Fueter regularity and slice regularity: meeting points for two function theories

Alessandro Perotti

Abstract We present some meeting points between two function theories, the Fueter theory of regular functions and the recent theory of quaternionic slice regular functions, which includes polynomials and power series with quaternionic coefficients. We show that every slice regular function coincides up to the first order with a unique regular function on the three-dimensional subset of reduced quaternions. We also characterize the regular functions so obtained. These relations have a higher dimensional counterpart between the theory of monogenic functions on Clifford algebras and the one of slice regular functions of a Clifford variable. We define a first order differential operator which extends the Dirac and Weyl operators to functions that can depend on all the coordinates of the algebra. The operator behaves well both w.r.t. monogenic functions and w.r.t. the powers of the (complete) Clifford variable. This last property relates the operator with the recent theory of slice monogenic and slice regular functions of a Clifford variable.

1 Introduction

The aim of this work is to illustrate some unexpected links between two function theories, one of which is well developed and dates back to the 1930's, while the other has been introduced recently but has seen rapid growth. The first is the Fueter theory of regular functions defined by means of the Cauchy-Riemann-Fueter differential operator, while the second is the theory of quaternionic slice regular functions, which comprises polynomials and power series with quaternionic coefficients on one side. These links have a higher dimensional counterpart between the theory of monogenic functions on Clifford algebras, defined in terms of the Dirac and

Alessandro Perotti
Department of Mathematics, University of Trento, Via Sommarive 14 I-38123 Italy, e-mail: perotti@science.unitn.it

Cauchy-Riemann operators, and the one of slice regular functions of a Clifford variable.

In order to obtain the promised relations between the two quaternionic function theories, we use a *modified* Cauchy-Riemann-Fueter operator $\mathcal{D}$, defined as:

$$\mathcal{D} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}\right),$$

where x_0, x_1, x_2, x_3 are the real coordinates of an element $q = x_0 + ix_1 + jx_2 + kx_3$ of the quaternionic space $\mathbb{H}$ w.r.t. the basic elements i, j, k. We refer e.g. to [31, 16] and [21] for some properties of this and other related differential operators on $\mathbb{H}$. The choice of $\mathcal{D}$ is justified by the fact that it behaves better than the standard Cauchy-Riemann-Fueter operator w.r.t. the powers of the quaternionic variable q. Moreover, the theory of functions defined by $\mathcal{D}$ has interesting relations with classes of holomorphic maps of two complex variables (see Sect. 2.1.1 for more details).

In general, the product of two Fueter regular functions is not Fueter regular. The same holds for functions in the kernel of $\mathcal{D}$. In particular, even if the identity function is regular, i.e. $\mathcal{D}(q) = 0$, the higher powers of q are not regular. Nevertheless, we are able to prove that for every positive power m, $\mathcal{D}(q^m)$ vanishes on the three-dimensional subset $\mathbb{H}_3 = \langle 1, i, j \rangle$ of reduced quaternions. This property extends to polynomials and convergent power series of the form $\sum_m q^m a_m$ and more generally to slice regular functions.

The concept of slice regularity for functions of one quaternionic, octonionic or Clifford variable has been introduced recently by Gentili and Struppa in [7, 8] and by Colombo, Sabadini and Struppa in [4] and further extended to real alternative *-algebras in [9, 10].

An application of the Cauchy-Kowalevski Theorem to the operator $\mathcal{D}$ assures that the restriction of any slice regular function to $\mathbb{H}_3$ has a unique regular (i.e. in the kernel of $\mathcal{D}$) extension to an open set. This extension gives an embedding of the space of slice regular functions into the space of regular functions.

The characterization of the image of this embedding is given by means of holomorphicity of the differentials w.r.t. the complex structures defined by left multiplication by imaginary reduced quaternions. It is based on a criterion for holomorphicity in the class of regular functions, which was proved in [22, 23] using the concept of the *energy quadric* of a function. The energy quadric is a positive semi-definite quadric, constructed by means of the Lichnerowicz homotopy invariants.

The second part of the paper is dedicated to the higher dimensional situation. We study some basic properties of a first order differential operator on the real Clifford algebra $\mathbb{R}_n$ of signature $(0, n)$ which generalizes the Weyl operator used in the theory of monogenic functions (for which we refer to [1], [3], [12]). While monogenic functions are usually defined on open subsets of the paravector space, the operator we consider acts on functions that can depend on all the coordinates of the algebra. This is similar to what happens in the quaternionic space $\mathbb{H} \simeq \mathbb{R}_2$, where the Cauchy-Riemann-Fueter operator acts on the whole space, not only on the reduced quaternions $\mathbb{H}_3$. Our starting point is the modified Cauchy-Riemann-Fueter operator

$\mathcal{D}$. When written in the notation of the Clifford algebra $\mathbb{R}_2$, $\mathcal{D}$ becomes the operator

$$\mathcal{D}_2 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} - e_{12}\frac{\partial}{\partial x_{12}}\right).$$

If $\mathcal{D}_1 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1}\right)$ and $\mathcal{D}_{1,2} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} + e_1\frac{\partial}{\partial x_{12}}\right)$ are the one-variable Cauchy–Riemann operators w.r.t. the complex variables $z_1 = x_0 + e_1x_1$, $z_2 = x_2 + e_1x_{12}$, then $\mathcal{D}_2 = \mathcal{D}_1 + e_2\mathcal{D}_{1,2}$. This observation suggests a recursive definition of a differential operator $\mathcal{D}_n$ on $\mathbb{R}_n$. Even if this definition of $\mathcal{D}_n$ is not symmetric w.r.t. the basis vectors, the operator we obtain *is* symmetric, and has the following explicit form:

$$\mathcal{D}_n = \frac{1}{2}\sum_K e_K^*\frac{\partial}{\partial x_K}$$

where $e_K^* = (-1)^{\frac{k(k-1)}{2}} e_K$ is obtained by applying to a basis element e_K the reversion anti-involution.

When restricted to functions of a paravector variable, $\mathcal{D}_n$ is equal (up to a factor $1/2$) to the Weyl (cf. e.g. [3, §4.2]), or Cauchy-Riemann (as in [12, §5.3]) operator of $\mathbb{R}_n$. Therefore every $\mathbb{R}_n$-valued monogenic function defined on an open domain of the paravector subspace $\mathbb{R}^{n+1}$ of $\mathbb{R}_n$ is in the kernel of $\mathcal{D}_n$. Moreover, the identity function x of $\mathbb{R}_n$ is in the kernel of $\mathcal{D}_n$, while its restriction to the paravector variable is *not* monogenic. The operator $\mathcal{D}_n$ behaves well also w.r.t. powers of the (complete) Clifford variable x. We show that every power x^m is in the kernel of $\mathcal{D}_n$ when n is odd. For even n, the same property holds on the so-called *quadratic cone* of the algebra (cf. [9, 10]). These properties link the operators $\mathcal{D}_n$ to the recent theory of *slice monogenic* [4] and *slice regular* functions on $\mathbb{R}_n$ [9, 10].

Operators similar to $\mathcal{D}_n$ have already been considered in the literature (e.g. in [28], [14] and [29]). However, it seems that the operators $\mathcal{D}_n$ are particularly well adapted to the theory of polynomials $\sum_m x^m a_m$ or more generally of slice regular functions on a Clifford algebra.

On the negative side, the operator $\mathcal{D}_n$ is not elliptic for $n > 2$ and its kernel is very large if we do not restrict the domains where functions are defined. In the last section, we focus on the case $n = 3$ and show a more strict relation of $\mathcal{D}_3$ with the Weyl operator. This suggests to consider a proper subspace of the kernel of $\mathcal{D}_3$, where the condition of *Cliffordian holomorphicity* [17] has a role. We get in this way the real analyticity in $\mathbb{R}_3$ and an integral representation formula on domains of polydisc type.

Some of the results of the present work have been presented in [26].

2 Fueter regularity and slice regularity

We begin recalling some results of the Fueter theory of regular functions. We then introduce some definitions of the recent theory of quaternionic slice regular functions. Our approach uses a modification of the Fueter construction based on stem functions. We then present some meeting points between the two function theories. We show that every slice regular function coincides, up to the first order, with a unique regular function on the three-dimensional subset $\mathbb{H}_3 = \langle 1, i, j\rangle$ of reduced quaternions.

2.1 Fueter regular functions

We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$. Given a bounded domain Ω in $\mathbb{H} \simeq \mathbb{C}^2$, a quaternionic function $f = f_1 + f_2 j$ of class C^1 on Ω will be called (left) *regular* on Ω if it is in the kernel of the *(modified) Cauchy-Riemann-Fueter operator*

$$\mathcal{D} = \frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}\right) \quad \text{on } \Omega\ . \tag{1}$$

We will denote by $\mathcal{R}(\Omega)$ the real vector space of regular functions on Ω. The space $\mathcal{R}(\Omega)$ contains the identity mapping and every holomorphic mapping (f_1, f_2) on Ω (w.r.t. the standard complex structure) defines a regular function $f = f_1 + f_2 j$.

Given the decomposition in *real* components $f = f^0 + i f^1 + j f^2 + k f^3$ of f, the operator $\mathcal{D}$ has the form:

$$\begin{aligned}2\mathcal{D}(f) =& \frac{\partial f^0}{\partial x_0} - \frac{\partial f^1}{\partial x_1} - \frac{\partial f^2}{\partial x_2} + \frac{\partial f^3}{\partial x_3} + i\left(\frac{\partial f^1}{\partial x_0} + \frac{\partial f^0}{\partial x_1} + \frac{\partial f^3}{\partial x_2} + \frac{\partial f^2}{\partial x_3}\right)\\ &+ j\left(\frac{\partial f^2}{\partial x_0} - \frac{\partial f^3}{\partial x_1} + \frac{\partial f^0}{\partial x_2} - \frac{\partial f^1}{\partial x_3}\right) + k\left(-\frac{\partial f^3}{\partial x_0} - \frac{\partial f^2}{\partial x_1} + \frac{\partial f^1}{\partial x_2} + \frac{\partial f^0}{\partial x_3}\right)\ .\end{aligned}$$

We recall some properties of regular functions, for which we refer to the papers of Naser [19], Nōno [20], Sudbery [33], Shapiro and Vasilevski [31], Kravchenko and Shapiro [16]:

1. Every regular function is harmonic: $\mathcal{D}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{D} = \frac{1}{4}\Delta_4$, where
$$\overline{\mathcal{D}} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right)\ . \tag{2}$$
2. The space $\mathcal{R}(\Omega)$ of regular functions on Ω is a *right* $\mathbb{H}$-module with integral representation formulas.
3. Given the decomposition in *complex* components $f = f_1 + f_2 j$, f is regular if and only if $\frac{\partial f_1}{\partial \bar{z}_1} = \frac{\partial \overline{f_2}}{\partial z_2}$, $\frac{\partial f_1}{\partial \bar{z}_2} = -\frac{\partial \overline{f_2}}{\partial z_1}$.
4. The complex components f_1, f_2 are both holomorphic or both non-holomorphic.
5. If Ω is pseudoconvex, every complex harmonic function is a complex component of a regular function on Ω.

The definition of regularity is equivalent to a notion introduced by Joyce [15] in the setting of hypercomplex manifolds. A hypercomplex structure on the manifold $\mathbb{H}$ is given by the complex structures J_1, J_2 on the tangent bundle $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on $T^*\mathbb{H} \simeq \mathbb{H}$ and set $J_3^* := J_1^* J_2^*$, which is equivalent to $J_3 = -J_1 J_2$. A function f is regular if and only if f is *q–holomorphic*, i.e. its differential $\mathrm{d}f$ satisfies the equation

$$\mathrm{d}f + iJ_1^*(\mathrm{d}f) + jJ_2^*(\mathrm{d}f) + kJ_3^*(\mathrm{d}f) = 0 \tag{3}$$

or, equivalently,

$$\mathrm{d}f^0 = J_1^*(\mathrm{d}f^1) + J_2^*(\mathrm{d}f^2) + J_3^*(\mathrm{d}f^3)\ . \tag{4}$$

In complex components $f = f_1 + f_2 j$, we can rewrite the equations of regularity as

$$\overline{\partial} f_1 = J_2^*(\partial \overline{f}_2), \tag{5}$$

where $\partial = \sum_{i=1}^2 \frac{\partial}{\partial z_i} dz_i$ and $\overline{\partial} = \sum_{i=1}^2 \frac{\partial}{\partial \bar{z}_i} d\bar{z}_i$ are the Cauchy-Riemann differential operators on $\mathbb{C}^2$ w.r.t. the standard complex structure.

Remark 1. The original definition of regularity given by Fueter (cf. e.g. [33, 12]) considered the differential operator

$$\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\ , \tag{6}$$

which differs from $\mathcal{D}$ in the sign of the last derivative. If γ denotes the real reflection of $\mathbb{C}^2 \simeq \mathbb{R}^4$ defined by $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$, then a C^1 function f is regular on the domain Ω if and only if $f \circ \gamma$ is Fueter–regular on $\gamma(\Omega) = \gamma^{-1}(\Omega)$, i.e. it satisfies the differential equation

$$\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right)(f \circ \gamma) = 0 \tag{7}$$

on $\gamma^{-1}(\Omega)$. The reflection γ has an algebraic interpretation. It can be seen as the *reversion* anti-involution $q \mapsto q^*$ of the Clifford algebra $\mathbb{R}_2 \simeq \mathbb{H}$, defined by

$$q^* = (x_0 + ix_1 + jx_2 + kx_3)^* = x_0 + ix_1 + jx_2 - kx_3\ . \tag{8}$$

2.1.1 Holomorphic functions w.r.t. a complex structure J_p

Let $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ be the *orthogonal complex structure* on $\mathbb{H}$ defined by a quaternion $p = p_1 i + p_2 j + p_3 k$ in the sphere $\mathbb{S} = \{p \in \mathbb{H} \mid p^2 = -1\} \simeq S^2$ of quaternionic imaginary units. In particular, J_1 is the standard complex structure of $\mathbb{C}^2 \simeq \mathbb{H}$. Let $\mathbb{C}_p = \langle 1, p \rangle$ be the complex plane spanned by 1 and p and let L_p be the complex structure defined on $T^*\mathbb{C}_p \simeq \mathbb{C}_p$ by left multiplication by p. Then $L_p = J_{p^*}$, where $p^* = p_1 i + p_2 j - p_3 k$.

Let $\mathrm{Hol}_p(\Omega,\mathbb{H})$ be the space of holomorphic maps between the (almost) complex manifolds (Ω,J_p) and $(\mathbb{H},L_p)$:

$$\mathrm{Hol}_p(\Omega,\mathbb{H}) = \{f:\Omega\to\mathbb{H} \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = \mathrm{Ker}\,\overline{\partial}_p\,, \tag{9}$$

where $\overline{\partial}_p$ is the Cauchy–Riemann operator

$$\overline{\partial}_p = \frac{1}{2}\left(\mathrm{d} + pJ_p^*\circ \mathrm{d}\right). \tag{10}$$

These functions will be called *J_p–holomorphic maps* on Ω. For any positive orthonormal basis $\{1,p,q,pq\}$ of $\mathbb{H}$ defined by a orthogonal pair $p,q\in\mathbb{S}^2$, let $f=f_1+f_2q$ be the decomposition of f with respect to the orthogonal sum

$$\mathbb{H} = \mathbb{C}_p \oplus (\mathbb{C}_p)q\,. \tag{11}$$

Let $f_1=f^0+pf^1$, $f_2=f^2+pf^3$, with f^0,f^1,f^2,f^3 the real components of f w.r.t. the basis $\{1,p,q,pq\}$. Then the equations of regularity can be rewritten in complex form as

$$\overline{\partial}_p f_1 = J_q^*(\partial_p \overline{f}_2), \tag{12}$$

where $\overline{f}_2 = f^2 - pf^3$ and $\partial_p = \frac{1}{2}\left(\mathrm{d} - pJ_p^*\circ\mathrm{d}\right)$. Therefore every $f\in\mathrm{Hol}_p(\Omega,\mathbb{H})$ is a regular function on Ω.

Remark 2. We refer to [22, 25] for the following properties of J_p-holomorphic maps.

1. The identity map belongs to the spaces $\mathrm{Hol}_i(\mathbb{H},\mathbb{H})$ and $\mathrm{Hol}_j(\mathbb{H},\mathbb{H})$ but not to $\mathrm{Hol}_k(\mathbb{H},\mathbb{H})$.
2. For every $p\in\mathbb{S}^2$, the spaces $\mathrm{Hol}_{-p}(\Omega,\mathbb{H})$ and $\mathrm{Hol}_p(\Omega,\mathbb{H})$ coincide.
3. Every $\mathbb{C}_p$–valued regular function is a J_p–holomorphic function.
4. If $f\in\mathrm{Hol}_p(\Omega,\mathbb{H})\cap\mathrm{Hol}_q(\Omega,\mathbb{H})$, with $p\neq\pm q$, then $f\in\mathrm{Hol}_r(\Omega,\mathbb{H})$ for every $r=\frac{\alpha p+\beta q}{\|\alpha p+\beta q\|}$ $(\alpha,\beta\in\mathbb{R})$ in the circle of $\mathbb{S}^2$ generated by p and q.

In [23] was proved that on every domain Ω there exist regular functions that are not J_p-holomorphic for any p. For example, the linear function $f=z_1+z_2+\bar{z}_1+(z_1+z_2+\bar{z}_2)j$ is regular on $\mathbb{H}$, but not holomorphic. The criterion for holomorphicity is based on an energy–minimizing property of holomorphic maps (see Sect. 2.4.4 for definitions and properties of the *energy quadric* of a quaternionic function f).

We can obtain regular functions also when considering non-constant (almost) complex structures. If $p=p(z)\in\mathbb{S}$ varies smoothly with $z\in\Omega$, the almost complex structures $J_{p(z)}$ and $L_{p(z)}$ are not constant, i.e. not compatible with the hyperkähler structure of $\mathbb{H}$. Note that in this case the structures are not necessarily integrable. Let $f\in C^1(\Omega)$. We shall say that p is *f-equivariant* if $f(z)=f(z')$ implies $p(z)=p(z')$ for $z,z'\in\Omega$. This property allows to define $\tilde{p}:f(\Omega)\to\mathbb{S}^2$ such that $\tilde{p}\circ f=p$ on Ω. In [22] was proved that $J_{p(z)}$-holomorphic maps $f:(\Omega,J_{p(z)})\to(\mathbb{H},L_{p(f(z))})$ give rise to regular functions:

Proposition 1. *[22, Proposition 1] If $f\in C^1(\Omega)$ satisfies the equation*

$$\overline{\partial}_{p(z)}f = \frac{1}{2}\left[df(z) + p(z)J^*_{p(z)} \circ df(z)\right] = 0 \qquad (13)$$

at every $z \in \Omega$, then f is a regular function on Ω. If, moreover, the structure p is f-equivariant and $\tilde{p}$ admits a continuous extension to an open set $U \supseteq f(\Omega)$, then f is a (pseudo)holomorphic map from (Ω, J_p) to $(U, L_{\tilde{p}})$.

For example, the function $f(z) = \bar{z}_1 + z_2^2 + \bar{z}_2 j$ is regular on $\mathbb{H}$. On the set $\Omega = \mathbb{H} \setminus \{z_2 = 0\}$ f is holomorphic w.r.t. the almost complex structure $J_{p(z)}$, where

$$p(z) = \frac{1}{\sqrt{|z_2|^2 + |z_2|^4}}\left(|z_2|^2 i - (\operatorname{Im} z_2) j - (\operatorname{Re} z_2) k\right) . \qquad (14)$$

2.2 Fueter construction

In 1934, Rudolf Fueter [6] proposed a simple method, which is now known as Fueter's Theorem, to generate quaternionic regular functions by means of complex holomorphic functions. Given a holomorphic "stem function"

$$F(z) = u(\alpha, \beta) + \mathrm{i}\, v(\alpha, \beta) \quad (z = \alpha + \mathrm{i}\beta \text{ complex, } u, v \text{ real-valued})$$

defined in the upper complex half-plane, real-valued when restricted to the real line, the formula:

$$f(q) := u\left(x_0, |\operatorname{Im}(q)|\right) + \frac{\operatorname{Im}(q)}{|\operatorname{Im}(q)|} v\left(x_0, |\operatorname{Im}(q)|\right) \qquad (15)$$

(with $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, $\operatorname{Im}(q) = x_1 i + x_2 j + x_3 k$) defines a *radially holomorphic* function on $\mathbb{H}$, whose Laplacian Δf is Fueter regular (now called the *Fueter transform* of F). Fueter's construction was later extended to higher dimensions by Sce [30], Qian [27] and Sommen [32] in the setting of octonionic and Clifford analysis. Fueter's Theorem and its generalizations provides a link between slice regular functions and Fueter regular (resp. monogenic) functions. In the next sections, we will present a relation of a different kind between these function theories.

2.3 Quaternionic slice regular functions

A modification of the Fueter construction can be applied to give a new approach (cf. [9, 10]) to the concept of "slice regularity" for functions of one quaternionic, octonionic or Clifford variable, which has been recently introduced by Gentili and Struppa in [7, 8] and by Colombo, Sabadini and Struppa in [4]. We start with a holomorphic function $F(z)$ with *quaternionic*-valued components u, v:

$$F(z) = u(\alpha, \beta) + \mathrm{i}\, v(\alpha, \beta) \quad (z = \alpha + \mathrm{i}\beta \text{ complex, } u, v \ \ \mathbb{H}\text{-}\textit{valued})$$

defined on a subset of the upper complex half-plane, real-valued on $\mathbb{R}$. Formula (15) defines then a *slice regular* (or *Cullen regular*) [7, 8] function on an open subset of the quaternionic space. We now make more precise this idea.

Let $q^c = x_0 - ix_1 - jx_2 - kx_4$ denote the *quaternionic conjugation*. Let $\mathbb{H}_\mathbb{C} = \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$ be the complexified quaternion algebra. We will use the representation

$$\mathbb{H}_\mathbb{C} = \{w = x + \mathrm{i}y \mid x, y \in \mathbb{H}\} . \tag{16}$$

$\mathbb{H}_\mathbb{C}$ is a complex algebra with unity w.r.t. the product defined as follows:

$$(x+\mathrm{i}y)(x'+\mathrm{i}y') = xx' - yy' + \mathrm{i}(xy' + yx') . \tag{17}$$

In $\mathbb{H}_\mathbb{C}$ two commuting operations are defined: the *anti-involution*

$$w \mapsto w^c = (x+\mathrm{i}y)^c = x^c + \mathrm{i}y^c \tag{18}$$

and the *complex conjugation*

$$w \mapsto \overline{w} = \overline{x+\mathrm{i}y} = x - \mathrm{i}y . \tag{19}$$

Definition 1. Let $D \subseteq \mathbb{C}$ be an open subset. If a function $F : D \to \mathbb{H}_\mathbb{C}$ is *complex intrinsic*, i.e. it satisfies the condition $F(\bar{z}) = \overline{F(z)}$ for every $z \in D$ such that $\bar{z} \in D$, then F is called a *stem function* on D.

Remark 3.

1. In the preceding definition, there is no restriction to assume that D is symmetric w.r.t. the real axis, i.e. $D = \mathrm{conj}(D) := \{z \in \mathbb{C} \mid \bar{z} \in D\}$.
2. A function F is a stem function if and only if the $\mathbb{H}$-valued components F_1, F_2 of $F = F_1 + \mathrm{i}F_2$ form an *even–odd pair* w.r.t. the imaginary part of z:

$$F_1(\bar{z}) = F_1(z) , \quad F_2(\bar{z}) = -F_2(z) \quad \text{for every } z \in D . \tag{20}$$

3. By means of a real basis $\mathcal{B}$ of $\mathbb{H}$, F can be identified with a complex intrinsic curve $F^\mathcal{B}$ in $\mathbb{C}^4$.

Given an open subset D of $\mathbb{C}$, let Ω_D be the open subset of $\mathbb{H}$ obtained by the action on D of the square roots of -1:

$$\Omega_D := \{q = \alpha + \beta p \in \mathbb{C}_p \mid \alpha, \beta \in \mathbb{R},\ \alpha + i\beta \in D,\ p \in \mathbb{S}\} . \tag{21}$$

Sets of this type will be called *circular sets* in $\mathbb{H}$.

Definition 2. Any stem function $F : D \to \mathbb{H}_\mathbb{C}$ induces a *left slice function* $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$. If $q = \alpha + \beta p \in D_p := \Omega_D \cap \mathbb{C}_p$, with $p \in \mathbb{S}$, we set

$$f(q) := F_1(z) + pF_2(z) \quad (z = \alpha + i\beta \in D) .$$

Note that if $q = x_0 + ix_1 + jx_2 + kx_3 = x_0 + \mathrm{Im}(q)$ and $\mathrm{Im}(q) \neq 0$, then $p = \pm\mathrm{Im}(q)/|\mathrm{Im}(q)|$. If $\mathrm{Im}(q) = 0$, then every choice of $p \in \mathbb{S}$ can be done. The complex intrinsicity of the stem function F assures that the definition of f is well posed.

There is an analogous definition for *right* slice functions when p is placed on the right of $F_2(z)$. In what follows, the term *slice functions* will always mean left slice functions.

We will denote the real vector space of (left) slice functions on Ω_D by $\mathcal{S}(\Omega_D)$. We will denote by $\mathcal{S}^1(\Omega_D) := \{f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D) \mid F \in C^1(D)\}$ the real vector space of slice functions with stem function of class C^1. It can be shown (cf. [10]) that every $f \in \mathcal{S}^1(\Omega_D)$ is of class C^1 on Ω_D.

Definition 3. Let $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$. We set

$$\frac{\partial f}{\partial q} := \mathcal{I}\left(\frac{\partial F}{\partial z}\right), \quad \frac{\partial f}{\partial q^c} := \mathcal{I}\left(\frac{\partial F}{\partial \bar{z}}\right) .$$

These functions are continuous slice functions on Ω_D.

We now introduce slice regularity. Left multiplication by i defines a complex structure on $\mathbb{H}_\mathbb{C}$. With respect to this structure, a C^1 function $F = F_1 + \mathrm{i}F_2 : D \to \mathbb{H}_\mathbb{C}$ is holomorphic if and only if its components F_1, F_2 satisfy the Cauchy–Riemann equations:

$$\frac{\partial F_1}{\partial \alpha} = \frac{\partial F_2}{\partial \beta}, \quad \frac{\partial F_1}{\partial \beta} = -\frac{\partial F_2}{\partial \alpha}, \quad \text{i.e.} \quad \frac{\partial F}{\partial \bar{z}} = 0 . \tag{22}$$

Definition 4. A (left) slice function $f \in \mathcal{S}^1(\Omega_D)$ is *(left) slice regular* if its stem function F is holomorphic. We will denote the real vector space of slice regular functions on Ω_D by $\mathcal{SR}(\Omega_D) := \{f \in \mathcal{S}^1(\Omega_D) \mid f = \mathcal{I}(F),\ F : D \to \mathbb{H}_\mathbb{C} \text{ holomorphic}\}$.

Polynomials $p(q) = \sum_{m=0}^{d} q^m a_m = \mathcal{I}(\sum_{m=0}^{d} z^m a_m)$ with right quaternionic coefficients can be considered as slice regular functions on $\mathbb{H}$. More generally, every convergent *power series* $\sum_m q^m a_m$ is a slice regular function on an open ball of $\mathbb{H}$ centered in the origin with (possibly infinite) positive radius.

Proposition 2. *[10, Proposition 8] Let $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$. Then f is slice regular on Ω_D if and only if for every $p \in \mathbb{S}$ the restriction $f_p := f_{|\mathbb{C}_p \cap \Omega_D} : D_p = \mathbb{C}_p \cap \Omega_D \to \mathbb{H}$ is holomorphic with respect to the complex structure J_p defined by left multiplication by p.*

Proposition 2 means that if D intersects the real axis, f is slice regular on Ω_D if and only if it is *Cullen regular* in the sense introduced by Gentili and Struppa in [7, 8].

2.4 A 3D-meeting point for two function theories

We start by computing the values of $\mathcal{D}(q^m)$. The crucial observation is that these values vanishes on the subspace of reduced quaternions. This property allows to extend the restriction to $\mathbb{H}_3$ of a slice regular function to a regular function.

2.4.1 Computation of $\mathcal{D}(q^m)$

In general, the product of two Fueter regular functions is not Fueter regular. The same holds for functions in the kernel of the modified Cauchy-Riemann-Fueter operator $\mathcal{D}$. In particular, even if the identity function is regular, i.e. $\mathcal{D}(q) = 0$, the higher powers of q are not regular. Nevertheless, we can show that $\mathcal{D}(q^m)$ vanishes on a three-dimensional subset of $\mathbb{H}$ for every positive power m.

Proposition 3. *Let $\mathcal{D}$ be the Cauchy-Riemann-Fueter operator of Sect. 2.1. Given two functions f, g of class C^1, the operator $\mathcal{D}$ satisfies the following product formula:*

$$\mathcal{D}(fg) = \mathcal{D}(f)g + \frac{1}{2}\left(f\frac{\partial g}{\partial x_0} + i\,f\frac{\partial g}{\partial x_1} + j\,f\frac{\partial g}{\partial x_2} - k\,f\frac{\partial g}{\partial x_3}\right) . \tag{23}$$

As a consequence, we get the following power formula for the (modified) Cauchy-Riemann-Fueter operator:

$$\mathcal{D}(q^m) = \mathcal{D}(q^{m-1})q + q^{m-1} - (q^{m-1})^* \tag{24}$$

where $q^* = x_0 + ix_1 + jx_2 - kx_3$ is obtained applying the reversion anti-involution to q (cf. Remark 1).

Proof. The product formula (23) follows immediately from the definition of $\mathcal{D}$. When applied to $q^m = q^{m-1}q$ it gives:

$$\mathcal{D}(q^m) = \mathcal{D}(q^{m-1})q + \frac{1}{2}\left(q^{m-1} + iq^{m-1}i + jq^{m-1}j - kq^{m-1}k\right) . \tag{25}$$

For every quaternion $p = x_0 + ix_1 + jx_2 + kx_3$, the sum $p + ipi + jpj - kpk$ is equal to $4kx_3 = 2(p - p^*)$, from which (24) follows. □

Let $\mathbb{H}_3 = \langle 1, i, j\rangle$ be the real vector subspace of *reduced quaternions*. It is well known (cf. e.g. [13]) that the powers of a reduced quaternion are still reduced quaternions. This follows easily from the fact that reduced quaternions are characterized by the condition $q = q^*$. Therefore, if $p, q \in \mathbb{H}_3$, $(pq)^* = q^*p^* = qp$, and then pq is still reduced if and only if p and q commute, i.e. $q \in \mathbb{C}_p$.

Corollary 1. *Let m be a positive integer. Then $\mathcal{D}(q^m)$ vanishes on $\mathbb{H}_3$.*

Proof. Since the power function $q \mapsto q^m$ maps $\mathbb{H}_3$ into $\mathbb{H}_3$, $q^{m-1} - (q^{m-1})^*$ vanishes on $\mathbb{H}_3$ for every $m \geq 1$. We conclude by induction on m starting from $\mathcal{D}(q) = 0$ and using (24). □

Corollary 2. *For every convergent power series $f(q)=\sum_m q^m a_m$ with quaternionic coefficients, $\mathcal{D}(f)$ vanishes on the intersection of the ball of convergence with the real vector space $\mathbb{H}_3 \simeq \mathbb{R}^3$ of reduced quaternions.* $\square$

Corollary 3. *Assume that Ω_D is connected and $\Omega_D \cap \mathbb{R} \neq \emptyset$. Let $f \in \mathcal{SR}(\Omega_D)$. Then $\mathcal{D}(f)$ vanishes at every point of $\widetilde{\Omega}_D := \Omega_D \cap \mathbb{H}_3$.*

Proof. Every slice regular function $f \in \mathcal{SR}(\Omega_D)$ has convergent power series expansions $\sum_m (q-r)^m a_m$ centered at real points r of Ω_D [7, 8]. Since every power $(q-r)^m$ is a polynomial in q, the thesis follows from Corollary 1. $\square$

We observe that the previous results are not true for the standard (not modified) Cauchy-Riemann-Fueter operator.

2.4.2 Regular extension of slice-regular functions

Let $f \in \mathcal{SR}(\Omega_D)$. Then f is real analytic on Ω_D (cf. [10, Proposition 7]). We can apply the Cauchy-Kowalevski Theorem (see for example [5]) to the initial values problem

$$\begin{cases} \mathcal{D}(g) = -\mathcal{D}(f) \quad \text{on } \Omega_D \\ g = 0 \quad \text{on } \widetilde{\Omega}_D = \Omega_D \cap \mathbb{H}_3 \end{cases} \tag{26}$$

and obtain a real analytic solution in the neighborhood of every point of the intersection of Ω_D with the hyperplane $\mathbb{H}_3$. By taking the union of all these neighborhoods, we get the existence of a solution of problem (26) on a open set $\Omega' \subset \mathbb{H}$ such that $\Omega' \cap \mathbb{H}_3 = \widetilde{\Omega}_D$.

Since $\mathcal{D}(f)=0$ and $g=0$ on $\widetilde{\Omega}_D$, also the normal derivative

$$\begin{aligned} \frac{\partial g}{\partial x_3} &= 2k\,\mathcal{D}(g) + k\left(\frac{\partial g}{\partial x_0} + i\frac{\partial g}{\partial x_1} + j\frac{\partial g}{\partial x_2}\right) \\ &= -2k\,\mathcal{D}(f) + k\left(\frac{\partial g}{\partial x_0} + i\frac{\partial g}{\partial x_1} + j\frac{\partial g}{\partial x_2}\right) \end{aligned} \tag{27}$$

vanishes on $\widetilde{\Omega}_D$. Therefore $g=0$, $\mathrm{d}g=0$ on $\widetilde{\Omega}_D$. This implies that the slice regular function $f \in \mathcal{SR}(\Omega_D)$ and the regular function $f+g \in \mathcal{R}(\Omega') \subset \mathrm{Ker}(\mathcal{D})$ coincide up to the first order on the three-dimensional set $\widetilde{\Omega}_D$. We summarize the result in the following statement.

Proposition 4. *Assume that Ω_D is connected and $\Omega_D \cap \mathbb{R} \neq \emptyset$. Let $f \in \mathcal{SR}(\Omega_D)$. Then there exists an open (relative to $\mathbb{H}$) neighborhood Ω' of $\Omega_D \cap \mathbb{H}_3$ and a regular function $\tilde{f} \in \mathcal{R}(\Omega')$ such that $f=\tilde{f}$ on $\Omega' \cap \mathbb{H}_3 = \Omega_D \cap \mathbb{H}_3$ up to the first order.* $\square$

For polynomials $f(q)=\sum_m q^m a_m$, the solution to problem (26) can be obtained explicitly in a finite number of steps by means of one-variable integrations w.r.t. the normal coordinate x_3. Consider the approximate problems:

$$\begin{cases} \dfrac{\partial g^{(h+1)}}{\partial x_3} = -2k\,\mathcal{D}(f + \sum_{l=1}^{h} g^{(l)}) \quad \text{on } \mathbb{H} \\ g^{(h+1)} = 0 \quad \text{on } \mathbb{H}_3 \end{cases} \tag{28}$$

for $h = 1, \ldots, \deg(f)$, starting with the function $g^{(1)} \equiv 0$. We are looking for polynomial solutions $g^{(h)}$ with $\deg(g^{(h)}) \leq h$.

At the first step, since x_3 divides the polynomial $\mathcal{D}(f)$ (from Corollary 1), x_3^2 divides the first solution $g^{(2)}$ of (28). Therefore x_3^2 divides also $\mathcal{D}(f + g^{(2)})$, since

$$2\mathcal{D}(f + g^{(2)}) = k\frac{\partial g^{(2)}}{\partial x_3} + 2\mathcal{D}(g^{(2)}) = \frac{\partial g^{(2)}}{\partial x_0} + i\frac{\partial g^{(2)}}{\partial x_1} + j\frac{\partial g^{(2)}}{\partial x_2} \tag{29}$$

and $g^{(2)} = 0$ on $\mathbb{H}_3$. By induction on h, we get by the same reasoning that the power x_3^h divides $g^{(h)}$ and $\mathcal{D}(f + \sum_{l=1}^{h} g^{(l)})$ for every $h = 1, \ldots, \deg(f)$. At the last step, we get that $x_3^{\deg(f)}$ divides $\mathcal{D}(f + \sum_{i=1}^{\deg(f)} g^{(i)})$, but then it has to be $\mathcal{D}(f + \sum_{i=1}^{\deg(f)} g^{(i)}) = 0$ for degree reasons. Observe that x_3^2 divides every partial solution $g^{(h)}$.

The polynomial $\tilde{f} := f + \sum_{i=1}^{\deg(f)} g^{(i)}$ is regular on the whole space, has the same degree as f, and coincides with f up to first order on $\mathbb{H}_3$.

As an illustration of this procedure, we compute the extension $\tilde{f}$ for the first three powers of q:

1. $f(q) = q$ is regular, so $\tilde{f} = f$.
2. $f(q) = q^2$ has regular extension $\tilde{f} = q^2 + 2x_3^2$.
3. $f(q) = q^3$ has regular extension $\tilde{f} = q^3 + x_3^2(6x_0 + 2x_1\,i + 2x_2\,j + \frac{2}{3}x_3\,k)$.

2.4.3 The embedding $\mathcal{SR} \hookrightarrow \mathcal{R}$

Definition 5. Assume that Ω_D is connected and $\Omega_D \cap \mathbb{R} \neq \emptyset$. Given $f \in \mathcal{SR}(\Omega_D)$, denote by $\mathrm{Reg}(f)$ the unique regular function defined on a maximal domain and satisfying

$$f = \mathrm{Reg}(f), \quad \mathrm{d}f = \mathrm{d}\,\mathrm{Reg}(f) \quad \text{at every point of } \Omega_D \cap \mathbb{H}_3\ .$$

Uniqueness follows from the identity principle for regular functions. Let

$$\widetilde{\mathcal{R}}(\Omega_D) := \{g \in \mathcal{R}(\Omega') \mid \Omega' \text{ open and connected in } \mathbb{H} \text{ s.t. } \Omega' \cap \mathbb{H}_3 = \Omega_D \cap \mathbb{H}_3\}\ .$$

In the space $\widetilde{\mathcal{R}}(\Omega_D)$ we identify two functions if they coincide on the intersection of the domains of definition. The mapping $f \mapsto \mathrm{Reg}(f)$ is an injective (right) $\mathbb{H}$-linear operator, giving an embedding $\mathcal{SR}(\Omega_D) \hookrightarrow \widetilde{\mathcal{R}}(\Omega_D)$.

2.4.4 Characterization of $\mathrm{Reg}(\mathcal{SR}(\Omega_D))$ in $\widetilde{\mathcal{R}}(\Omega_D)$

Assume $\Omega_D \cap \mathbb{R} \neq \emptyset$. Let $f \in \mathcal{SR}(\Omega_D)$ be slice regular and $x \in \mathbb{H}_3 \setminus \mathbb{R}$, let $p := \mathrm{Im}(x)/|\mathrm{Im}(x)| \in \mathbb{S}$. From Proposition 2 we get that the restriction

$$f_p := f_{|\mathbb{C}_p \cap \Omega_D} : \mathbb{C}_p \cap \Omega_D \to \mathbb{H}$$

is holomorphic with respect to the complex structure J_p defined by left multiplication by p. Moreover, at every $x \in \widetilde{\Omega}_D = \Omega_D \cap \mathbb{H}_3$, because of Proposition 4 the differential map

$$\mathrm{d}f_x = \mathrm{d}\,\mathrm{Reg}(f)_x$$

is a regular linear function on $\mathbb{H}$. These two properties can be strengthened in the sense explained by the next statement.

Theorem 1. *If $f \in \mathcal{SR}(\Omega_D)$ and $x \in \widetilde{\Omega}_D \setminus \mathbb{R} = (\Omega_D \cap \mathbb{H}_3) \setminus \mathbb{R}$, then the differential map $\mathrm{d}f_x$ belongs to the space $\mathrm{Hol}_{J_p}(\mathbb{H},\mathbb{H}) \subset \mathcal{R}(\mathbb{H})$, where $p := \mathrm{Im}(x)/|\mathrm{Im}(x)|$, i.e. the linear map*

$$\mathrm{d}f_x \ :\ (\mathbb{H}, J_p) \to (\mathbb{H}, J_p)$$

is holomorphic.

The proof of Theorem 1 is based on a criterion for holomorphicity in the space $\mathcal{R}(\Omega_D)$, which was proved in [22, 23] using the concept of the *energy quadric* of a function. The energy quadric of a regular function f is a positive semi-definite quadric, constructed by means of the Lichnerowicz homotopy invariants.

We recall some definitions and results from [22, 23]. The *energy density* of a map $f : \Omega \to \mathbb{H}$, of class $C^1(\Omega)$, w.r.t. the euclidean metric, is the function

$$\mathcal{E}(f) := \tfrac{1}{2}\|\mathrm{d}f\|^2 = \tfrac{1}{2}\,\mathrm{tr}(Jac(f)\overline{Jac(f)}^T)\ ,$$

where $Jac(f)$ is the Jacobian matrix of f. Assume Ω relatively compact. The *energy* of $f \in C^1(\overline{\Omega})$ on Ω is the integral defined by

$$\mathcal{E}_\Omega(f) := \int_\Omega \mathcal{E}(f)\mathrm{d}V\ .$$

Let $A = (a_{\alpha\beta})$ be the 3×3 matrix with entries the real functions

$$a_{\alpha\beta} = -\langle J_\alpha, f^* L_{i_\beta}\rangle, \text{ where } (i_1, i_2, i_3) = (i, j, k)\ .$$

(these numbers are the analogues of the Lichnerowicz invariants (cf. [18] and [2]) For $f \in C^1(\overline{\Omega})$, we set

$$A_\Omega := \int_\Omega A\,dV \quad \text{and} \quad M_\Omega := \tfrac{1}{2}\left((\mathrm{tr}A_\Omega)I_3 - A_\Omega\right)\ ,$$

where I_3 denotes the identity matrix.

Theorem 2 ([23]). *If $f \in C^1(\overline{\Omega})$ is regular on Ω, then it minimizes energy in its homotopy class (relative to $\partial\Omega$).*

Theorem 3 ([23]). *Let $f \in C^1(\overline{\Omega})$. The following facts hold:*

1. *f is regular on Ω if and only if $\mathcal{E}_\Omega(f) = \operatorname{tr} M_\Omega$.*
2. *If $f \in \mathcal{R}(\Omega)$, then M_Ω is symmetric and positive semidefinite.*
3. *If $f \in \mathcal{R}(\Omega)$, then f belongs to some space $\operatorname{Hol}_p(\Omega,\mathbb{H})$ (for a constant structure J_p) if and only if $\det M_\Omega = 0$. More precisely, $X_p = (p_1, p_2, p_3)$ is a unit vector in the kernel of M_Ω if and only if $f \in \operatorname{Hol}_p(\Omega,\mathbb{H})$.*

The criterion of holomorphicity holds also pointwise: let Ω be connected and $f \in C^1(\Omega)$. Consider the matrix of real functions on Ω:

$$M := \tfrac{1}{2}\left((\operatorname{tr} A) I_3 - A\right) .$$

Theorem 4 ([22]). *Let $f \in C^1(\Omega)$. The following facts hold:*

1. *f is regular on Ω if and only if $\mathcal{E}(f) = \operatorname{tr} M$ at every point $z \in \Omega$.*
2. *If $f \in \mathcal{R}(\Omega)$, then M is a 3×3 symmetric and positive semidefinite matrix.*
3. *If $f \in \mathcal{R}(\Omega)$, then $\det M = 0$ on Ω if and only if there exists an open, dense subset $\Omega' \subseteq \Omega$ such that f is a (pseudo)holomorphic map from $(\Omega', J_{p(z)})$ to $(\mathbb{H}, L_{p(f(z))})$ for some $p(z)$.*

Now we come to the *proof of Theorem 1*. To this end, we compute the energy quadric of f at $x \in \Omega_D \cap \mathbb{H}_3$. Since f coincides with $\operatorname{Reg}(f)$ up to first order on $\Omega_D \cap \mathbb{H}_3$, from Theorem 4 we get that $\mathcal{E}(f) = \operatorname{tr} M(f)$ on $\Omega_D \cap \mathbb{H}_3$ and that $M(f)$ is positive semidefinite at every point $x \in \Omega_D \cap \mathbb{H}_3$. Let us denote by $\pi_1(f) = f_1$ and $\pi_2(f) = f_2$ the complex components of f. From [22] we get the following expression of the energy quadric $M(f)$ at $x \in \Omega_D \cap \mathbb{H}_3$ in terms of f_1 and f_2 and their complex derivatives:

$$M(f) = \begin{bmatrix} 2|c|^2 & \operatorname{Im}\langle c, a-b\rangle & \operatorname{Re}\langle c, a+b\rangle \\ \operatorname{Im}\langle c, a-b\rangle & \frac{1}{2}|a-b|^2 & -\operatorname{Im}\langle a,b\rangle \\ \operatorname{Re}\langle c, a+b\rangle & -\operatorname{Im}\langle a,b\rangle & \frac{1}{2}|a+b|^2 \end{bmatrix} , \tag{30}$$

where

$$a = \left(\frac{\partial f_1}{\partial z_1}, \frac{\partial f_2}{\partial z_1}\right),\ b = \left(\frac{\partial \bar f_2}{\partial \bar z_2}, -\frac{\partial \bar f_1}{\partial \bar z_2}\right),\ c = \left(\frac{\partial \bar f_2}{\partial z_1}, -\frac{\partial \bar f_1}{\partial z_1}\right)$$

are all computed at $x \in \Omega_D \cap \mathbb{H}_3$. We now show that the vector $(x_1, x_2, 0)$ belongs to the null space of the matrix $M(q^m)$. If $(x_1, x_2, 0) \neq (0,0,0)$, it is possible to find a similarity of the space $\langle i, j, k\rangle \simeq \mathbb{R}^3$, with rotational component induced by a reduced quaternion $a \in \mathbb{H}_3$, which sends $(x_1, x_2, 0)$ in $(1,0,0)$. The transformation property of the energy quadric w.r.t. rotations (see [24, Theorem 4]) implies that $(x_1, x_2, 0) \in \operatorname{Ker}(M(q^m))$ at $x = x_0 + ix_1 + jx_2$ if and only if $(1,0,0) \in \operatorname{Ker}(M(q^m))$ at $x_0 + i$.

In view of (30), in order to show that the vector $(1,0,0)$ belongs to the null space of $M(q^m)$ at $x_0 + i$ it suffices to prove that $\bar c = \left(\frac{\partial \pi_2(q^m)}{\partial \bar z_1}, -\frac{\partial \pi_1(q^m)}{\partial \bar z_1}\right)$ vanishes at $x_0 + i$. This is a consequence of the following Lemma.

Lemma 1. *For every positive integer m, z_2 divides on the left the partial derivative $\frac{\partial q^m}{\partial \bar{z}_1}$, i.e. $\frac{\partial q^m}{\partial \bar{z}_1} = z_2\, g$ for a quaternionic function g.*

Proof. For $m > 1$ the following product formula can be easily obtained:

$$\frac{\partial q^m}{\partial \bar{z}_1} = \frac{\partial q^{m-1}}{\partial \bar{z}_1} q + \frac{1}{2}\left(q^{m-1} + iq^{m-1} i\right) = \frac{\partial q^{m-1}}{\partial \bar{z}_1} q + \pi_2(q^{m-1})\, j\ . \tag{31}$$

Using (31) and the equation

$$\pi_2(q^m) = \pi_2\left(\left(\pi_1(q^{m-1}) + \pi_2(q^{m-1})\, j\right)(z_1 + z_2 j)\right) = z_2 \pi_1(q^{m-1}) + \pi_2(q^{m-1})\bar{z}_1\ , \tag{32}$$

we get the thesis by induction on m. □

Let $x \in \mathbb{H}_3 \setminus \mathbb{R}$ be fixed. From Theorem 3 we get that $(\mathrm{d}q^m)_x \in \mathrm{Hol}_p(\mathbb{H},\mathbb{H})$, where $p := \mathrm{Im}(x)/|\mathrm{Im}(x)|$. Since $f \in \mathcal{SR}(\Omega_D)$ has a series expansion, we get that also $\mathrm{d}f_x$ belongs to $\mathrm{Hol}_p(\mathbb{H},\mathbb{H})$ for every $x \in (\Omega_D \cap \mathbb{H}_3) \setminus \mathbb{R}$. To finish the proof of Theorem 1, we observe that $J_p = L_p$, since $x \in \mathbb{H}_3$. Therefore the linear map $\mathrm{d}f_x\ :\ (\mathbb{H}, J_p) \to (\mathbb{H}, J_p)$ is holomorphic. □

From the holomorphycity of the differentials $\mathrm{d}f_x$ at a reduced quaternion x, we get immediately the following properties for the real Jacobian matrix of f.

Corollary 4. *If $f \in \mathcal{SR}(\Omega_D)$ and $x \in \widetilde{\Omega}_D \setminus \mathbb{R} = (\Omega_D \cap \mathbb{H}_3) \setminus \mathbb{R}$, then*

1. $\det(Jac(f)) \geq 0$ *at x.*
2. $\mathrm{rank}(Jac(f))$ *is even at x.*

The characterization of $\mathrm{Reg}(\mathcal{SR}(\Omega_D))$ in $\widetilde{\mathcal{R}}(\Omega_D)$ is completed by the following converse statement.

Proposition 5. *Assume Ω_D connected and $\Omega_D \cap \mathbb{R} \neq \emptyset$. Let $f \in \mathcal{R}(\Omega_D)$. If $\mathrm{d}f_x \in \mathrm{Hol}_p(\mathbb{H},\mathbb{H})$ for every $x \in (\Omega_D \cap \mathbb{H}_3) \setminus \mathbb{R}$, with $p := \mathrm{Im}(x)/|\mathrm{Im}(x)|$ then there exists a (unique) slice regular function g on Ω_D, such that g and f are equal up to the first order on $\Omega_D \cap \mathbb{H}_3$.*

Proof. For every $p \in \mathbb{S} \cap \mathbb{H}_3$ and any $x \in (\Omega_D \cap \mathbb{C}_p) \setminus \mathbb{R}$, the restriction $f_p = f_{|\mathbb{C}_p} : \mathbb{C}_p \to \mathbb{H}$ is holomorphic w.r.t. the structure J_p, since $\mathbb{C}_p$ is a complex subspace of $\mathbb{H}$ w.r.t. J_p and the differential $\mathrm{d}f_x \in \mathrm{Hol}_p(\mathbb{H},\mathbb{H})$ by hypothesis. As proven in [10, Corollary 9], this implies that the restriction of f to $\Omega_D \cap \mathbb{H}_3$ is a *slice monogenic* function (cf. [4]) when $\mathbb{H}$ is identified with the Clifford algebra $\mathbb{R}_2$ and $\mathbb{H}_3$ is identified with the subspace of *paravectors* in $\mathbb{R}_2$. As seen in [10], every slice monogenic function on $\Omega_D \cap \mathbb{H}_3$ can be uniquely extended to a slice regular function on Ω_D. Since $\mathcal{D}(g - f) = 0$ on $\Omega_D \cap \mathbb{H}_3$, $g = f$ up to the first order on $\Omega_D \cap \mathbb{H}_3$. □

3 The full Dirac operators

We now look at the higher dimensional situation. Our starting point is the modified Cauchy-Riemann-Fueter operator $\mathcal{D}$. If we consider the quaternionic space as the real Clifford algebra $\mathbb{R}_2$, we can give a new look at $\mathcal{D}$ in terms of the algebraic involutions of the algebra. This reinterpretation of $\mathcal{D}$ suggests to study a new first order differential operator on the Clifford algebras $\mathbb{R}_n$, which behaves well w.r.t. monogenic functions and also w.r.t. the powers of the (complete) Clifford variable. This last property relates the operator with the theory of slice monogenic and slice regular functions.

3.1 The operators $\mathcal{D}_n$

Denote by $e_1,\dots,e_n$ the generators of $\mathbb{R}_n$. Let $x=\sum_K x_K e_K$ be a Clifford number, where $K=(i_1,\dots,i_k)$ is a multiindex, with $0\le |K| := k\le n$, the coefficients x_K are real numbers and e_K is the product of basis elements $e_K=e_{i_1}\cdots e_{i_k}$.

Definition 6. Let $\mathcal{D}_1=\frac{1}{2}\left(\frac{\partial}{\partial x_0}+e_1\frac{\partial}{\partial x_1}\right)$ and $\mathcal{D}_{1,2}=\frac{1}{2}\left(\frac{\partial}{\partial x_2}+e_1\frac{\partial}{\partial x_{12}}\right)$. For $n>1$, define recursively

$$\mathcal{D}_n := \mathcal{D}_{n-1}+e_n\mathcal{D}_{n-1,n}\ . \tag{33}$$

Here we consider $\mathbb{R}_{n-1}$ embedded in $\mathbb{R}_n$ and $\mathcal{D}_{n-1,n}$ is the operator defined as $\mathcal{D}_{n-1}$ w.r.t. the 2^{n-1} variables $x_n, x_{1n}, x_{2n},\dots,x_{12n},\dots,x_{12\cdots n}$. Since $\mathcal{D}_n$ depends on all the basis coordinates of $\mathbb{R}_n$, we call it the *full Dirac operator* on $\mathbb{R}_n$.

Remark 4. The operator $\mathcal{D}_1$ is the standard Cauchy–Riemann operator on the complex plane $\mathbb{C}\simeq\mathbb{R}_1$. The operator $\mathcal{D}_2$ is the same as the modified Cauchy-Riemann-Fueter operator $\mathcal{D}$ on $\mathbb{H}\simeq\mathbb{R}_2$:

$$\mathcal{D}_2=\frac{1}{2}\left(\frac{\partial}{\partial x_0}+e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}-e_{12}\frac{\partial}{\partial x_{12}}\right)\ . \tag{34}$$

$\mathcal{D}_3=\mathcal{D}_2+e_3\mathcal{D}_{2,3}$ has the following expression

$$\mathcal{D}_3=\frac{1}{2}\left(\frac{\partial}{\partial x_0}+e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}+e_3\frac{\partial}{\partial x_3}-e_{12}\frac{\partial}{\partial x_{12}}-e_{13}\frac{\partial}{\partial x_{13}}-e_{23}\frac{\partial}{\partial x_{23}}-e_{123}\frac{\partial}{\partial x_{123}}\right).$$

Despite its recursive definition, the operator $\mathcal{D}_n$ is symmetric w.r.t. the basis elements $e_1,\dots,e_n$. More precisely, it has the following expression involving the *reversion* anti-involution of $\mathbb{R}_n$.

Proposition 6. *The operator $\mathcal{D}_n$ can be written in the following form:*

$$\mathcal{D}_n=\frac{1}{2}\sum_{|K|\le n} e_K^*\frac{\partial}{\partial x_K} \tag{35}$$

where $e_K^*=(-1)^{\frac{k(k-1)}{2}}e_K$ is obtained by applying to e_K the reversion anti-involution $x\mapsto x^*$. Moreover,

$$\mathcal{D}_{n-1,n}=\frac{1}{2}\sum_{H\not\ni n} e_H^*\frac{\partial}{\partial x_{(Hn)}}\ . \tag{36}$$

Proof. $\mathcal{D}_1$ and $\mathcal{D}_{1,2}$ have the required form. Equations (35) and (36) follows from an easy inductive argument. □

On functions depending only on paravectors, the operator $\mathcal{D}_n$ acts as $\frac{1}{2}\mathcal{W}_n$, where $\mathcal{W}_n$ is the *Weyl* (or *Cauchy-Riemann) operator* on $\mathbb{R}_n$:

$$\mathcal{W}_n = \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} . \tag{37}$$

Corollary 5. *Every monogenic function (i.e. in the kernel of $\mathcal{W}_n$) defined on an open subset of the paravector subspace $\mathbb{R}^{n+1} \subset \mathbb{R}_n$ can be identified with an element of* $\ker \mathcal{D}_n$.

We can define also the conjugated operator $\overline{\mathcal{D}}_n$ and the auxiliary operator $\mathcal{D}^*_n$.

Definition 7.

$$\begin{cases} \overline{\mathcal{D}}_1 = \frac{\partial}{\partial z_1} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1}\right), \quad \mathcal{D}^*_1 = \frac{\partial}{\partial \bar{z}_1} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1}\right) \\ \overline{\mathcal{D}}_n = \overline{\mathcal{D}}_{n-1} - e_n \mathcal{D}^*_{n-1,n}, \quad \mathcal{D}^*_n = \mathcal{D}^*_{n-1} + e_n \overline{\mathcal{D}}_{n-1,n} \end{cases}$$

where $\mathcal{D}^*_{n-1,n}$ and $\overline{\mathcal{D}}_{n-1,n}$ are defined as $\mathcal{D}^*_{n-1}$ and $\overline{\mathcal{D}}_{n-1}$ w.r.t. the 2^{n-1} variables $x_n, x_{1n}, \ldots, x_{12n}, \ldots, x_{12\cdots n}$.

Still by induction, we obtain the following explicit forms for the operators $\overline{\mathcal{D}}_n$ and $\mathcal{D}^*_n$, now involving the *principal involution* of $\mathbb{R}_n$.

Proposition 7.

$$\overline{\mathcal{D}}_n = \frac{1}{2} \sum_{|K| \le n} \tilde{e}_K \frac{\partial}{\partial x_K} , \tag{38}$$

where $\tilde{e}_K = (-1)^k e_K$ is obtained by applying to e_K the principal involution $x \mapsto \tilde{x}$. Moreover,

$$\mathcal{D}^*_n = \frac{1}{2} \sum_{|K| \le n} e_K \frac{\partial}{\partial x_K} \quad \textit{and} \quad \mathcal{D}^*_{n-1,n} = \frac{1}{2} \sum_{H \not\ni n} e_H \frac{\partial}{\partial x_{Hn}} . \tag{39}$$

The differential operator

$$\mathcal{D}^*_n = \frac{1}{2} \sum_{|K| \le n} e_K \frac{\partial}{\partial x_K}$$

has already been considered in the literature (cf. [28] and [14]). The behavior of $\mathcal{D}_n$ w.r.t. power functions (see below Theorem 5) and the property stated in the next remark indicate that the operators $\mathcal{D}_n$ are better suited than $\mathcal{D}^*_n$ or $\overline{\mathcal{D}}_n$ to the theory of polynomials or more generally *slice regular* functions on a Clifford algebra. In [29] Dirac operators on the subspace of l-vectors have been studied. They coincide (up to sign) with the restriction of $\mathcal{D}_n$ to l-vectors. Since we are interested in the global behavior of the operator on the algebra, the choice of the grade-depending sign for the coefficients of $\mathcal{D}_n$ is essential.

Remark 5. The identity function x of $\mathbb{R}_3$ belongs to the kernel of $\mathcal{D}_2$ and $\mathcal{D}_3$ (and of course of the Cauchy-Riemann operator $\mathcal{D}_1$). Starting from $\mathcal{D}_1 x = 0$, $\mathcal{D}_{1,2} x = 0$ on $\mathbb{R}_2$, we get recursively that $\mathcal{D}_n x = 0$ on $\mathbb{R}_n$ for every n. Observe that even if $\mathcal{D}_1^* x = \mathcal{D}_{1,2}^* x = 0$, the identity function does not belong to the kernels of $\mathcal{D}_n^*$ or $\overline{\mathcal{D}}_n$ for every n.

3.2 Slice regularity and the full Dirac operators

We are interested in the values of $\mathcal{D}_n$ on polynomials $\sum_m x^m a_m$ in the complete Clifford variable x. We start from the powers of x. To express our computation, we need some definitions and results from the theory of *slice regular* functions on $\mathbb{R}_n$ (see [9, 10] where the theory of slice regularity is constructed in a greater generality, for functions defined on a real alternative *-algebra).

Definition 8. Let $t(x) = x + \bar{x}$ be the *trace* of x and $n(x) = x\bar{x}$ the (squared) *norm* of $x \in \mathbb{R}_n$. The *quadratic cone* of $\mathbb{R}_n$ is the subset

$$\mathcal{Q}_n := \mathbb{R} \cup \{x \in \mathbb{R}_n \mid t(x) \in \mathbb{R},\ n(x) \in \mathbb{R}, 4n(x) > t(x)^2\} .$$

(It can be seen that the last condition is automatically satisfied on $\mathbb{R}_n$)
Let $\mathbb{S}_n := \{J \in \mathcal{Q}_n \mid J^2 = -1\} = \{x \in \mathbb{R}_n \mid t(x) = 0,\ n(x) = 1\}$. The elements of $\mathbb{S}_n$ are called *square roots of* -1 in the algebra $\mathbb{R}_n$.

Proposition 8 ([9, 10]). *The quadratic cone $\mathcal{Q}_n$ satisfies the following properties:*

1. *$\mathcal{Q}_n = \mathbb{R}_n$ only for $n = 1, 2$.*
2. *$\mathcal{Q}_n$ contains (properly) the subspace of paravectors*

$$\mathbb{R}^{n+1} := \{x = \textstyle\sum_K x_K e_K \in \mathbb{R}_n \mid x_K = 0 \text{ for every } K \text{ such that } |K| > 1\} .$$

3. *$\mathcal{Q}_n$ is the real algebraic subset (proper for $n > 2$) of $\mathbb{R}_n$ defined by the equations*

$$x_K = 0,\ x \cdot (x e_K) = 0 \quad \forall e_K \neq 1 \text{ such that } e_K^2 = 1 , \tag{40}$$

where $x \cdot y$ denotes the euclidean scalar product on $\mathbb{R}_n \simeq \mathbb{R}^{2^n}$.

4. *For $J \in \mathbb{S}_n$, let $\mathbb{C}_J = \langle 1, J \rangle \simeq \mathbb{C}$ be the subalgebra generated by J. Then*

$$\mathcal{Q}_n = \bigcup_{J \in \mathbb{S}_n} \mathbb{C}_J \tag{41}$$

and $\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}$ for every $I, J \in \mathbb{S}_n$, $I \neq \pm J$. As a consequence, if x belongs to $\mathcal{Q}_n$, also the powers x^m belong to the quadratic cone $\mathcal{Q}_n$.

Slice regular functions are defined only on subdomains of the quadratic cone (we refer to [9, 10] for full details). However, if the domain intersects the real axis, then the class of slice regular functions coincides with the one of functions having local power series expansion centered at real points.

Now we compute the values of $\mathcal{D}_n(x^m)$. We already know the result for $n=2$ (cf. Corollary 1): $\mathcal{D}_2(x^m)=0$ on the subset of reduced quaternions $\mathbb{H}_3\subset\mathbb{H}\simeq\mathbb{R}_2$. We can show that the behavior of $\mathcal{D}_n$ on the powers depends on the parity of n (as many other properties of $\mathbb{R}_n$). In this scheme the quaternions ($n=2$) are in some sense exceptional.

Theorem 5. *Let $x=\sum_{|K|\le n}x_Ke_K$ be the complete Clifford variable in $\mathbb{R}_n$. The following facts hold:*

1. *If n is an odd integer, then $\mathcal{D}_n(x^m)=0$ on the whole algebra $\mathbb{R}_n$ for every integer $m\ge 1$.*
2. *If n is an even integer, $n>2$, then $\mathcal{D}_n(x^m)=0$ on the quadratic cone $\mathcal{Q}_n$ of $\mathbb{R}_n$ for every integer $m\ge 1$.*
3. *$\mathcal{D}_2(x^m)=0$ on the subset of reduced quaternions $\mathbb{H}_3\subset\mathbb{R}_2$ for every integer $m\ge 1$.*

In the proof of Theorem 5 we will apply the following algebraic Lemma:

Lemma 2. *Let $N=(1,\dots,n)$. For every $x\in\mathbb{R}_n$, it holds*

$$\sum_{H\not\ni n} e_H^* x e_H = 2^{n-1}\left(x_n e_n + x_{(1\cdots n-1)}e_{(1\cdots n-1)}\right) \quad \textit{for odd } n\,, \tag{42}$$

$$\sum_{H\not\ni n} e_H^* x e_H = 2^{n-1}\left(x_n e_n + x_N e_N\right) \quad \textit{for even } n\,. \tag{43}$$

Proof. Let $h=|H|$, $k=|K|$ and let $\sigma_{H,K}$ be the sign such that $e_He_K=\sigma_{H,K}e_Ke_H$. Then it holds

$$\begin{aligned} e_H^* x e_H &= (-1)^{\frac{h(h-1)}{2}}\sum_K x_K\sigma_{H,K}e_Ke_H^2 = (-1)^{\frac{h(h-1)}{2}}\sum_K x_K\sigma_{H,K}e_K(-1)^{\frac{h(h+1)}{2}}\\ &= (-1)^h\sum_K x_K\sigma_{H,K}e_K\,. \end{aligned} \tag{44}$$

If i is the cardinality of $H\cap K$, then $\sigma_{H,K}=(-1)^{hk+i}$. Therefore

$$\sum_{H\not\ni n} e_H^* x e_H = \sum_K\left(\sum_{H\not\ni n}(-1)^{h(k+1)+i}\right)x_Ke_K\,. \tag{45}$$

If k is even, then $\sum_{H\not\ni n}(-1)^{h(k+1)+i}=\sum_{H\not\ni n}(-1)^{h+i}=\sum_{H\not\ni n}(-1)^{h-i}$ counts the difference between the number of the even and the odd subsets of the set $\{1,\dots,n-1\}\setminus K$. Therefore the sum is zero unless n is odd and $K=(1,\dots,n-1)$ or n is even and $K=(1,\dots,n)$. In both cases the sum is equal to 2^{n-1}.

If k is odd, then $\sum_{H\not\ni n}(-1)^{h(k+1)+i}=\sum_{H\not\ni n}(-1)^i$ counts the difference between the number of even and odd subsets of $K\cap\{1,\dots,n-1\}$. Then it is zero unless $K=(n)$. In this case, the sum is 2^{n-1}.

From these and (45) we get the statement of the lemma. $\square$

Proof of Theorem 5.
The third case ($n=2$) has already been proved in Corollary 1.
Case (1): *n odd*. We show by induction on m that $\mathcal{D}_{n-1}x^m=-e_n\mathcal{D}_{n-1,n}x^m$. Since $\mathcal{D}_n x=0$ (cf. Remark 5), the equality is valid for $m=1$. Take $m>1$ and assume that $\mathcal{D}_{n-1}x^{m-1}=-e_n\mathcal{D}_{n-1,n}x^{m-1}$. We have the following product formula (obtained in a way similar to the $n=2$ case of Proposition 3):

$$\begin{aligned}\mathcal{D}_{n-1,n}x^m&=(\mathcal{D}_{n-1,n}x^{m-1})x+\frac{1}{2}\sum_{H\not\ni n}e_H^*x^{m-1}e_{(Hn)}\\&=(\mathcal{D}_{n-1,n}x^{m-1})x+\frac{1}{2}\left(\sum_{H\not\ni n}e_H^*x^{m-1}e_H\right)e_n\ .\end{aligned}\tag{46}$$

Since, from Lemma 2,

$$\left(\sum_{H\not\ni n}e_H^*xe_H\right)e_n=2^{n-1}(-x_n+x_{(1\cdots n-1)}e_N)\ ,\tag{47}$$

the last term in equation (46) belongs to the center $\langle 1,e_N\rangle$ of $\mathbb{R}_n$. Therefore, from (46) we get

$$-e_n\mathcal{D}_{n-1,n}x^m=-e_n(\mathcal{D}_{n-1,n}x^{m-1})x+\frac{1}{2}\sum_{H\not\ni n}e_H^*x^{m-1}e_H\ .\tag{48}$$

On the other hand, we also have

$$\mathcal{D}_{n-1}x^m=(\mathcal{D}_{n-1}x^{m-1})x+\frac{1}{2}\sum_{H\not\ni n}e_H^*x^{m-1}e_H\tag{49}$$

and then the inductive hypothesis gives the equality $\mathcal{D}_{n-1}x^m=-e_n\mathcal{D}_{n-1,n}x^m$, which is equivalent to $\mathcal{D}_nx^m=0$.
Case (2): *n even*, greater than 2. We show that

$$\mathcal{D}_nx^m=(\mathcal{D}_nx^{m-1})x+2^{n-1}\left[x^{m-1}\right]_N e_N\ ,\tag{50}$$

where $[a]_N$ denotes the coefficient of the *pseudoscalar* e_N of the element $a\in\mathbb{R}_n$. If $m=1$, the equality (50) is true since $\mathcal{D}_nx=0$. Let $m>1$. Then it holds

$$
\begin{aligned}
\mathcal{D}_n x^m &= \mathcal{D}_{n-1} x^m + e_n \mathcal{D}_{n-1,n} x^m \\
&= (\mathcal{D}_{n-1} x^{m-1}) x + \tfrac{1}{2} \sum_{H \not\ni n} e_H^* x^{m-1} e_H + e_n ((\mathcal{D}_{n-1,n} x^{m-1}) x \\
&\quad + \tfrac{1}{2} \sum_{H \not\ni n} e_H^* x^{m-1} e_H e_n) \, . \qquad (51)
\end{aligned}
$$

From Lemma 2, since n is even we have

$$
\sum_{H \not\ni n} e_H^* x^{m-1} e_H + e_n \sum_{H \not\ni n} e_H^* x^{m-1} e_H e_n = 2^n \left[x^{m-1} \right]_N e_N \qquad (52)
$$

and therefore, from (51) and (52)

$$
\mathcal{D}_n x^m = (\mathcal{D}_n x^{m-1}) x + 2^{n-1} \left[x^{m-1} \right]_N e_N \, . \qquad (53)
$$

Now we prove by induction on m that $\mathcal{D}_n x^m$ vanishes on the quadratic cone $\mathcal{Q}_n$. For $m = 1$, $\mathcal{D}_n x = 0$ on the whole algebra. Let $m > 1$ and assume that $\mathcal{D}_n x^{m-1} = 0$ at every point of $\mathcal{Q}_n$. Since the power function maps $\mathcal{Q}_n$ in $\mathcal{Q}_n$, for every $x \in \mathcal{Q}_n$ we have $\left[x^{m-1} \right]_N = 0$. The equality (53) and the inductive hypothesis allow to conclude that $\mathcal{D}_n x^m = 0$ at $x \in \mathcal{Q}_n$. $\square$

From the right linearity of the operators $\mathcal{D}_n$, we get the following result.

Corollary 6. *Let $n \geq 3$. Let $p(x) = \sum_{m=0}^d x^m a_m$ be a polynomial in the complete Clifford variable $x = \sum_{|K| \leq n} x_K e_K$ with right Clifford coefficients. If n is odd, then p is in the kernel of $\mathcal{D}_n$. If n is even, then the restriction of $\mathcal{D}_n(p)$ to the quadratic cone $\mathcal{Q}_n$ vanishes.*

Polynomials $p(x) = \sum_{m=0}^d x^m a_m$ and convergent power series $\sum_k x^k a_k$ with right Clifford coefficients are examples of *slice regular* functions on the intersection of $\mathcal{Q}_n$ with a ball centered in the origin (cf. [9, 10]. If $n \geq 3$, slice regularity generalizes the concept of *slice monogenic functions* introduced in [4]: if f is slice regular on a domain which intersects the real axis, then the restriction of f to the set of paravectors is a slice monogenic function. Conversely, every slice monogenic function is the restriction of a unique slice regular function. Since every slice monogenic function has a power expansions in the paravector variable, centered at points of the real axis (cf. [4]), every slice monogenic f function has an extension $\tilde{f}$ to on open domain in $\mathbb{R}_n$ which satisfies the property stated in Corollary 6: if n is odd, then $\tilde{f}$ is in the kernel of $\mathcal{D}_n$. If n is even, then the restriction of $\mathcal{D}_n(\tilde{f})$ to the quadratic cone $\mathcal{Q}_n$ vanishes.

The same property holds for slice regular functions on a domain Ω in $\mathcal{Q}_n$ with non empty intersection with $\mathbb{R}$. If f is slice regular then it has local power series expansion (in the complete Clifford variable) on an open neighborhood of every real point. This can be seen using the *Clifford operator norm* (see [11, 7.20]) of $\mathbb{R}_n$, which reduces to the Clifford norm on the quadratic cone $\mathcal{Q}_n$.

Remark 6. For $n = 1, 2$ the operators $\mathcal{D}_n$ are elliptic, since in this case

$$
4 \overline{\mathcal{D}}_n \mathcal{D}_n = 4 \mathcal{D}_n \overline{\mathcal{D}}_n = \Delta_{\mathbb{R}^{2^n}} \, . \qquad (54)
$$

For $n=3$ it holds

$$4\overline{\mathcal{D}}_3\mathcal{D}_3 = 4\mathcal{D}_3\overline{\mathcal{D}}_3 = \Delta_{\mathbb{R}^8} + \mathcal{L}_3, \tag{55}$$

where

$$\mathcal{L}_3 = -2\left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_{123}} - \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_{23}} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_{13}} - \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_{13}}\right)e_{123}\ . \tag{56}$$

For $n \geq 4$,

$$4\overline{\mathcal{D}}_n\mathcal{D}_n = \Delta_{\mathbb{R}^{2^n}} + \mathcal{L}_n \quad \text{and} \quad 4\mathcal{D}_n\overline{\mathcal{D}}_n = \Delta_{\mathbb{R}^{2^n}} + \mathcal{L}'_n\ , \tag{57}$$

where

$$\mathcal{L}_n = \sum_{H\neq K} t(e^*_H\tilde{e}_K)\frac{\partial}{\partial x_H}\frac{\partial}{\partial x_K} \quad \text{and} \quad \mathcal{L}'_n = \sum_{H\neq K} t(\tilde{e}_H e^*_K)\frac{\partial}{\partial x_H}\frac{\partial}{\partial x_K} \tag{58}$$

(the summations are made over multindices H,K without repetitions). For $n \geq 4$ the operators $\mathcal{L}_n$ and $\mathcal{L}'_n$ are different. In particular, for $n \geq 3$ the operators $\mathcal{D}_n$ are not elliptic. Note that the symbol of the differential operator $\mathcal{L}_3$ is, up to a multiplicative constant, the polynomial $x_0x_{123} - x_1x_{23} + x_2x_{13} - x_3x_{12}$ whose zero set is the *normal cone* of the Clifford algebra $\mathbb{R}_3$ (cf. [10] for its definition). A similar relation holds for the symbols of $\mathcal{L}_n$ and $\mathcal{L}'_n$ and the equations of the normal cone of $\mathbb{R}_n$ for $n > 3$.

3.3 The case of $\mathcal{D}_3$

In $\mathbb{R}_3$ can be introduced a particular algebraic decomposition in terms of paravector variables. Denote by $I = e_{123}$ the pseudoscalar of $\mathbb{R}_3$. The central idempotents $I_\pm = \frac{1}{2}(1 \pm I)$ satisfy the properties

$$I_+^2 = I_+, \quad I_-^2 = I_-, \quad I_+I_- = I_-I_+ = 0, \quad I_+ + I_- = 1\ . \tag{59}$$

Let $X = x_0 + x_1e_1 + x_2e_2 + x_3e_3$ be the paravector variable and $X' = x - X = x_{12}e_{12} + x_{13}e_{13} + x_{23}e_{23} + x_{123}e_{123}$. We can define two new (rotated) paravector variables $Y = y_0 + y_1e_1 + y_2e_2 + y_3e_3$ and $Z = z_0 + z_1e_1 + z_2e_2 + z_3e_3$ by setting

$$Y = \tfrac{1}{2}(X + X'I), \quad Z = \tfrac{1}{2}(X - X'I)\ , \tag{60}$$

from which we get the decomposition

$$x = X + X' = Y + Z + (Y - Z)I = 2YI_+ + 2ZI_-\ . \tag{61}$$

Since the multiplication by $I_\pm$ gives two orthogonal projections, for every positive integer m it holds

$$x^m = (2Y)^m I_+ + (2Z)^m I_-\ , \tag{62}$$

and therefore for every polynomial, power series or in general for a slice regular function f on a domain which intersects the real axis, we can write

$$f(x) = f(2Y)I_+ + f(2Z)I_- \ . \tag{63}$$

The operator $\mathcal{D}_3$ decomposes as $\mathcal{D}_3 = \frac{1}{2}(\partial_X - \partial_{X'})$, where $\partial_X = \partial_{x_0} + e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3}$ is the Weyl operator of $\mathbb{R}_3$ and $\partial_{X'} = e_{12}\partial_{x_{12}} + e_{13}\partial_{x_{13}} + e_{23}\partial_{x_{23}} + e_{123}\partial_{x_{123}}$. Denote by ∂_Y and ∂_Z the Weyl operators w.r.t. Y and Z respectively. Then

$$\partial_X = \tfrac{1}{2}(\partial_Y + \partial_Z), \quad \partial_{X'} = \tfrac{1}{2}(\partial_Y - \partial_Z)I \ , \tag{64}$$

and therefore in the variables Y,Z the operator $\mathcal{D}_3$ has the following form:

$$\mathcal{D}_3 = I_-\partial_Y + I_+\partial_Z = \partial_Y I_- + \partial_Z I_+ \ . \tag{65}$$

This decomposition implies that a function f belongs to the kernel of $\mathcal{D}_3$ if and only if its projections $f_- := fI_-$ and $f_+ := fI_+$ belong to the kernels of the Weyl operators ∂_Y and ∂_Z respectively. In particular, every pair of arbitrary functions $g(Y)$, $h(Z)$ define a function $f(Y,Z) = I_-h(Z) + I_+g(Y)$ in the kernel of $\mathcal{D}_3$. This property shows again that $\mathcal{D}_3$ is not an elliptic operator, as can be seen also when formula (55) is expressed in the variables Y,Z:

$$4\overline{\mathcal{D}}_3\mathcal{D}_3 = 4\mathcal{D}_3\overline{\mathcal{D}}_3 = \tfrac{1}{2}(\Delta_Y + \Delta_Z) - \tfrac{1}{2}(\Delta_Y - \Delta_Z)I = I_-\Delta_Y + I_+\Delta_Z \ , \tag{66}$$

where Δ_Y is the Laplacian w.r.t. the variables y_0, y_1, y_2, y_3 and similarly for Δ_Z.

3.4 The space $\mathcal{F}(\Omega)$

In view of the non-ellipticity of $\mathcal{D}_3$, we consider a proper subspace of $\ker\mathcal{D}_3$. As we will see in Corollary 7, this space extends the one of monogenic functions. Consider the Laplacians $\Delta_X = \partial_X\overline{\partial}_X$, $\Delta_{X'} = \partial_{X'}\overline{\partial}_{X'}$ and $\Delta = \partial_X\overline{\partial}_X + \partial_{X'}\overline{\partial}_{X'} = \Delta_{\mathbb{R}^8}$.

Definition 9. Let Ω be an open subset of $\mathbb{R}_3$. We define

$$\mathcal{F}(\Omega) := \{f \in C^1(\Omega) \,|\, \mathcal{D}_3 f = 0, \Delta_X \partial_X f = 0 \text{ on } \Omega\} \ .$$

The space $\mathcal{F}(\Omega)$ can be expressed in the paravector variables Y,Z in the way described by the next Proposition.

Proposition 9. *Let $\Omega \subseteq \mathbb{R}_3$ be open. Then*

$$\mathcal{F}(\Omega) = \{f \in C^1(\Omega) \,|\, \mathcal{D}_3 f = 0, \Delta_Y\partial_Y f = \Delta_Z\partial_Z f = 0 \text{ on } \Omega\} \ .$$

Every $f \in \mathcal{F}(\Omega)$ is biharmonic on Ω (i.e. $\Delta^2 f = 0$) and also biharmonic w.r.t. the variables Y and Z separately. In particular, it is real analytic on Ω. Moreover, $f = f_- + f_+ \in \mathcal{F}(\Omega)$ if and only if its projections f_- and f_+ satisfy

$$\partial_Y f_- = \Delta_Z \partial_Z f_- = 0\ , \quad \partial_Z f_+ = \Delta_Y \partial_Y f_+ = 0\ . \tag{67}$$

Proof. If $\mathcal{D}_3 f = 0$, then $\partial_X f = \partial_{X'} f$. Therefore $\Delta f = (\partial_X \overline{\partial}_X + \partial_{X'} \overline{\partial}_{X'}) f = 2\Delta_X f$. Moreover, from (64) it follows that $\partial_Z f = (\partial_X - I\partial_{X'}) f = (\partial_X - I\partial_X) f = 2I_- \partial_X f$. Then

$$\Delta_Z f = \partial_Z \overline{\partial}_Z f = 4I_- \partial_X \overline{\partial}_X f = 4\Delta_X f_- = 2\Delta f_- \tag{68}$$

and therefore $\Delta_Z \partial_Z f = 8\Delta_X \partial_X f_-$. A similar computation gives $\Delta_Y \partial_Y f = 8\Delta_X \partial_X f_+$. Then $\Delta_X \partial_X f = 0$ if and only if $\Delta_Z \partial_Z f = \Delta_Y \partial_Y f = 0$.

If $f \in \mathcal{F}(\Omega)$, then $0 = \overline{\partial}_Z \partial_Z \Delta_Z f = \Delta_Z^2 f$ and $0 = \overline{\partial}_Y \partial_Y \Delta_Y f = \Delta_Y^2 f$. From these equalities we get $4\Delta^2 f_- = \Delta_Z^2 f = 0$, $4\Delta^2 f_+ = \Delta_Y^2 f = 0$ and then $\Delta^2 f = 0$: f is biharmonic on Ω.

The last statement is immediate from the decomposition (65) of $\mathcal{D}_3$. □

Remark 7. The preceding Proposition tells that every function in the space $\mathcal{F}(\Omega)$ is (separately) *holomorphic Cliffordian* [17] in the paravector variables X, Y and Z.

Corollary 7. *Every polynomial $p(x) = \sum_{m=0}^{d} x^m a_m$ in the complete Clifford variable $x = \sum_{|K|\le 3} x_K e_K$ belongs to $\mathcal{F}(\mathbb{R}_3)$. The same holds for every slice regular function on a domain in the quadratic cone $\mathcal{Q}_3$ intersecting the real axis. If $f(X)$ is a function depending only on the paravector variable X of $\mathbb{R}_3$, then $f \in \mathcal{F}$ if and only if it is monogenic, i.e. $\partial_X f = 0$.*

Proof. From the algebraic decomposition (62), every power of x can be expressed by means of powers of Y and Z. Since every power of a paravector variable X is holomorphic Cliffordian (cf. [17]), i.e. $\Delta_X \partial_X f = 0$, the first two statements follow from Theorem 5 and Corollary 6. The last statement is an immediate consequence of Corollary 5. □

Let B denote the eight-dimensional unit ball in $\mathbb{R}_3$. Let $T \simeq S^3 \times S^3$ be the subset of the unit sphere ∂B defined by $T := \{|Y| = |Z| = 1/2\}$ and $P := \{|Y| < 1/2\} \cap \{|Z| < 1/2\}$. Since $|x|^2 = 2|Y|^2 + 2|Z|^2$, $P \subset B$ and $T \subset \partial P$. We will call T the *distinguished boundary* of P. Note that T is contained in the normal cone $\mathcal{N}_3$ of $\mathbb{R}_3$ (cf. [10]), which has equation $|Y| = |Z|$ in the variables Y, Z.

Proposition 10 (Integral Representation Formula). *There is an integral representation formula for functions $f \in \mathcal{F}(P) \cap C^2(\overline{P})$ with the distinguished boundary T as domain of integration. The values of f on P are determined by the values on T of f, $\partial_X f$ and the second derivatives $\frac{\partial}{\partial x_K}(\partial_X f)$ for multiindices K with $|K| \le 3$.*

Proof. Consider the component $f_- \in \mathcal{F}(P)$. Since $\partial_Y f_- = 0$, we can apply the representation formula for the Weyl operator ∂_Y (cf. [1]) and reconstruct f_- on the set $\{|Y| < 1/2, |Z| = 1/2\}$. On functions in the class $\mathcal{F}(P)$, the operators ∂_Y and ∂_Z commute (see the proof of Proposition 9). Since $\partial_Y \partial_Z f_- = \partial_Z \partial_Y f_- = 0$, we can reconstruct also $\partial_Z f_-$ on the set $\{|Y| < 1/2, |Z| = 1/2\}$. Since $\Delta_Z \partial_Z f_- = 0$, we can now apply the integral representation formula for holomorphic Cliffordian functions (see [17]) w.r.t. the paravector variable Z and obtain the values of f_- on P. A similar reasoning for f_+ gives the result. □

Acknowledgements This work was partially supported by MIUR (PRIN Project "Proprietà geometriche delle varietà reali e complesse") and GNSAGA of INdAM.

References

1. Brackx, F., Delanghe, R., Sommen, F.: Clifford analysis, *Research Notes in Mathematics*, vol. 76. Pitman (Advanced Publishing Program), Boston, MA (1982)
2. Chen, J., Li, J.: Quaternionic maps between hyperkähler manifolds. J. Differential Geom. **55**(2), 355–384 (2000).
3. Colombo, F., Sabadini, I., Sommen, F., Struppa, D.C.: Analysis of Dirac systems and computational algebra, *Progress in Mathematical Physics*, vol. 39. Birkhäuser Boston Inc., Boston, MA (2004)
4. Colombo, F., Sabadini, I., Struppa, D.C.: Slice monogenic functions. Israel J. Math. **171**, 385–403 (2009)
5. Folland, G.B.: Introduction to partial differential equations, second edn. Princeton University Press, Princeton, NJ (1995)
6. Fueter, R.: Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen. Comment. Math. Helv. **7**(1), 307–330 (1934)
7. Gentili, G., Struppa, D.C.: A new approach to Cullen-regular functions of a quaternionic variable. C. R. Math. Acad. Sci. Paris **342**(10), 741–744 (2006)
8. Gentili, G., Struppa, D.C.: A new theory of regular functions of a quaternionic variable. Adv. Math. **216**(1), 279–301 (2007)
9. Ghiloni, R., Perotti, A.: A new approach to slice regularity on real algebras. In: Hypercomplex analysis and its Applications, Trends Math. Birkhäuser, Basel (2010)
10. Ghiloni, R., Perotti, A.: Slice regular functions on real alternative algebras. Adv. Math. **226**(2), 1662–1691 (2011)
11. Gilbert, J.E., Murray, M.A.M.: Clifford algebras and Dirac operators in harmonic analysis, *Cambridge Studies in Advanced Mathematics*, vol. 26. Cambridge University Press, Cambridge (1991). DOI 10.1017/CBO9780511611582
12. Gürlebeck, K., Habetha, K., Sprößig, W.: Holomorphic functions in the plane and n-dimensional space. Birkhäuser Verlag, Basel (2008)
13. Hengartner, W., Leutwiler, H.: Hyperholomorphic functions in $\mathbb{R}^3$. Adv. Appl. Clifford Algebras **11**(S1), 247–259 (2001)
14. Hestenes, D., Sobczyk, G., Clifford algebra to geometric calculus. A unified language for mathematics and physics. Fundamental Theories of Physics. Reidel, Dordrecht (1984)
15. Joyce, D.: Hypercomplex algebraic geometry. Quart. J. Math. Oxford Ser. (2) **49**(194), 129–162 (1998). DOI 10.1093/qjmath/49.194.129
16. Kravchenko, V.V., Shapiro, M.V.: Integral representations for spatial models of mathematical physics, *Pitman Research Notes in Mathematics Series*, vol. 351. Longman, Harlow (1996)
17. Laville, G., Ramadanoff, I.: Holomorphic Cliffordian functions. Adv. Appl. Clifford Algebras **8**(2), 323–340 (1998). DOI 10.1007/BF03043103.
18. Lichnerowicz, A.: Applications harmoniques et variétés kähleriennes. In: Symposia Mathematica, Vol. III (INDAM, Rome, 1968/69), pp. 341–402. Academic Press, London (1968/1969)
19. Naser, M.: Hyperholomorphe funktionen. Sib. Math. J. **12**, 959–968 (1971)
20. Nōno, K.: Characterization of domains of holomorphy by the existence of hyper-conjugate harmonic functions. Rev. Roumaine Math. Pures Appl. **31**(2), 159–161 (1986)
21. Perotti, A.: Quaternionic regularity and the $\overline{\partial}$-Neumann problem in $\mathbb{C}^2$. Complex Var. Elliptic Equ. **52**(5), 439–453 (2007). DOI 10.1080/17476930601178392
22. Perotti, A.: Every biregular function is a biholomorphic map. Adv. Appl. Clifford Algebr. **19**(2), 441–451 (2009). DOI 10.1007/s00006-009-0161-8

23. Perotti, A.: Holomorphic functions and regular quaternionic functions on the hyperkähler space $\mathbb{H}$. In: MORE PROGRESSES IN ANALYSIS: Proceedings of the 5th International Isaac Congress, Trends Math., pp. 1009–1018. World Scientific, Singapore (2009)
24. Perotti, A.: Regular quaternionic functions and conformal mappings. Cubo **11**(1), 123–143 (2009)
25. Perotti, A.: On directional Hilbert operators for regular quaternionic functions on $\mathbb{R}^3$. Adv. Appl. Clifford Algebr. **20**(3/4), 803–817 (2010)
26. Perotti, A.: The full Dirac operator on a Clifford algebra. In: Electronic Proceedings of the 9th International Conference on Clifford Algebras and their Applications (ICCA9) 2011. Gürlebeck, Klaus (ed.) Weimar, Germany
27. Qian, T.: Generalization of Fueter's result to $\mathbf{R}^{n+1}$. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **8**(2), 111–117 (1997)
28. Rizza, G.B.: Funzioni regolari nelle algebre di Clifford. Rend. Mat. e Appl. (5) **15**, 53–79 (1956)
29. Sabadini, I., Sommen, F.: Clifford analysis on the space of vectors, bivectors and l-vectors. In: Advances in analysis and geometry, Trends Math., pp. 161–185. Birkhäuser, Basel (2004)
30. Sce, M.: Osservazioni sulle serie di potenze nei moduli quadratici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23**, 220–225 (1957)
31. Shapiro, M.V., Vasilevski, N.L.: Quaternionic ψ-hyperholomorphic functions, singular integral operators and boundary value problems. I. ψ-hyperholomorphic function theory. Complex Variables Theory Appl. **27**(1), 17–46 (1995)
32. Sommen, F.: On a generalization of Fueter's theorem. Z. Anal. Anwendungen **19**(4), 899–902 (2000)
33. Sudbery, A.: Quaternionic analysis. Math. Proc. Cambridge Philos. Soc. **85**(2), 199–224 (1979)