Q^2 - \pi J^2/M^2 \tag{6.21}$$

Therefore the thermodynamic probability that a small fluctuation brings the black hole in a state with charge in the range dQ around $Q \ll 1$ and angular momentum in the range dJ around $J \ll 1$, is proportional to

$$e^{-2\pi Q^2 - \pi J^2/M^2} J^2 dJ dQ \tag{6.22}$$

where $d^3J = 4\pi J^2 dJ$ has been used. One may conclude that charge and angular momentum fluctuations are statistically independent. It does not make sense in this context to discuss the fluctuation of M, since there is no maximum of S_H as a function of M. Physically it does not make sense because the specific heat of a Schwarzschild black hole is negative

$$C = \frac{\partial M}{\partial T} = -8\pi M^2 \tag{6.23}$$

and this makes unstable the equilibrium of the black hole with an infinite environment of thermal radiation. The mass fluctuations can only be consistently considered for black holes in a large box.

From the first law, the minimum reversible work needed to change the charge from zero to Q and the angular momentum from zero to J is, up to quadratic orders

$$R_{min} = \frac{1}{2}\Omega J + \frac{1}{2}\Phi Q = \frac{J^2}{8M^3} + \frac{Q^2}{4M} \tag{6.24}$$

Therefore $-R_{min}/T$ is just the exponent of Eq.(6.22), once more in agreement with the general theory of thermodynamic fluctuations [29].

According to statistical thermodynamics, the entropy also determines the density of black hole states, the number of such states in the range $J^2 dJ dQ dM$ of black hole parameters being proportional to

$$dN = dM \int_{-M}^{M} dQ \int_0^{M^2} \exp(S_H) J^2 dJ \tag{6.25}$$

For $M \gg 1$ this is easily seen to be

$$dN \simeq M^3 e^{4\pi M^2} dM \tag{6.26}$$

In principle it is possible for a black hole to emit also small black holes with masses $m \ll M$, since a black hole radiates every matter configuration with equal probability [64]. Not every matter configuration arrives at infinity with equal probability, however, since the penetration factor of the potential barrier depends on the configuration. The degeneracy of the emitted black holes being given by Eq.(6.26), with m replacing M, and the thermal factor governing the emission being $\exp(-8\pi Mm)$ for large M, the probability to emit a black hole is proportional to $\exp(4\pi m(m-2M))$. Classically bifurcation is forbidden whereas quantum effects make it possible but with an exponentially small probability. Entropy considerations would forbid bifurcation either, because S_H is a quadratic function of M. Thus if a black hole could splits into two black holes with masses M_1 and M_2 without energy loss, the final entropy, $4\pi(M_1^2 + M_2^2)$, would be less than the initial entropy, $4\pi(M_1 + M_2)^2$.

6.3 The quantum gravity argument

We know that quantum field theory has an alternative formulation in terms of path integrals over classical field configurations in phase space. It is certainly tempting to push this description so far so as to include gravity [112]. To get complete consistency and a well defined canonical ensemble, however, one has to enclose the black hole in a box of finite size and then fix the temperature at the boundary [113].
Then one is led to formulate gravitational thermodynamics by giving the partition function for matter and gravity in terms of the path integral

$$Z_\beta = \sum_{\mathcal{M}} \int \mathcal{D}(g)\mathcal{D}(\phi) e^{-(I_g+I_m)} \tag{6.27}$$

over all manifolds $\mathcal{M}$ together with matter fields ϕ and metric tensors such that

(i) $\mathcal{M}$ is asymptotically flat with boundary $\partial\mathcal{M} = S^2(r_0) \times S^1_\beta$ at some large radius r_0,

(ii) g is positive definite and periodic with period β near the boundary,

(iii) the matter fields are periodic or anti-periodic with period β near the boundary.

The gravitational Euclidean action is

$$I_g = -\frac{1}{16\pi}\int_{\mathcal{M}} R(g)^{1/2} d^4x - \frac{1}{8\pi}\int_{\partial\mathcal{M}} [K](h)^{1/2} d^3x \tag{6.28}$$

where h is the determinant of the metric, h_{ab}, induced on the boundary, K is the trace of the second fundamental form of the boundary and $[K] = K - K_0$, K_0 being the trace as if the boundary were embedded in flat euclidean space. This boundary term may seem a nuisance but is actually necessary in order to have a well posed variational problem [20, 114]. Also it is natural that in the flat space limit the action vanishes. This is the meaning of the subtraction of K_0 from K, since the boundary remains curved in the flat space limit. The contribution to the partition function from a given manifold and matter field will be given to first order by the stationary value of the action functional, that is to say

$$\log Z_\beta \simeq -[I_g(\tilde{g}) + I_m(\tilde{\phi})] \tag{6.29}$$

where $\tilde{g}$ and $\tilde{\phi}$ is the stationary point of the action. One such stationary point is just flat euclidean space plus matter fields obeying the usual $SO(4)$-invariant field equations. This has $\beta = \infty$ and $I_g = 0$. For free fields either $I_m = 0$ or $I_m = \infty$, so the former solution gives zero contribution to the partition function. A more interesting solution is the manifold $\mathcal{M} = \mathbb{R}^2 \times S^2$ with no matter fields, for which a stationary point occurs at the Euclidean Schwarzschild metric. This is obtained by the Wick rotation $t \to -i\tau$ and is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\ell^2 \tag{6.30}$$

where $d\ell^2$ is the line element of the unit 2-sphere. Putting $x = 4M(1 - 2M/r)^{1/2}$ gives

$$ds^2 = \left(\frac{x}{4M}\right)^2 d\tau^2 + \left(\frac{r^2}{4M^2}\right)^2 dx^2 + r^2 d\ell^2 \tag{6.31}$$

The manifold is the region $x \geq 0$. Near $x = 0$ ($r = 2M$) the (τ, x) section of the metric describes a singularity free space if and only if τ is identified with period $\beta = 8\pi M$, namely with the Hawking temperature! In this case the solution, called by now the Gibbons-Hawking's instanton, is perfectly regular.In the contrary case instead, there is a conical singularity on the axis $r = 2M$ and the metric cannot be a stationary point of the action. Hence the solution gives a contribution to the partition function for $\beta = 8\pi M$. The boundary is the manifold $r = r_0$, $r_0 \gg 2M$, whose induced metric is

$$h_{ab} dx^a dx^b = \left(1 - \frac{2M}{r_0}\right) d\tau^2 + r_0^2 d\ell^2$$

The normal vector field to the boundary is

$$n^a = \left(1 - \frac{2M}{r_0}\right)^{1/2} \delta_1^a$$

and the trace of the second fundamental form is the covariant divergence of n^a [19, 20, 115], which gives

$$K = \nabla_a n^a = \frac{2}{r_0} - \frac{M}{r_0^2} + O(r_0^{-3})$$

Now $R = 0$ for the Schwarzschild solution and so the non trivial contribution comes from the boundary term. Also $K_0 = 2/r_0$ and therefore

$$\log Z_\beta = -I_g = -\frac{\beta M}{2} = -\frac{\beta^2}{16\pi} = -\beta M + \frac{A}{4} \tag{6.32}$$

Thus the contribution to the free energy is $F(\beta) = -\beta^{-1} \log Z_\beta = M/2$. The energy is $E = -\partial_\beta \log Z_\beta = M$, as might be expected, and the entropy is

$$S = \beta(E - F) = \frac{A}{4} = 4\pi M^2 \tag{6.33}$$

This is the entropy that was assigned to black holes on the basis of their thermal emission. This is surprising since classical solutions of matter field equations do not normally contribute to the entropy. If there were only one state with energy E contributing to the partition function one would expect $\log Z = -\beta E$ and not $\log Z = -\beta E/2$, as it was found for the black hole solution. The two main reasons for this behaviour are that (i) the gravitational action is not scale invariant and that (ii) the gravitational field can admit different topologies for a given metric [112]. The treatment can be extended to include other black hole solutions. For example, the Euclidean action of the Reissner-Nordström solution is, including the electromagnetic contribution

$$I = \beta \left(\frac{M}{2} + \frac{Q^2}{2r_+}\right), \qquad \beta = \frac{2\pi}{k} \tag{6.34}$$

as can be verified with a simple calculation. From the integral mass formula [41, 111]

$$M = \frac{Q^2}{r_+} + \frac{k}{4\pi} A \tag{6.35}$$

one obtains $\log Z = -I = -\beta M + A/4$ and therefore the entropy is $S = A/4$.
The path integral approach to black hole entropy also suggest to go beyond the "zero loop" formula and to investigate the first quantum ("one-loop") corrections. Unfortunately these turn out to be intolerably divergent [116] so as to frustrate many attempts in this direction, and the issue is still far from a complete solution [15, 117, 16].

6.4 The entropy of extreme black holes

We noticed above the breaks down of thermodynamics arguments for extreme black holes. However, the path integral approach still applies, as was first discussed in [118].
The Euclidean extreme Reissner-Nordström solution is

$$ds^2 = r^{-2}(r-M)^2 d\tau^2 + r^2(r-M)^{-2}dr^2 + r^2 d\ell^2 \tag{6.36}$$

This may be transformed to a new form by setting $x = \log(r - M)$, for $r > M$, and one obtains

$$ds^2 = (e^x + M)^{-2} e^{2x} d\tau^2 + (e^x + M)^2 dx^2 + (e^x + M)^2 d\ell^2 \tag{6.37}$$

This is a perfectly regular metric on the manifold $\mathbb{R} \times \mathbb{R} \times S^2$, where $\tau \in \mathbb{R}$ and $x \in \mathbb{R}$. Since τ belongs to the $\mathbb{R}$ factor, it can be identified with any period β. Therefore one considers the Euclidean theory on the manifold $S^1 \times \mathbb{R} \times S^2$, where the circumference of the S^1 factor is an arbitrary β. The boundary is the submanifold $r = r_0 \gg M$, that is $S^1 \times S^2$. The euclidean action is just Eq.(6.28) to which one adds the Euclidean electromagnetic contribution

$$I_f = \frac{1}{16\pi} \int_0^\beta \int F_{ab} F^{ab} \sqrt{g} d^3x d\tau = \frac{M\beta}{2} \tag{6.38}$$

in the limit $r_0 \to \infty$. Now the trace of the second fundamental form is

$$K = 2r_0^{-1} - Mr_0^{-2} \tag{6.39}$$

and the Euclidean action becomes $I = \beta M$, of which one half is the gravitational contribution, the other half is the electromagnetic contribution. Therefore the partition function is $\log Z = -\beta M$, which being a linear function of β gives zero entropy [118, 119]. In this approach, this may be interpreted as the fact that extreme black holes are pure states in quantum gravity.

7 Conclusions

The long journey into the realm of black holes has revealed many deep aspects of the connection between gravity, quantum mechanics and thermodynamics but also a great deal of unexpected and unsolved problems. Among the most urgent there are: (i) the information problem, or does black holes destroy information? After a black hole has formed it will decay by Hawking emission and we have the final state problem (ii), or what is the ground state of a decaying black hole? Hawking radiation, remnants, a naked singularity? The question mark in Fig. [7] just represents this uncertainty. (It has also been suggested that black holes never form due to thermal emission [7]).

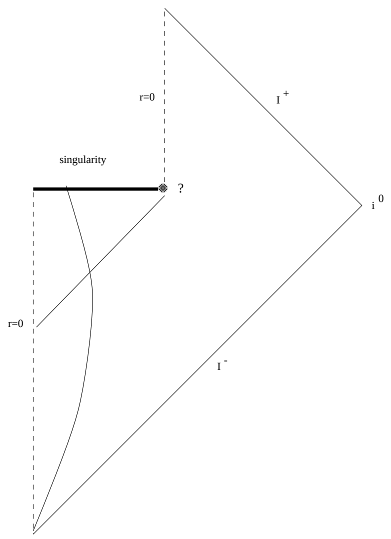

Fig.7 The Penrose's diagram of a black hole which decays to an uncertain end point by the Hawking's evaporation process.

The black hole entropy is universal since it does not depend on how many species of elementary particles exist, so there is an entropy problem (iii), or why the entropy is $A/4$ and how it is related to Boltzmann statistical formula given that the density of states near the horizon is divergent? Let us mention also the possibility that the spectrum of black hole states might be discrete [116] and that baryon number conservation must be violated in the presence of black holes [116, 8, 120]. These problems are not mutually independent, but they show that black holes do represent a major stumbling-block to progress in the way. For example, it is argued in [116] that the density of states cannot be infinite if a lightest stable black hole fails to exist. Hence the final state problem is tightly connected with the entropy problem.
Thanks to the work of many peoples, it is now a widespread belief that the solution of these questions will open the door for a deeper understanding of the quantum aspects of gravity, but the possibility exists that quantum mechanics itself may show its own limitations, so that our picture of the microphysical world may drastically change [121].

References

[1] S. W. Hawking. Commun. Math. Phys. **43**, 199 (1975).

[2] R.M. Wald. Commun. Math. Phys. **45**, 9 (1975).

[3] B.S. DeWitt. Phys. Rep. **19**, 295 (1975).

[4] L. Parker. Phys. Rev. **D12**, 1519 (1975).

[5] W.G. Unruh. Phys. Rev. **D14**, 870 (1976).

[6] W. Israel. Phys. Lett. **A57**, 107 (1976).

[7] U.H. Gerlach. Phys. Rev. **D14**, 1479 (1976).

[8] J.D. Bekenstein. Phys. Rev. **D5**, 1239 (1972).

[9] J.D. Bekenstein. Phys. Rev. **D7**, 2333 (1973).

[10] J.D. Bekenstein. Phys. Rev. **D12**, 3077 (1975).

[11] B. Punsly. Phys. Rev. **46**, 1288 (1992).

[12] B. Punsly. Phys. Rev. **46**, 1312 (1992).

[13] B. Carter. Phys. Rev. Lett. **26**, 331 (1971).

[14] D.C. Robinson. Phys. Rev. Lett. **34**, 905 (1975).

[15] D.N. Page. Int. J. Mod. Phys. **D3**, 93 (1994).

[16] S. B. Giddings. *Quantum Mechanics of Black Holes.* hep-th/9412138 (lectures at 1994 Trieste Spring School, (1994).

[17] A. Strominger. *Les Houches Lectures on Black Holes.* to appear, (1994).

[18] R. Brout, S. Massar, R. Parentani and Ph. Spindel. Phys. Rep. **260**, 329 (1995).

[19] S.W. Hawking and G.F. Ellis. *The Large Scale Structure of Space-Time.* Cambridge University Press, Cambridge, England, (1973).

[20] R.M. Wald. *General Relativity.* The University of Chicago Press, (1984).

[21] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, Inc., (1983).

[22] S. Weinberg. Rev. Mod. Phys. **61**, 1 (1989).

[23] A. Ashtekar and R.O. Hansen. J. Math. Phys. **19**, 1542 (1978).

[24] Black Holes. *ed. by C. DeWitt and B. S. DeWitt, Gordon and Breach, N.Y.* , (1973).

[25] B. H. Harrison, K. S. Thorne, M. Wakano and J. A. Wheeler. *Gravitational Theory and Gravitational Collapse.* University of Chicago Press, Chicago, (1965).

[26] C. W. Misner, K. S. Thorne and J. A. Wheeler. *Gravitation.* San Francisco: Freeman, (1973).

[27] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* John Wiley and Sons, Inc., New York, (1972).

[28] Ya.B. Zel'dovich and I.D. Novikov. *Relativistic Astrophysics, Volume 1.* The University of Chicago Press, (1971).

[29] L.D. Landau and E. M. Lifshitz. *Statistical Physics.* Pergamon Press, London, (1958).

[30] L.D. Landau and E. M. Lifshitz. *The Classical Theory of Fields.* Oxford: Pergamon, (1962).

[31] H.A. Bethe. Rev. Mod. Phys. **62**, 801 (1990).

[32] M. D. Kruskal. Phys. Rev. **119**, 1743 (1960).

[33] R. Oppenheimer and H. Snyder. Phys. Rev. **56**, 455 (1939).

[34] R. Penrose. Rivista del N. Cimento **1**, 252 (1969).

[35] P.C.W. Davies. Proc. R. Soc. Lond. **A353**, 499 (1977).

[36] W. Israel. Phys. Rev. **164**, 1776 (1967).

[37] R.H. Price. Phys. Rev. **D5**, 2419 (1972).

[38] S. Chandrasekhar. *The Mathematical Theory of Black Holes*. Oxford: Clarendon Press, (1983).

[39] H. Reissner. Ann. d. Physik **50**, 106 (1916).

[40] G. Nordström. Proc. Kon. ned. Akad. Wet. **20**, 1238 (1918).

[41] J.M. Bardeen, B. Carter and S.W. Hawking. Commun. Math. Phys. **31**, 161 (1973).

[42] J. C. Graves and D. R. Brill. Phys. Rev. **120**, 1507 (1960).

[43] D.G. Boulware. Phys. Rev. **D8**, 2363 (1973).

[44] R. P. Kerr. Phys. Rev. Lett. **11**, 237 (1963).

[45] R. H. Boyer and R. W. Lindquist. J. Math. Phys. **8**, 265 (1967).

[46] B. Carter. Phys. Rev. **141**, 1262 (1966).

[47] B. Carter. Phys. Rev. **174**, 1559 (1968).

[48] E. T.. Newman, E. Couch, K. Chinnapared et al. J. Math. Phys. **6**, 918 (1965).

[49] J. G. Taylor and N.D. Birrell. J. Math. Phys. **21**, 1740 (1980).

[50] N.D. Birrell and P.C.W. Davies. *Quantum Fields on Curved Space*. Cambridge University Press, Cambridge, England, (1982).

[51] B.S. Kay. Commun. Math. Phys. **62**, 55 (1978).

[52] I.S. Gradshteyn and I.M Ryzhik. *Table of Integrals, Series and Products*. Academic, New York, (1980).

[53] R.M. Wald. Commun. Math. Phys. **54**, 1 (1977).

[54] S. A. Fulling, F. J. Narcowich and R.M. Wald. Ann. Phys. **136**, 243 (1981).

[55] S. A. Fulling, M. Sweeny and R.M. Wald. Commun. Math. Phys. **63**, 257 (1978).

[56] R. Haag, N.M. Hugenholtz and M. Winnink. Commun. Math. Phys. **5**, 215 (1967).

[57] S.A. Fulling and S.N.M. Ruijsenaars. Phys. Rep. **152**, 135 (1987).

[58] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics II*. Springer, New York, (1981).

[59] V. Moretti and L. Vanzo. Phys. Lett. **B375**, 54 (1996).

[60] S.A. Fulling. Phys. Rev. **D7**, 2850 (1973).

[61] S.A. Fulling. *Aspects of Quantum Field Theory in Curved Space-Time*. Cambridge University Press, (1989).

[62] R. K. Sachs. Phys. Rev. **128**, 2851 (1962).

[63] R. P. Geroch. J. Math. Phys. **13**, 556 (1972).

[64] S.W. Hawking. Phys. Rev. **D14**, 2460 (1976).

[65] S. Takagi. Prog. of Theor. Phys. Supplement **88**, 1 (1986).

[66] D.W. Sciama, P. Candelas and D. Deutsch. Adv. Phys. **30**, 327 (1981).

[67] J. S. Bell and J.M. Leinaas. Nucl. Phys. **b212**, 131 (1983).

[68] W. Rindler. Amer. J. Phys. **34**, 1174 (1966).

[69] P.C.W. Davies. J. Phys. **A8**, 609 (1975).

[70] S.A. Fulling. J. Phys. A: Math. Gen. **10**, 917 (1977).

[71] D.G. Boulware. Phys. Rev. **D11**, 1404 (1975).

[72] P. Candelas and D.J. Raine. J. of Math. Phys. **17**, 2101 (1976).

[73] K. Freese, C.T. Hill and M. Mueller. Nucl. Phys. **B255**, 693 (1985).

[74] C.R. Stephens. Phys. Rev. **D33**, 2813 (1986).

[75] J.S. Dowker. J. Phys. **A10**, 115 (1977).

[76] J.S. Dowker. Phys. Rev. **18**, 1856 (1978).

[77] B.S. Kay and R.M. Wald. Phys. Rep. **207**, 49 (1991).

[78] G.W. Gibbons and M.J. Perry. Proc. R. Soc. Lond **A358**, 467 (1978).

[79] W. G. Unruh and R.M. Wald. Phys. Rev. **29**, 1047 (1984).

[80] J.B. Hartle and S.W. Hawking. Phys. Rev. **D13**, 2188 (1976).

[81] P. C. W. Davies. Rep. Prog. Phys. **41**, 1313 (1978).

[82] P. Candelas. Phys. Rev. **D21**, 2185 (1980).

[83] D.G. Boulware. Phys. Rev. **D13**, 2169 (1976).

[84] T. Elster. Class. Quantum Grav. **1**, 43 (1984).

[85] V. P. Frolov and N. Sanchez. Phys. Rev. **D33**, 1604 (1986).

[86] M.R. Brown, A.C. Ottewill and Don N. Page. Phys. Rev. **D33**, 2840 (1986).

[87] G.W. Gibbons. Commun. Math. Phys. **44**, 245 (1975).

[88] Ya.B. Zeldovich. Sov. Phys.-JETP **35**, 1085 (1972).

[89] A. A. Starobinsky. Soviet Phys.-JETP **37**, 28 (1973).

[90] W.G. Unruh. Phys. Rev. **D10**, 3194 (1974).

[91] W. H. Press and S. A. Teukolsky. Nature **238**, 211 (1972).

[92] R. Penrose and R. M. Floyd. Nature Phys. Sci. **229**, 177 (1971).

[93] S. M. Wagh and N. Dadhich. Phys. Rep. **183**, 137 (1989).

[94] D. Christodoulou. Phys. Rev. Lett. **25**, 1596 (1970).

[95] D. Christodoulou and R. Ruffini. Phys. Rev. **D4**, 3552 (1971).

[96] W. T. Zaumen. Nature **247**, 530 (1974).

[97] B. Carter. Phys. Rev. Lett. **33**, 558 (1974).

[98] D.N. Page. Phys. Rev. **D14**, 3260 (1976).

[99] D. N. Page. Phys. Rev. **D13**, 198 (1976).

[100] A. A. Starobinsky and S. M. Churilov. Soviet Phys.-JETP **38**, 1 (1974).

[101] W. H. Press and S. A. Teukolsky. Astrophys. J. **193**, 443 (1974).

[102] W. Israel and V. de la Cruz. Nuovo Cimento **51**, 774 (1967).

[103] J.D. Bekenstein. Phys. Rev. **D4**, 2185 (1971).

[104] J.E. Chase. Nuovo Cimento **67B**, 136 (1970).

[105] W. Israel. Nuovo Cimento **44B**, 1 (1966).

[106] K. Kuchar. Czech. J. Phys. **B18**, 435 (1968).

[107] L.L. Lindblom and D.R. Brill. Phys. Rev. **D10**, 3151 (1974).

[108] S.W. Hawking. Phys. Rev. **D13**, 191 (1976).

[109] P. Hut. Mon. Not. R. Astron. Soc. **180**, 379 (1977).

[110] I. Okamoto and O. Kaburaki. Phys. Rev. **43**, 340 (1991).

[111] L. Smarr. Phys. Rev. Lett. **30**, 71 (1973).

[112] S.W. Hawking. Phys. Rev. **D16**, 1747 (1978).

[113] J.M. York. Phys. Rev. **D33**, 2092 (1986).

[114] G.W. Gibbons and S.W. Hawking. Phys. Rev. **D15**, 2572 (1977).

[115] L. P. Eisenhart. *Riemannian Geometry.* Princeton University Press, Princeton, (1926).

[116] G. t'Hooft. Nucl. Phys. **B256**, 727 (1985).

[117] J. Bekenstein. *Do We Understand Black hole Entropy?* gr-qc/9409015, (1994).

[118] S.W. Hawking, G. Horowitz and S.F. Ross. Phys. Rev. **D51**, 4302 (1995).

[119] G.W. Gibbons and R.E. Kallosh. Phys. Rev. **D51**, 2839 (1995).

[120] Ya.B. Zeldovich. Sov. Phys.-JETP **45**, 9 (1977).

[121] R. Penrose. *The Emperor's New Mind.* Oxford University Press, Oxford, England, (1989).