Some Nottingham algebras

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Thin algebras

An infinite-dimensional graded Lie algebra

$$L = \bigoplus_{i=1}^{\infty} L_i$$

over a field $\mathbb{F}$ of characteristic $p > 0$ is said to be thin if $\dim(L_1) = 2$ and the following *covering property* holds:

$$[uL_1] = L_{i+1} \quad \text{for all } 0 \neq u \in L_i, \text{ for all } i \geq 1$$

An equivalent formulation of the covering property is the following: for every (graded) ideal I in L, $L^{i+1} \subseteq I \subseteq L^i$, for some i.

[A. Caranti, S. Mattarei, M. Newman, C. Scoppola, *Thin Lie algebras and thin p-groups*, 1996]

Thin groups

A pro p-group G is said to be thin if $\gamma_i(G)/\gamma_{i+1}(G)$ is elementary abelian of order p or p^2 and for every nontrivial normal closed subgroup N of G

$$\gamma_{i+1}(G) \leq N \leq \gamma_i(G)$$

for some i.

[R. Brandl, *The Dilworth number of subgroup lattices*, 1988]

[A. Caranti, S. Mattarei, M. Newman, C. Scoppola, *Thin Lie algebras and thin p-groups*, 1996]

Pro p-groups of *width* two and *obliquity* zero.

[G. Klass, C. R. Leedham - Green, W. Plesken, *Linear pro p-groups of finite width*, 1997]

The *Nottingham group* over the field $\mathbb{F}_p$ (p odd):

$$J = \{t + \sum_{i=2}^{\infty} a_i t^i : a_i \in \mathbb{F}_p\}$$

with formal substitution as binary operation.

Let L be a thin Lie algebra.

- L is generated as Lie algebra by L_1.
- Every homogeneous component of L has dimension 1 or 2.
- L is centerless.

A homogeneous component of dimension two is termed a *diamond* of L. In particular L_1 is a diamond, the first diamond of L.

Thin Lie algebras which have no diamonds except for L_1 are *(graded) Lie algebras of maximal class (generated in degree one)*.
(Note that $L^k = \bigoplus_{i \geq k} L_i$, hence L/L^k is a nilpotent algebra of dimension k and nilpotency class $k-1$).
Completely classified.

[A. Caranti, M. Newman, *Graded Lie algebras of maximal class II*, 2000] p odd
[G. Jurman, *Graded Lie algebras of maximal class III*, 2005] $p = 2$

Thin algebras

The second diamond of a thin algebra can occur in degree k where

- $k = 3, 5, p^n$ or $2p^n - 1$, if $\mathrm{char}(\mathbb{F}) = p$, p odd, for some $n > 0$,
- $k = 2p^n - 1$, if $\mathrm{char}(\mathbb{F}) = p = 2$, for some $n > 0$,
- $k = 3, 5$ if $\mathrm{char}(\mathbb{F}) = 0$.

Thin algebras associated with groups: $k \leq p$ (and p is odd)

[A. Caranti, S. Mattarei, M. Newman, C. Scoppola, *Thin Lie algebras and thin p-groups*, 1996]
[M. A., G. Jurman, *Diamonds in thin Lie algebras*, 2001]
[M. A., G. Jurman, S. Mattarei, *The structure of thin Lie algebras with characteristic two*, 2010]

Nottingham algebras

Thin Lie algebras with second diamond in degree p^n (p odd) have been referred to as *Nottingham (Lie) algebras*.

The Lie algebra associated to the Nottingham group with respect to its lower central series has second diamond in degree p.

To each diamond of a Nottingham algebra (except the first) can be attached a parameter, the *type* of the diamond, which takes values in $\mathbb{F} \cup \{\infty\}$.

The types and the locations of the diamonds determine the algebra up to isomorphism.

[A. Caranti, S. Mattarei *Nottingham Lie algebras with diamonds of finite type*, 2004]

Nottingham algebras

Let $L = \bigoplus L_i$ be a Nottingham Lie algebra with second diamond in degree $q = p^n$ and assume that $\dim(L_3) = 1$.
Choose $0 \neq y \in L_1$ such that $[L_2 y] = 0$. The element y centralizes every homogeneous component

$$L_2, \dots L_{q-2}.$$

For any $x \in L_1 \setminus \mathbb{F}y$, the element

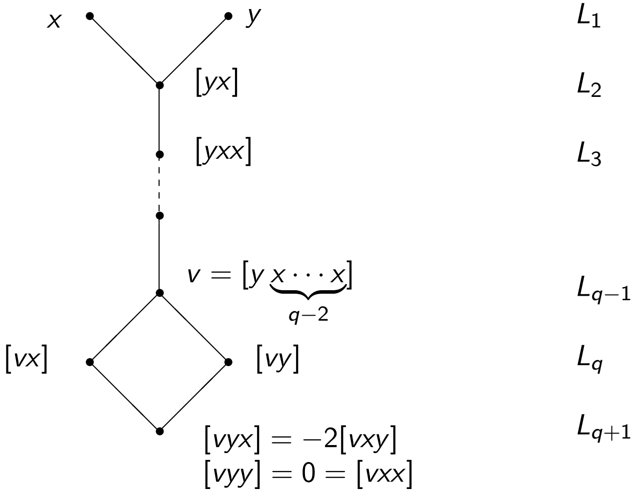
$$v = [y \underbrace{x \dots x}_{q-2}]$$

spans the homogeneous component L_{q-1} and therefore $[vx]$ and $[vy]$ span the diamond L_q. In degree L_{q+1} one has the elements

$$[vxx], [vxy], [vyx], [vyy]$$

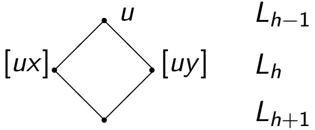
such that $[vyy] = 0$, $[vyx] = -2[vxy]$. Assume (eventually replacing x by $x + \alpha y$) that $[vxx] = 0$.

Nottingham algebras



Type of a diamond

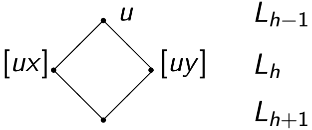
Let $\dim(L_h) = 2$, $h > 1$ thus $\dim(L_{h-1}) = 1$. Let $0 \neq u \in L_{h-1}$.



$[uxx] = 0 = [uyy]$

$\mu[uyx] = (1-\mu)[uxy]$, $\mu \in \mathbb{F}$

diamond of finite type μ



$[uxx] = 0 = [uyy]$

$[uyx] + [uxy] = 0$,

diamond of infinite type

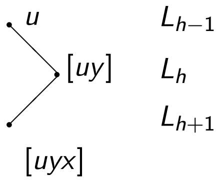
The second diamond has type -1:

$$[vyx] = -2[vxy] \quad \text{and} \quad [vxx] = 0 = [vyy]$$

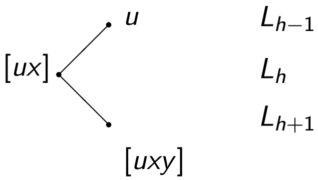
Nottingham algebras

Fake diamonds

For $\mu = 0$ we have $[uxy] = 0$ and $[uxx] = 0$, thus $[ux] = 0$
For $\mu = 1$ we have $[uyx] = 0$ and $[uyy] = 0$, thus $[uy] = 0$



$\mu = 0$



$\mu = 1$

Nottingham algebras can be interpreted also for $p = 2$ by regarding the second diamond of type $-1 \equiv 1 \bmod 2$ as a fake diamond.

Nottingham algebras

Diamonds of infinite type

Two families of Nottingham algebras with all diamonds, except for the second one, of infinite type. Each family is in one-to-one correspondance with a certain subclass of graded Lie algebras of maximal class.

Completely classify Nottingham algebras with third diamond (if any) in degree greater than $2q-1$.

[D. Young, *Thin Lie algebras with long second chain*, 2001]

Diamonds of finite type

Let L be a Nottingham algebra and suppose that L has a third diamond in degree $2q-1$ of finite type μ. Then L is uniquely determined. It has diamonds of finite type in all degrees congruent to 1 mod $q-1$. The types of the diamonds follow an arithmetic progression (counting the fake ones).

[A. Caranti, S. Mattarei *Nottingham Lie algebras with diamonds of finite type*, 2004]

Explicit constructions:

- $\mu = -1$ ({types of the diamonds} $= \{-1\}$)
  [A. Caranti, *Presenting the graded Lie algebra associated to the Nottingham group*, 1997]
- $\mu \in \mathbb{F}_p$ ({types of the diamonds} $= \mathbb{F}_p$)
  [M. A., *Some loop algebras of Hamiltonian Lie algebras*, 2002]
- $\mu \in \mathbb{F} \setminus \mathbb{F}_p$ ({types of the diamonds} $\not\subseteq \mathbb{F}_p$)
  [M. A., S. Mattarei, *Thin loop algebras of Albert-Zassenhaus algebras*, 2007]

Loop algebras of certain simple, finite-dimensional Lie algebras of Cartan type.

Definition

Let S be a finite-dimensional Lie algebra over the field $\mathbb{F}$ with a cyclic grading $S = \bigoplus_{k \in \mathbb{Z}/m\mathbb{Z}} S_k$, let U be a subspace of $S_{\bar{1}}$ and t be an indeterminate over $\mathbb{F}$. The loop algebra of S (w.r.t. U and the given cyclic grading) is the Lie subalgebra of $S \otimes \mathbb{F}[t]$ generated by $U \otimes t$.

Diamonds of both finite and infinite type

Nottingham algebras with diamonds in all degrees congruent to 1 mod $q-1$. The diamonds occur in sequences of p^s-1 diamonds of infinite type separated by single occurrences of diamonds of finite type, for some $s \geq 1$. The types of the finite diamonds follow an arithmetic progression. Let μ the type of the third diamond of finite type in order of occurence (that is $L_{(p^s+1)(q-1)+1}$)

- $\mu = -1$ ($\{$types of the finite diamonds$\} = \{-1\}$)
  [M. A., S. Mattarei, *Thin loop algebras of Albert-Zassenhaus algebras*, 2007]
- $\mu \in \mathbb{F}_p$, ($\{$types of the finite diamonds$\} = \mathbb{F}_p$)
  constructions follow easily from some results in
  [S. Mattarei, *Artin-Hasse exponentials of derivations*, J. Algebra (2005)]
- $\mu \in \mathbb{F} \setminus \mathbb{F}_p$ ($\{$types of the finite diamonds$\} \not\subseteq \mathbb{F}_p$)
  [M. A., S. Mattarei, *Nottingham algebras with diamonds of finite and infinite type*, in preparation] p odd
  [C. Scarbolo, *Some Nottingham algebras in characteristic two*, 2010] $p = 2$

Gradings

Definition

Let A be a (finite-dimensional) non-associative algebra over a field $\mathbb{F}$. A grading of A over an abelian group G is a direct sum decomposition:

$$A = \bigoplus_{g \in G} A_g$$

such that $A_g A_h \subseteq A_{g+h}$.

Let D be a derivation of A, with all its characteristic roots in $\mathbb{F}$. The direct sum decomposition of A into generalized eigenspaces for D

$$A = \bigoplus_{\alpha \in \mathbb{F}} A_\alpha$$

(where $A_\alpha = \{x \in A : (D - \alpha \, \mathrm{Id})^i(x) = 0, \text{for some } i > 0\}$) is a grading of A over $(\mathbb{F}, +)$.
Let σ be an automorphism of A, with all its characteristic roots in $\mathbb{F}$. The direct sum decomposition of A into generalized eigenspaces for σ

$$A = \bigoplus_{\alpha \in \mathbb{F}} A_\alpha$$

is a grading of A over $(\mathbb{F}^*, \cdot)$.

The exponential of a derivation

Let $\mathrm{char}(\mathbb{F}) = 0$ and D be a nilpotent derivation of A, with $D^n = 0$. Then the exponential map $\exp(D) = \sum_{i=0}^{n-1} \frac{D^i}{i!}$ defines an automorphism of A.

Let $\mathrm{char}(\mathbb{F}) = p > 0$.

[S. Mattarei, *Artin-Hasse exponentials of derivations*, J. Algebra (2005)]

The exponential series does not make sense, in general, because the denominators vanish, except for the first p terms of the series.

- Let D be a nilpotent derivation of A with $D^p = 0$, then

$$\exp(D) = \sum_{i=0}^{p-1} \frac{D^i(x)}{i!}$$

  and it defines a bijective linear map on A.

- Define the truncated exponential of D as

$$E(D) = \sum_{i=0}^{p-1} \frac{D^i(x)}{i!}.$$

Direct computation shows that

$$E(D)x \cdot E(D)y - E(D)(xy) = \sum_{t=p}^{2p-2} \sum_{i=t+1-p}^{p-1} \frac{(D^i x)(D^{t-i} y)}{i!(t-i)!}.$$

If p is odd and $D^{\frac{p+1}{2}} = 0$ then each term in the sum vanishes and $E(D)$ is an automorphism of A, but in general $E(D)$ it is not an automorphism of A even if $D^p = 0$.

The exponential of a derivation

However, under certain hypotheses, the (truncated) exponential of a derivation has the property of sending a grading of A into another grading of A.

Lemma (S. Mattarei)

Let A be a non-associative algebra over a field of positive characteristic p, with derivation D such that $D^p = 0$. Then

$$\exp(D)x \cdot \exp(D)y - \exp(D)(xy) = \exp(D)\left(\sum_{i=1}^{p-1} \frac{(-1)^i}{i} D^i x \cdot D^{p-i} y\right)$$

for all $x, y \in A$.

The exponential of a derivation

Let $A = \oplus A_i$ be a grading of A over the integers modulo m. A derivation D of A is graded of degree d if $D(A_i) \subseteq A_{i+d}$ for every i.

Theorem (S. Mattarei)

Let $A = \oplus A_i$ be a non-associative algebra over a field of positive characteristic p, graded over the integers modulo m. Suppose that A has a graded derivation D of degree d, with $m \mid pd$, such that $D^p = 0$. Then the direct sum decomposition $A = \oplus \exp(D)A_i$ is a grading of A over the integers modulo m.

Let $x \in A_s$ and $y \in A_t$. Then $D^i(x) \in A_{s+di}$ and $D^{p-i}(y) \in A_{t+d(p-i)}$ thus $D^i(x) \cdot D^{p-i}(y) \in A_{s+t+pd} = A_{s+t}$.
The previous lemma yields

$$\underbrace{\exp(D)x \cdot \exp(D)y}_{\exp(D)(A_s)\,\exp(D)(A_t)} - \underbrace{\exp(D)(xy)}_{\exp(D)(A_{s+t})} = \underbrace{\exp(D)\left(\sum_{i=0}^{p-1} \frac{(-1)^i}{i} D^i x \cdot D^{p-i} y\right)}_{\exp(D)(A_{s+t})}$$

The Artin-Hasse exponential of a derivation

The assumption that $D^p = 0$ can be relaxed by considering the Artin-Hasse exponential series which is defined as

$$E_p(X) = \exp\left(\sum_{i=0}^{\infty} \frac{X^{p^i}}{p^i}\right) = \prod_{i=0}^{\infty} \exp\left(\frac{X^{p^i}}{p^i}\right) \in \mathbb{Z}_p[[X]].$$

Theorem (S. Mattarei)

Let $A = \oplus A_i$ be a non-associative algebra over a field of positive characteristic p, graded over the integers modulo m. Suppose that A has a nilpotent graded derivation D of degree d, with $m \mid pd$. Then the direct sum decomposition $A = \oplus E_p(D)A_i$ is a grading of A over the integers modulo m.

Laguerre polynomials

The classical (generalized) Laguerre polynomial of degree $n \geq 0$ is defined as

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \frac{(-x)^k}{k!}$$
$$= \sum_{k=0}^{p-1} \frac{(\alpha+n)(\alpha+n-1)\cdots(\alpha+k+1)}{(n-k)!k!}(-x)^k$$

where α is a complex number. Instead of $L_n^{(0)}(x)$ it is usual to write $L_n(x)$. Laguerre polynomials are solutions of second order differential equations and they are related to the confluent hypergeometric series.

Laguerre polynomials

Let $\mathbb{F}$ be a field and $\alpha \in \mathbb{F}$. For $m \in \mathbb{N}$ we denote by $\langle \alpha \rangle_m$ the *falling factorial* where

$$\langle \alpha \rangle_0 = 1 \quad \text{and} \quad \langle \alpha \rangle_m = \alpha(\alpha - 1) \cdots (\alpha - m + 1), \; m = 1, 2 \ldots$$

thus

$$\frac{\langle \alpha \rangle_m}{m!} = \binom{\alpha}{m}.$$

The Laguerre polynomial $L_n^{(\alpha)}(x)$ can be rewritten as

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{\langle \alpha + n \rangle_{n-k}}{k!(n-k)!} (-x)^k.$$

Laguerre polynomials

Let now $\mathrm{char}(\mathbb{F}) = p > 0$.
The Laguerre polynomial $L^{(\alpha)}_{p-1}(x)$, for $\alpha \in \mathbb{F}$, can be regarded as a polynomial in $\mathbb{F}[x]$.
Explicitly we have

$$L^{(\alpha)}_{p-1}(x) = \sum_{k=0}^{p-1} \frac{\langle \alpha + p - 1 \rangle_{p-1-k}}{k!(p-1-k)!}(-x)^k$$
$$= -\sum_{k=0}^{p-1} \langle \alpha + p - 1 \rangle_{p-1-k}\, x^k,$$

since $\binom{p-1}{k} \equiv (-1)^k \bmod p$. For our purposes $\alpha = 0$ or $\alpha \in \mathbb{F} \setminus \mathbb{F}_p$.

In the special case $\alpha = 0$ we have that $L_{p-1}(x) = \mathrm{E}(x) = \sum_{k=0}^{p-1} \frac{x^k}{k!}$ the truncated exponential of x.

Laguerre polynomials

Let D be a derivation of A, such that $D^{p^2} = D^p$ (thus D is semisimple with eigenvalues the elements of $\mathbb{F}_p$).
Let $u, v \in A$ such that $D^p(u) = au$ and $D^p(v) = bv$, with $a, b \in \mathbb{F}_p$.
Suppose there exist α, β in $\mathbb{F}$ such that

$$\alpha^p - \alpha = \langle \alpha \rangle_p = a \quad \text{and} \quad \beta^p - \beta = \langle \beta \rangle_p = b.$$

We compute the product $L^{(\alpha)}_{p-1}(D)(u) \cdot L^{(\beta)}_{p-1}(D)(v)$ obtaining

$$\begin{aligned} L^{(\alpha)}_{p-1}(D)(u) \cdot L^{(\beta)}_{p-1}(D)(v) = & c_1(\alpha, \beta)\, L^{(\alpha+\beta)}_{p-1}(D)(uv) + \\ & + c_2(\alpha, \beta)\, L^{(\alpha+\beta)}_{p-1}(D)(T_p), \end{aligned}$$

where $c_i(\alpha, \beta) \in \mathbb{F}$, where

$$T_p = \sum_{i=1}^{p-1} \langle \alpha + p - 1 \rangle_{p-1-i} \langle \beta + p - 1 \rangle_{i-1} D^i(u) \cdot D^{p-i}(v).$$

Laguerre polynomials

In the special case $\alpha = 0 = \beta$ we recover that for $D^p = 0$

$$\exp(D)(u)\exp(D)(v) - \exp(D)(uv) = \exp(D)\left(\sum_{i=1}^{p-1} \frac{(-1)^i}{i} D^i(u) D^{p-i}(v)\right).$$

Let A_b be the eigenspace for D^p with respect to the eigenvalue $b \in \mathbb{F}_p$ and let $\beta \in \mathbb{F}$ such that $\langle\beta\rangle_p = b$. The map $L_{p-1}^{(\beta)}(D)$ induces a bijective linear map on A_b with inverse the map

$$G_\beta(D) = \sum_{i=0}^{p-1} \frac{D^i}{\langle\beta + p - 1\rangle_i}.$$

For $\beta = 0$ we recover that the map $L_{p-1}(D) = \exp(D)$ with $D^p = 0$ has inverse $G_0(D) = \sum_{i=0}^{p-1} \frac{D^i}{\langle p-1\rangle_i} = \exp(-D)$.

The result obtained prove the following

Theorem (M. A, S. Mattarei)

Let $A = \bigoplus A_k$ be a non-associative algebra over the field $\mathbb{F}$ of characteristic $p > 0$, graded over the integers modulo m. Suppose that A has a graded derivation D of degree d with $m|pd$, such that $D^{p^2} = D^p$. Assume there exists $\gamma \in \mathbb{F}$ with $\gamma^p - \gamma = 1$. Let $A_k = \bigoplus A_{(k,a)}$ be the decomposition of A_k into direct sum of eigenspaces for D^p, so that $A = \bigoplus_{(k,a)} A_{(k,a)}$ is a grading of A over $\mathbb{Z}/m\mathbb{Z} \times \mathbb{F}_p$. Then

$$A = \bigoplus_{(k,a)} L_{p-1}^{(a\gamma)}(D)(A_{(k,a)})$$

is a grading of A over $\mathbb{Z}/m\mathbb{Z} \times \mathbb{F}_p$.

Toral switching

Replace a torus T of a restricted Lie algebra L with another torus T_x by applying to T a map in $\operatorname{ad} x$ for a certain $x \in L$. This techinique goes back to Winter and it has been generalized by Block, Wilson and Premet. A crucial step in the toral switching process is the construction of linear maps mapping the root spaces with respect to T bijectively onto the root spaces with respect to T_x.

- In the case x is p-nilpotent one of the result in [S. Mattarei, *Artin-Hasse exponentials of derivations*, 2005] is that this map is (a variation of) the Artin-Hasse exponential of $\operatorname{ad} x$.
- In the case $x^{[p]^2} = x^{[p]}$ the map can be intrepreted in terms of Laguerre polynomials in $\operatorname{ad} x$.