On the minimal permutation degrees of abelian quotients of finite p-groups

Clara Franchi

Trento, June 8th 2012

Definitions

All groups are finite.

Definition

For a group G, the *minimal faithful permutation degree* is

$$\mu(G) := \min\{n \mid G \hookrightarrow \mathrm{Sym}(n)\}.$$

Definition

A *minimal (faithful) permutation representation* of G is a faithful permutation representation of G of degree $\mu(G)$

A general formula

$$\mu(G) = \min\{\sum_{i=1}^{k} |G : H_i| \mid H_i \leq G \text{ and } \bigcap_{i=1}^{k} \bigcap_{g \in G} H_i^g = 1\}$$

Theorem (D.L. Johnson (1971))

We can always choose a minimal representation of G where point stabilizers are meet-irreducible subgroups.

Definition

A subgroup H of a group G is called *meet-irreducible* if it is not the intersection of two proper subgroups of G containing H properly.

Bounds on $\mu(G)$

- $\mu(G) \leq |G| \leq \mu(G)!$ (Cayley's theorem)
- $|G| = \mu(G)! \iff G = \mathrm{Sym}(\mu(G))$.

Theorem (D.L. Johnson (1971))

$\mu(G) = |G| \iff$ *G is one of the following*

1. *cyclic of prime power order,*
2. *generalized quaternion 2-group,*
3. *Klein 4-group.*

Abelian groups

A. Povsner (1937), O. Ore (1939), G.I. Karpilovsky (1970), D.L. Johnson (1971)

$$A = \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_r^{\alpha_r}} \implies \mu(A) = p_1^{\alpha_1} + \cdots + p_r^{\alpha_r}.$$

Direct products

For any groups G and H we have

$$\mu(G \times H) \leq \mu(G) + \mu(H)$$

equality holds provided

- $(|G|, |H|) = 1$ (D.L. Johnson 1971)
- G and H are nilpotent non-trivial groups (D. Wright 1975)

If T_i are simple groups, $\mu(T_1 \times \ldots \times T_k) = \mu(T_1) + \ldots + \mu(T_k)$.
(Easdown, Praeger 1989)

Subgroups and quotients

Clearly

$$H \leq G \implies \mu(H) \leq \mu(G)$$

What about quotients?

If G is **abelian**, for any $N \leq G$ we have

$$\mu(G/N) \leq \mu(G).$$

But, this bound does not hold in general!

Examples

- $$G = < x, y \mid x^8 = y^4 = 1, x^y = x^{-1} >$$

$$Z(G) = < x^4, y^2 > \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$N := < x^4 y^2 > \leq Z(G)$$

$$G/N = < \overline{x}, \overline{y} \mid \overline{x}^8 = 1, \overline{x}^4 = \overline{y}^2, \overline{x}^{\overline{y}} = \overline{x}^{-1} >$$

is a generalized quaternion 2-group of order 16.
We have that $\mu(G) = 12$, in fact a faithful representation of degree 12 is given by

$$x \mapsto (1, 2, 3, 4, 5, 6, 7, 8)$$

$$y \mapsto (1, 8)(2, 7)(3, 6)(4, 5)(9, 10, 11, 12)$$

while $\mu(G/N) = |G/N| = 16$.

• (P. M. Neumann,1987)
Let

$$G = \underbrace{D_8 \times D_8 \times \ldots \times D_8}_{m} \leq \mathrm{Sym}(4m).$$

G has a normal subgroup N such that G/N is extraspecial and $\mu(G/N) = 2^{m+1}$.

'Good' cases

$$\mu(G/N) \leq \mu(G)$$

provided:

- G is abelian;
- G/N is cyclic

(if $G/N = < gN >$, then $\mu(G/N) \leq \mu(< g >) \leq \mu(G)$);

- G/N is an elementary abelian p-group (Kovács & Praeger, 1989)
- G/N has no non-trivial abelian normal subgroups (Kovàcs & Praeger 2000)

Theorem (D.F. Holt & J. Walton, 2002)

There exists a constant c, ($c \sim 5.34$) such that in any finite group G we have

$$\mu(G/N) \leq c^{\mu(G)-1}.$$

A conjecture

Conjecture (D. Easdown, C. Praeger, 1987)
$\mu(G/N) \leq \mu(G)$ *whenever* G/N *is abelian.*

Call G a *minimal counterexample* if it is a counterexample to the conjecture with minimal degree and minimal order and let N be a normal subgroup of G such that $\mu(G/N) > \mu(G)$. Then

- G is a non-abelian p-group (D. Easdown, C. Praeger 1988)
- $N = G'$
- $\mu(G/G') = \mu(G) + p$ (L. Kovàcs, C. Praeger 2000).

Related results

Theorem (M. Aschbacher, R. Guralnick, 1989)

Let G be a primitive permutation group of degree n. Then $|G/G'| \leq n$.

Theorem (M. Aschbacher, R. Guralnick, 1989)

If n is an odd power of 2, there exists a transitive 2-group G of degree n such that $|G/G'| \geq 2^{n/2\log_2 n}$.

Theorem (R. Guralnick, 2000)

Let G be a transitive group of degree $n > 1$ and let N be a normal subgroup of G with G/N cyclic. Then $|G/N| \leq n$.

p-groups with an abelian maximal subgroup

Theorem (F. 2011)

Let G be a non-abelian finite p-group with an abelian maximal subgroup. Then $\mu(G/G') \leq \mu(G)$.

Proof.

Ingredients

- If G is transitive of degree p^n with meet-irreducible point stabilizer, then
$$\mu(G/G') \leq p^{n-1} + p$$
and every section of G' which is central in G has order at most p.
- G has no abelian transitive constituent.
- Good knowledge of subdirects products.
- Induction on the number of orbits.

□

p-groups with a 'large' abelian normal subgroup

Proposition (F. 2012)

Let P be a non-abelian p-group which is a transitive permutation group of degree p^n, $n \geq 2$, with meet-irreducible point stabilizer.

Suppose that P contains a normal abelian subgroup M such that G/M is cyclic of order p^k, $k \leq n$. Then

1. *every section of P' which is central in P has order at most p^k;*
2. $|P/P'| \leq p^n$;
3. $\mu(P/P') \leq p^{n-1} + p$.

Lemma

Let $W_1 = C_{p^m} \wr C_{p^k}$, and let B denote the base subgroup of W_1. Then

1. *every section of B which is central in W_1 has order at most p^m;*
2. *every section of W_1' which is central in W_1 has order at most*

$$|W_1' \cap Z(W_1)| \leq min\{p^k, p^m\} \leq p^k.$$

Proposition (F. 2012)

Let G be a non-abelian p-group with a normal abelian subgroup M such that G/M is cyclic of order at most p^2.

Suppose that G has a faithful representation of degree $d = k_2p^2 + k_3p^3 + \ldots + k_np^n$, with k_i orbits of length p^i, for each $i = 2, \ldots, n$.

Suppose that each point stabilizer is meet-irreducible and that G has no abelian transitive constituent. Then

$$|G/G'| \leq p^{2k_2+3k_3+\ldots+nk_n}.$$

Proposition (F. 2012)

Let P be a non-abelian p-group which is a transitive permutation group of degree p^n, $n \geq 2$, with meet-irreducible point stabilizer.

Suppose that P contains a normal abelian subgroup M such that G/M is elementary abelian of order p^2. Then

$$\mu(P/P') \leq p^n.$$

It is well known that the base subgroup B of the wreath product

$$W = C_{p^m} \wr (C_p \times C_p)$$

has a ring structure isomorphic to the group ring $A = \mathbb{Z}_{p^m}[C_p \times C_p]$ and the isomorphism sends subgroups of B that are normal in W to ideals of A.

The group ring $\mathbb{Z}_{p^m}[C_p \times C_p]$

Theorem (Okon, Rush, Vicknair, 2000)

Every ideal of the group ring $\mathbb{Z}_{p^m}[C_p \times C_p]$ can be generated by at most:

1. $p + 2$ *elements, if* $m > 2$;
2. $p + 1$ *elements, if* $m = 2$;
3. p *elements, if* $m = 1$.

Corollary

Every proper section of B which is central in W has rank at most

1. $p + 2$, *if* $m > 2$;
2. $p + 1$, *if* $m = 2$;
3. p, *if* $m = 1$.

In particular, every proper section of the base subgroup of $\mathbb{Z}_p \wr (C_p \times C_p)$ which is central in whole group, is elementary abelian of order at most p^p.

Lemma (F. 2012)

Let $W = C_{p^m} \wr (C_p \times C_p)$, $m \geq 2$, *and let* B *be its base subgroup. Then*

1. B/W' *is cyclic of order* p^m*;*
2. $\gamma_i(W)/\gamma_{i+1}(W)$ *is elementary abelian of rank at most* $p+2$ *(respectively* $p+1$ *when* $m=2$*), for every* $i \geq 2$*;*
3. *every section of* B *which is central in* W *has order at most* p^{mp+1}*;*
4. *every section of* W' *which is central in* W *has order at most* $p^{m(p-1)+1}$*.*

Questions

1. *Find best possible bound for the size and (possibly) the exponent of sections of* B *which are central in* W*.*
2. *Find best possible bound for the size and (possibly) the exponent of sections of* W' *which are central in* W*.*