

## THE PLATONIC SOLIDS AND THEIR SYMMETRY GROUPS

A regular polyhedron in the ordinary three-dimensional space (or platonic solid) is a convex polyhedron (that is, the intersection of a finite number of half-spaces), such that all faces are regular polygons having the same number of sides, and such that the same number of faces meet in each vertex.

A simple combinatorial argument (running like, a vertex can only be the meeting point of three, four or five equilateral triangles, or three squares, or three regular regular pentagons) shows that there are exactly five regular polyhedra, up to similarity: the tetrahedron (four triangular faces), the cube (six square faces), the octahedron (eight triangular faces), the dodecahedron (twelve pentagonal faces) and the icosahedron (twenty triangular faces).

	$F$	$E$	$V$	Schläfli symbol	proper symmetries	all symmetries
Tetrahedron	4	6	4	{3,3}	12	24
Cube	6	12	8	{4,3}	24	48
Octahedron	8	12	6	{3,4}	24	48
Dodecahedron	12	30	20	{5,3}	60	120
Icosahedron	20	30	12	{3,5}	60	120

The table lists the number of faces, edges and vertices of each polyhedron. For each polyhedron, each of these three numbers can be computed from the other two provided we know the number of sides of each face.

Moreover, these numbers are completely determined by the *Schläfli symbol*  $\{p, q\}$  of the polyhedron, where  $p$  is the number of sides of each face, and  $q$  is the number of faces which meet in each vertex, via the formulas

$$pF = 2E = qV \quad \text{and} \quad V - E + F = 2.$$

The former is obvious, and the latter is Euler's formula, due to the topological fact that each platonic solid (or rather, the union of its faces) is homeomorphic to the surface of a sphere.

Note that the cube and the octahedron are dual of each other, and so are the dodecahedron and the icosahedron (while the tetrahedron is self-dual). The centres of the faces of a polyhedron are the vertices of another polyhedron, obtained by joining by an edge the centres of each pair of adjacent faces. Note that each edge in the new polyhedron is orthogonal to the corresponding one in the original polyhedron. Repeating the process once again gives a polyhedron similar to the original one (for regular polyhedra), and the two polyhedra involved (which are possibly similar) are called dual of each other.

Note also that Euler's formula  $V - E + F = 2$  holds. However, this is a topological formula due to the fact that all our polyhedra are homeomorphic to a sphere, and has nothing to do with the fact that they are regular.

The last two columns of the table give the number of proper symmetries and the total number of symmetries (proper and improper).

The symmetries of each polyhedron can be considered as orthogonal transformations of the whole space fixing the center of the polyhedron, and so they can be classified as proper

or improper according to whether the corresponding orthogonal transformation of the three-dimensional Euclidean (vector) space has determinant 1 or  $-1$ . Together they form a group, the *general symmetry group* of the polyhedron, and the former constitute a subgroup of order two, hence normal, which we call the *proper symmetry group*. Note that each proper symmetry is a rotation of the space (as in the two-dimensional case), but not every improper symmetry is a reflection (in fact, it can be expressed in general as the composition of a reflection with respect to a plane and a rotation around an axis orthogonal to the plane).

The order of the subgroup of proper symmetries can be easily seen to be twice the number of edges (any proper symmetry is determined by knowing to which edge it sends a given edge, and with which orientation). But it can also be computed in other ways, using faces or vertices instead of edges. Note that each of these calculations is an application of the orbit-stabilizer theorem, by letting the (proper or general) symmetry group acting on the set of vertices, or faces, or edges.

Since dual polyhedra share the same symmetry groups, we can limit ourselves to studying the tetrahedron, the cube and the dodecahedron (for example).

**Tetrahedron.** The general symmetry group  $G$  permutes the four vertices, and this gives a homomorphism of  $G$  into  $S_4$ . Since the action is clearly faithful and the two groups have the same order, the homomorphism is actually an isomorphism. We identify  $G$  and  $S_4$  from now on. The proper symmetry group  $G^+$ , being a subgroup of index two (or by direct verification), can only be  $A_4$ .

The non-identity elements of  $G^+$ , being all rotations, can be divided into four *face* rotations of order three (or *vertex* rotations, since a rotation of a face is also a rotation, of the opposite angle, around the opposite vertex), and six *edge* rotations of order two. Each of these two sets forms a conjugacy class of  $S_4$  (and this fact is now visible geometrically!) contained in  $A_4$ ; as we know, two more conjugacy classes of  $S_4$  lie outside  $A_4$ , but only one of them is made of reflections.

However, not all face rotations are conjugated in  $G^+ = A_4$ . In fact, the conjugates in  $G^+$  of a face rotation, say of  $120^\circ$  clockwise, are rotation of other faces, but still of  $120^\circ$  clockwise. These four form a conjugacy class of  $A_4$ , and the four face rotations of  $120^\circ$  anti-clockwise form another conjugacy class.

Since  $G$  permutes the six edges of the tetrahedron we get a homomorphism of  $G$  into  $S_6$  (actually, into  $A_6$ ). Thus, its image is a subgroup of  $S_6$  isomorphic with  $S_4$ .

More interestingly,  $G$  permutes transitively the three pairs of opposite edges of the tetrahedron, and this gives a surjective homomorphism of  $G = S_4$  onto  $S_3$ , proving once again that  $S_4$  has a normal subgroup of order four (the kernel of the action,  $\{e, (12)(34), (13)(24), (14)(23)\}$ ) with quotient isomorphic with  $S_3$ .

**Cube.** In this case (as well as in the case of the dodecahedron which we consider later), vertices of the cube come in opposite pairs, meaning symmetric pairs with respect to the centre  $O$  of the cube. Consequently, the central symmetry with respect to  $O$  is a symmetry of the cube (an improper one), and it commutes with any other symmetry (in matrix formulation it would be given by the scalar matrix  $\text{diag}(-1, -1, -1)$ , which commutes with any other  $3 \times 3$ -matrix). It follows that  $G_c$ , the general symmetry group of the cube (rather than the group  $G$  of the tetrahedron considered earlier, which we will denote by  $G_t$  from now on), is the direct product of its subgroup  $G_c^+$  of proper symmetries and the subgroup of order two generated by that particular symmetry (which will turn out to coincide with the centre of  $G_c$ ).

We may then restrict our attention to the proper symmetry group  $G_c^+$ . Note that a tetrahedron can be inscribed in a cube, and in exactly two ways. Since  $G_c^+$  permutes them, the

stabilizer of either of them is a subgroup of index two in  $G_c^+$ , and induces proper symmetries of that inscribed tetrahedron, actually all 12 of them. Therefore, the proper symmetry group  $G_c^+$  of the cube contains a subgroup of index two, hence normal, isomorphic with the proper symmetry group  $G_t^+$  of the tetrahedron, which we already know to be isomorphic with  $A_4$ . It would be possible to conclude from this that  $G_c^+$  is isomorphic with  $S_4$ , but we take a different route.

The proper symmetry group  $G_c^+$  permutes the four main diagonal of the cube (or, if one prefers, the four pairs of opposite vertices, which is the same), and this gives a homomorphism of  $G_c^+$  into  $S_4$ . It is easy to see that the action is faithful, and so the homomorphism is an isomorphism, because both groups have the same order 24.

The nonidentity classes of  $G_c^+ = S_4$  can be found again geometrically: the eight vertex rotations of order three (of  $120^\circ$  in either direction, all conjugate here); the six edge rotations of order two; the six face rotations of order four (of  $90^\circ$  in either direction); the three face rotations of order two (of  $180^\circ$ ).

Note, in particular, that the the vertex rotations of the cube, that is, elements of order three in  $G_c^+$  (which is  $\cong S_4$ ) are all conjugate, because each clockwise vertex rotation of  $120^\circ$  is an anti-clockwise rotation of the same angle with respect to the opposite vertex. That was not the case for the tetrahedron, where not all vertex rotations were conjugate in  $G_t^+$  ( $\cong A_4$ ).

One may consider now various transitive actions of  $G_c^+$ . This is a good way of constructing subgroups, as the corresponding stabilisers. For example,  $G_c^+$  acts transitively on the six faces of the cube, and the stabiliser of a face in this action, which must have order four according to the orbit-stabiliser theorem, is clearly the subgroup of rotations of that face, hence cyclic of order four. But one can also note that opposite faces move in pairs, and so consider the action of  $G_c^+$  on the (unordered) three pairs of opposite faces. In this way we find once again the surjective homomorphism  $G_c^+ \cong S_4 \rightarrow S_3$  already seen with the tetrahedron. The stabiliser of one pair of opposite faces, which must have order eight, consists of all permutation that either rotate each of the two faces or interchange them. It is then easy to see that it is isomorphic with the dihedral group of order eight.

**Dodecahedron.** As was the case for the cube, the general symmetry group  $G_d$  of the dodecahedron is the direct product of the proper symmetry group  $G_d^+$  and a cyclic group of order two (the centre of  $G_d$ ). Hence we restrict our attention to the proper symmetry group  $G_d^+$ .

Note that exactly five cubes can be inscribed in a dodecahedron: we may take one diagonal of each pentagonal face of the dodecahedron to be one of the 12 edges of one particular inscribed cube; note that once we choose as one edge one of the five diagonals of a particular face of the dodecahedron we have no other choice left and we obtain exactly one inscribed cube. Now the proper symmetry group  $G_d^+$  of the dodecahedron permutes those five inscribed cubes transitively, and this gives a homomorphism of  $G_d^+$  into  $S_5$ . There are now various way of checking that the action is faithful.

For example, one sees that the stabiliser in  $G_d^+$  of any one of these cubes induces exactly half of the proper symmetries of that cube. (This is best seen by looking at an actual model of the dodecahedron with an inscribed cube: by rotating the dodecahedron we may bring any edge of the chosen inscribed cube into any other edge, but in only one of the two possible orientations.) Hence the stabiliser must be isomorphic with the only subgroup of index two of  $G_c^+ \cong S_4$ , namely, with  $A_4$ . We look at the action of this stabiliser on the remaining four inscribed cubes. Any element of order two in the stabiliser is a rotation of  $180^\circ$  around the center of a face of the cube, and one sees that such a rotation interchanges the remaining four cubes in pairs. Any element of order three in the stabiliser is a rotation of  $120^\circ$  around a vertex of the cube, and one sees that such a rotation fixes one of the remaining four inscribed cubes, and permutes the other

three. Therefore, the only element of the stabiliser of the inscribed cube under consideration, which fixes the remaining four inscribed cubes, is the identity. Hence the action is faithful.

Thus, the associated homomorphism  $G_d^+ \rightarrow S_5$  maps  $G_d^+$  isomorphically onto a subgroup of  $S_5$  of order 60 (hence normal), which can only be the alternating group  $A_5$ . So we have  $G_d^+ \cong A_5$ .

Note that  $A_5$  has two conjugacy classes of elements of order 5 (that is, 5-cycles). We can see why geometrically: those elements correspond to rotations of  $72^\circ$  or  $144^\circ$  in either direction, and rotations of a different angle cannot be conjugated in  $G_d^+$ .

The actions of  $G_d^+$  on various elements of the dodecahedron can be used to produce various subgroups of  $A_5 \cong G_d^+$  as the corresponding stabilisers, and various embeddings of  $G_d^+$  into symmetric groups. For example, the (transitive) action of  $G_d^+$  on the set of 12 faces has cyclic groups of order five as stabilisers but, more interestingly, the action on pairs of opposite faces (transitive as well) has dihedral groups of order ten as stabilisers. To this faithful action corresponds an embedding (that is, injective homomorphism)  $G_d^+ \rightarrow S_6$ . One can actually check that the image consists of even permutations of the six pairs of faces, and so the embedding is actually  $G_d^+ \rightarrow A_6$ . (This can also be seen by group-theoretic reasons: otherwise the image of the homomorphism would intersect  $A_6$  in a subgroup of index 2, which we know not to exist.) This embedding is interesting because its image, being transitive on the six objects, cannot be conjugate to the most obvious subgroup of  $A_6$  isomorphic with  $A_5$ , say the stabiliser of the sixth object. This shows that  $A_6$  contains at least two (in fact, exactly two) conjugacy classes of subgroups isomorphic with  $A_5$ , the alternating group on one object less. This is an exceptional situation among the alternating groups. (In fact,  $A_6$  even has an exceptional automorphism of order two which interchanges those two conjugacy classes of subgroups.)

The 20 vertices of the dodecahedron can also be grouped together in several ways to obtain interesting actions. For example, they can be grouped into five sets of four vertices, each of which is the set of vertices of an inscribed tetrahedron. (In fact, the ten tetrahedrons inscribed in the five inscribed cubes form two distinct orbits of length five under the symmetry group; take one of those.) We may have used these from the start instead of the five inscribed cubes to produce the isomorphism  $G_d^+ \cong A_5$ .

The 30 edges can be also grouped into five sets of six edges each, each set containing all edges parallel or orthogonal to a given one (and, in turn, parallel or orthogonal to the faces of a given inscribed cube). This, of course, is exactly the same action as our initial one on the set of five inscribed cubes.

One can also, for example, suitably divide the 30 edges into 10 groups, of three each, transitively permuted by  $G_d^+$ . The stabiliser must therefore have order six. In fact, it is a subgroup of  $A_5$  isomorphic with  $S_3$  (say  $\{e, (123), (132), (12)(45), (13)(45), (23)(45)\}$ ).