Character degrees of finite p-groups by coclass

Martin Couson

TU Braunschweig

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Overview

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Coclass theory
(Leedham-Green & Newman, 1980)

- Coclass of a finite p-group G with $|G| = p^n$ and $\mathrm{cl}(G) = c$:

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$$G = \gamma_1(G) \geq \gamma_2(G) \geq \gamma_3(G) \geq \dots$$

- Coclass of an infinite pro-p-group G:

$$\mathrm{cc}(G) := \lim_{i \to \infty} \mathrm{cc}(G/\gamma_i(G)).$$

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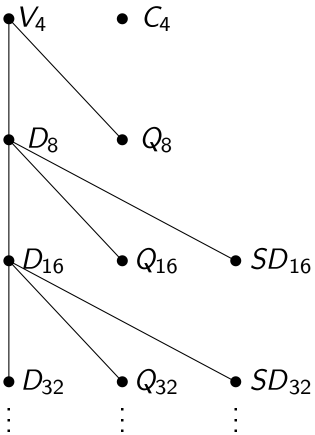
- vertices: isomorphism types of finite p-groups of coclass r
- there is an edge from G to H if there is $N \triangleleft H$ with $|N| = p$ and $H/N \cong G$.

Example $\mathcal{G}(2, 1)$ (2-groups of maximal class):



Infinite paths in $\mathcal{G}(p, r)$

Infinite paths correspond 1-1 to pro-p-groups of coclass r.

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For fixed p and r there are only finitely many isomorphism classes of pro-p-groups of coclass r. These pro-p-groups are soluble.

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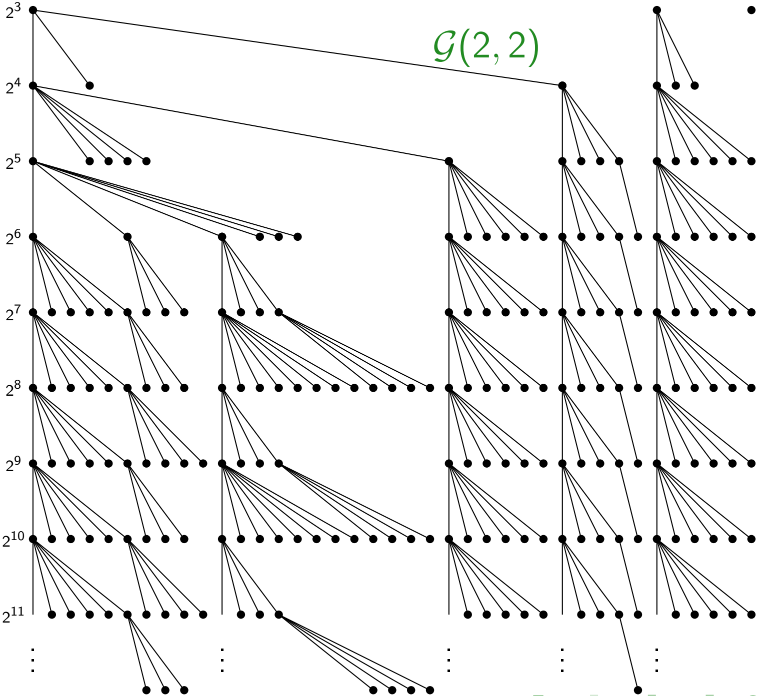
Corollary 2

There are only finitely many infinite paths in $\mathcal{G}(p, r)$.


$\mathcal{G}(2,2)$
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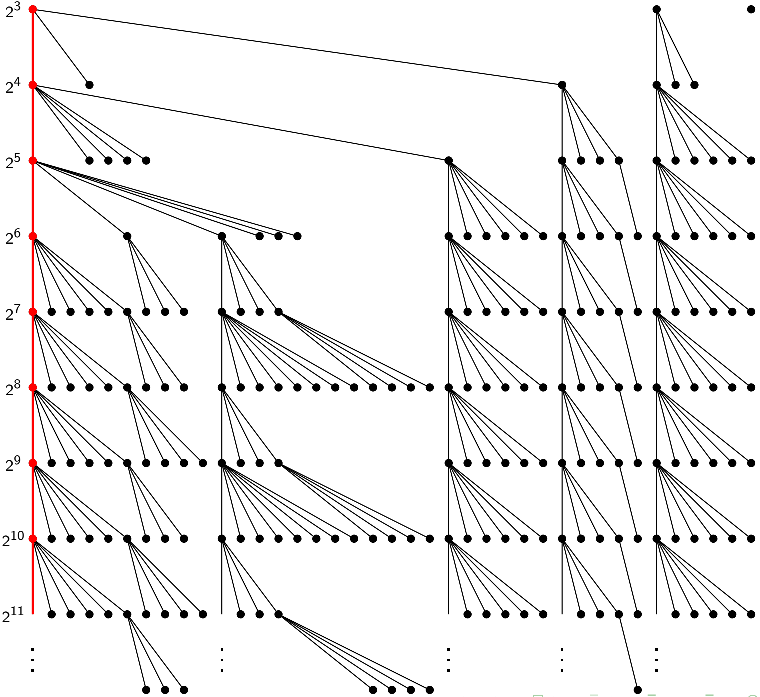
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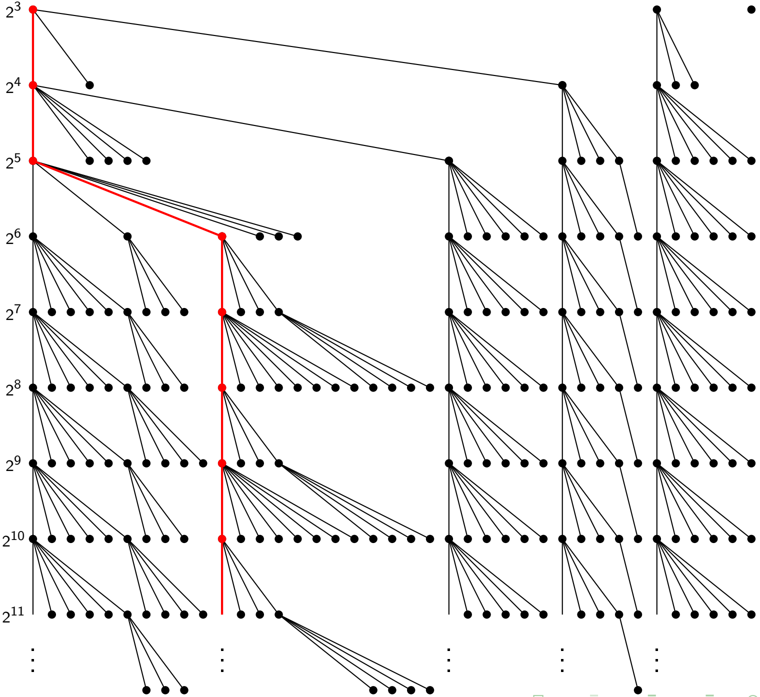
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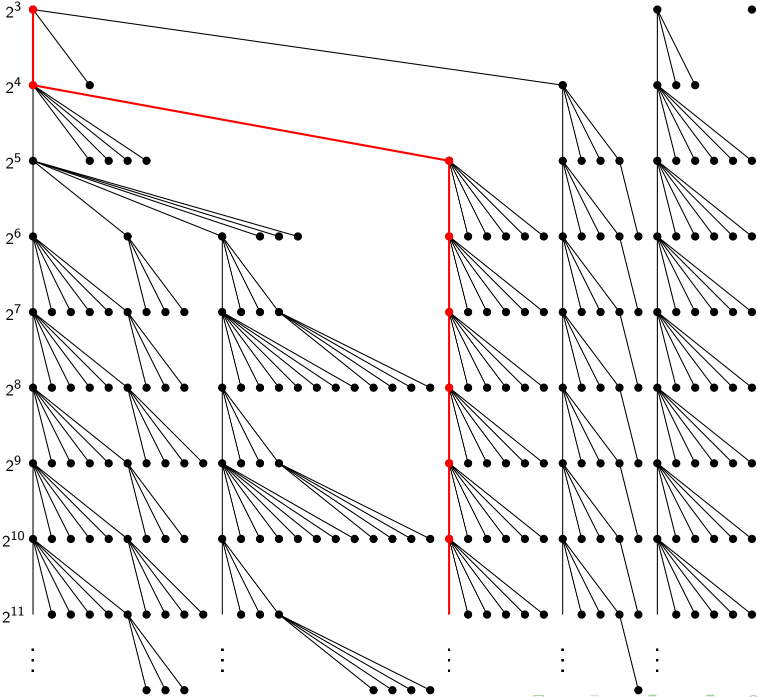
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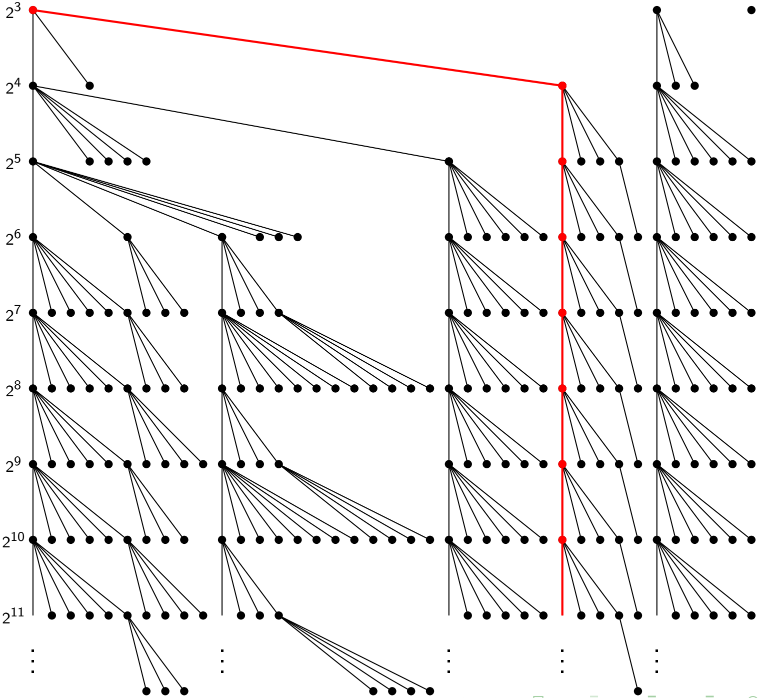
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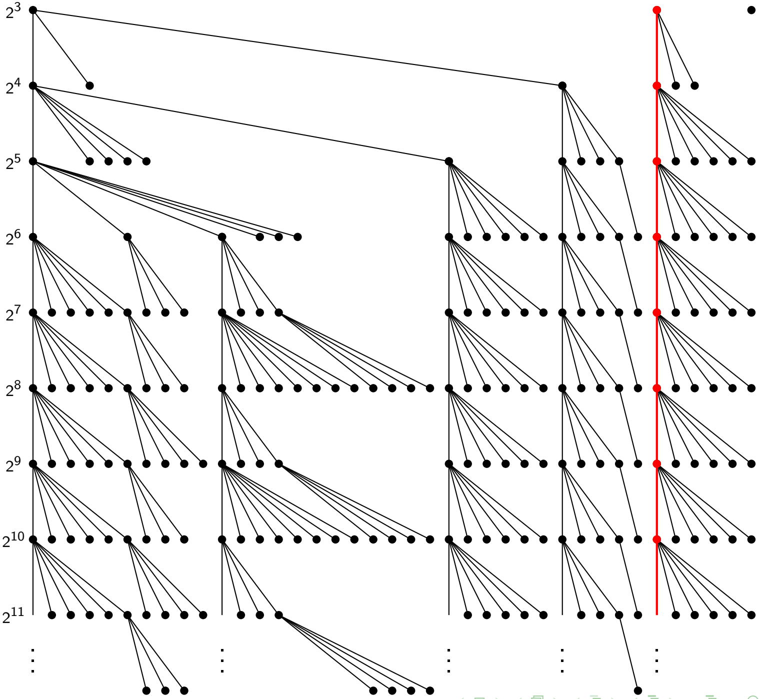
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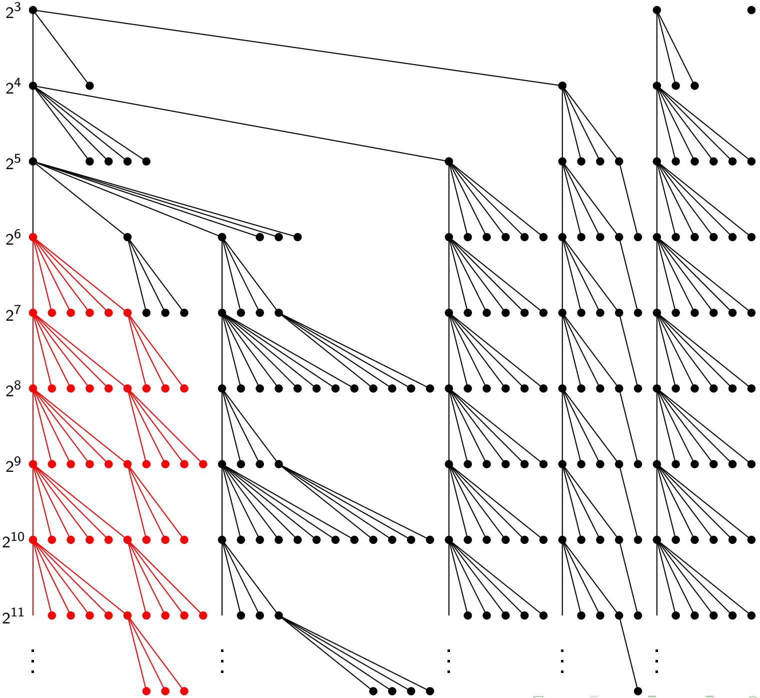
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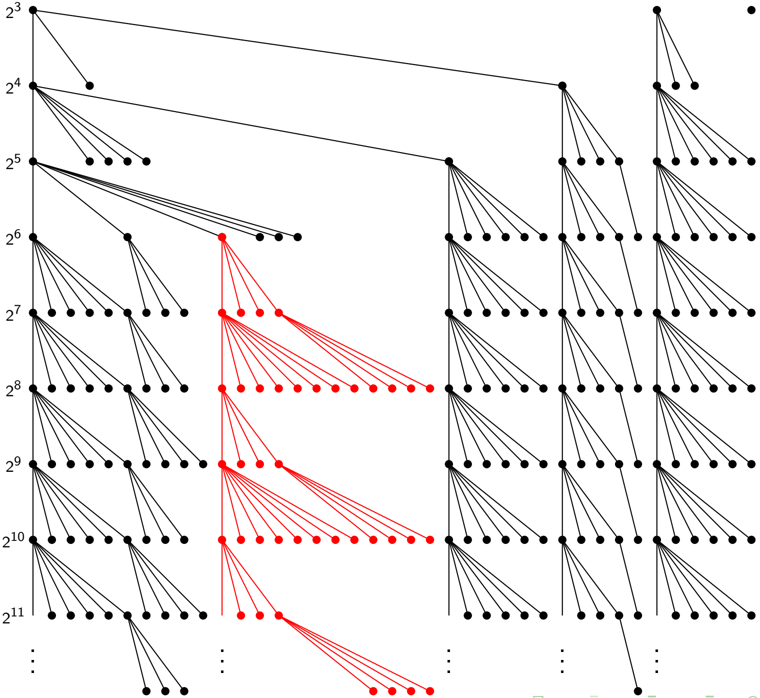
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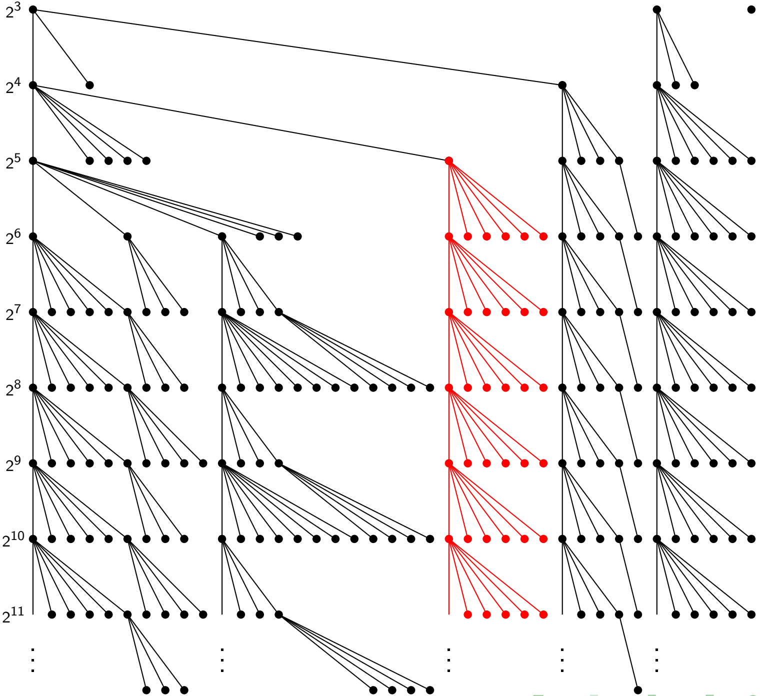
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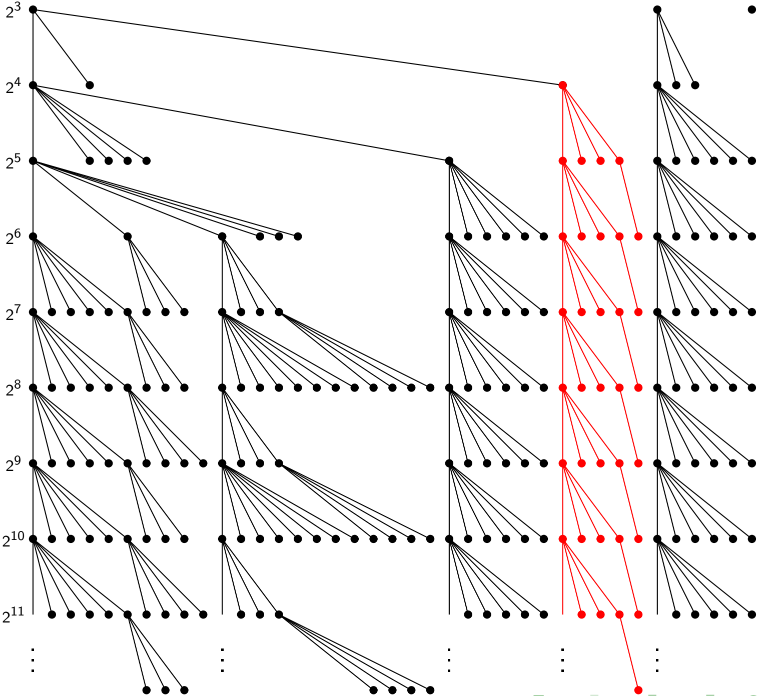
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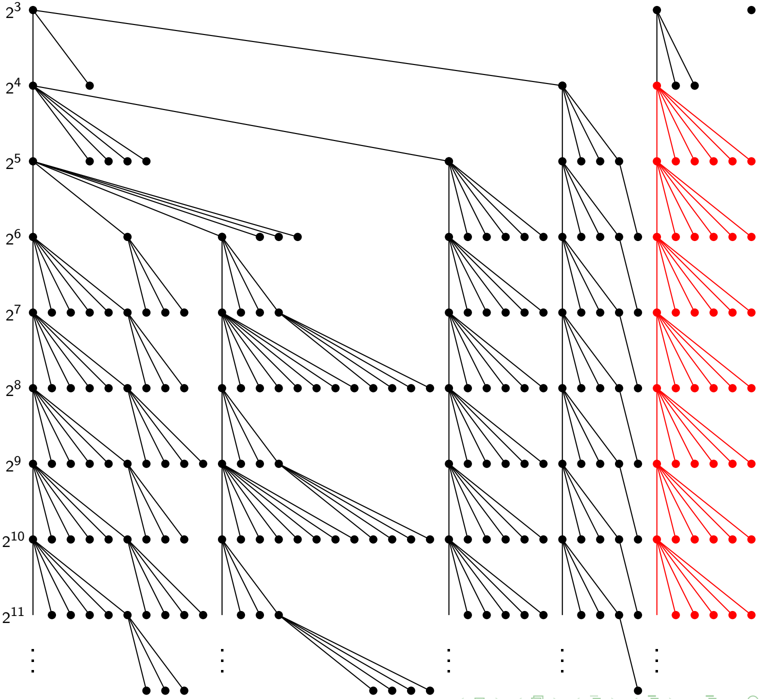
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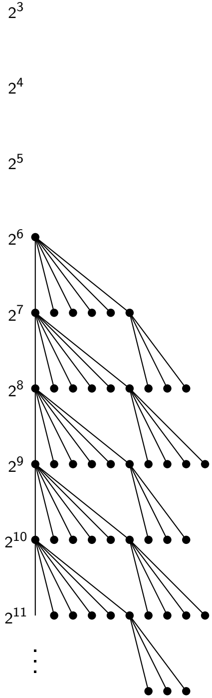
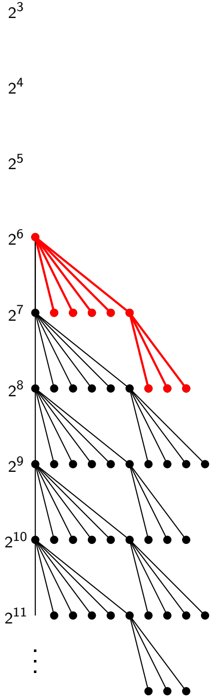
Coclass Tree





Coclass Tree

- Coclass tree is ultimately periodic for $p = 2$ (du Sautoy, 2001 and Eick & Leedham-Green, 2008).

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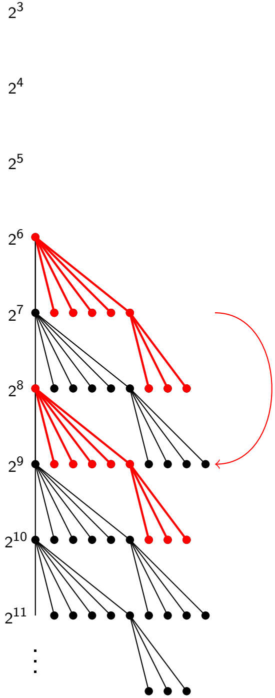

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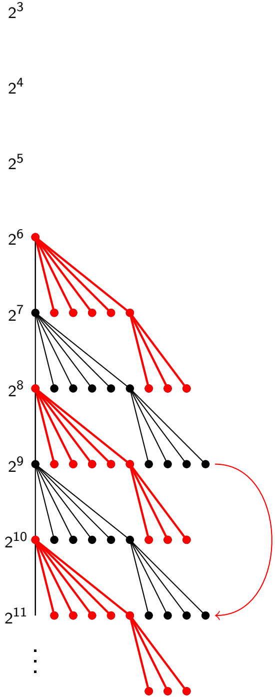

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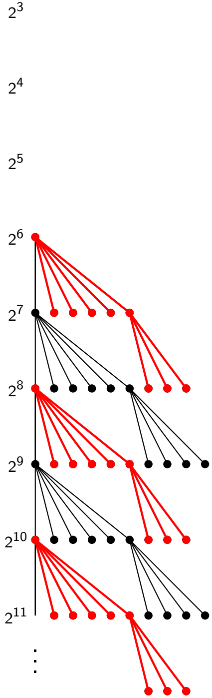

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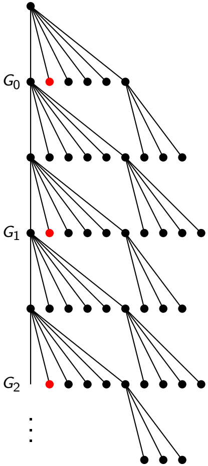
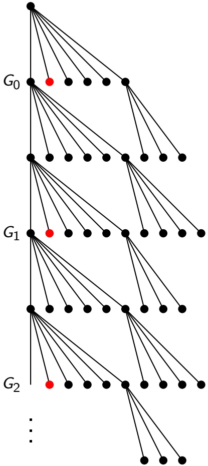
- p odd: in general the result holds only for shaved coclass trees.

Coclass Family



Coclass Family
Parametrized Presentations
(Eick & Leedham-Green, 2008)



$$
\begin{aligned}
G_{\mathbf{k}} = \langle g_1, \ldots, g_5, t_1, t_2 \mid \\
& g_1^2 = g_4, \\
& g_2^{g_1} = g_2 g_3, g_2^2 = 1, \\
& g_3^{g_1} = g_3 g_5, g_3^{g_2} = g_3 t_1^{1+2^{\mathbf{k}}} t_2, g_3^2 = t_1^{-1+2^{\mathbf{k}}} t_2^{-1}, \\
& g_4^{g_1} = g_4, g_4^{g_2} = g_4 g_5 t_1^{2^{\mathbf{k}}} t_2, \ldots, g_4^2 = 1, \\
& g_5^{g_1} = g_5 t_2, g_5^{g_2} = g_5 t_1^{-1}, \ldots, g_5^2 = t_1, \\
& t_1^{g_1} = t_1 t_2^2, t_1^{g_2} = t_1^{-1}, \ldots, t_1^{2^{\mathbf{k}+1}} = 1, \\
& t_2^{g_1} = t_1^{-1} t_2^{-1}, t_2^{g_2} = t_2^{-1}, \ldots, \mathbf{t_2^{t_1} = t_2}, t_2^{2^{\mathbf{k}+1}} = 1 \rangle
\end{aligned}
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Schur Multiplier

Let $\mathcal{F} = (G_k \mid k \in \mathbb{N}_0)$ be a coclass family.
Let $M(G)$ denote the Schur multiplier of a group G.

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Theorem 3 (Eick & Feichtenschlager, 2010)

There are $m \in \mathbb{N}_0$, and for $1 \leq i \leq m$ integers $r_i \in \mathbb{N}_0$ and $s_i \in \mathbb{Z}$ such that for every large enough $k \in \mathbb{N}_0$ it follows that $d_i(k) = p^{r_i k + s_i} \in \mathbb{N}$ and

$$M(G_k) \cong C_{d_1(k)} \times \ldots \times C_{d_m(k)}.$$

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Corollary 4

There is $f(X) = f_{\mathcal{F}} \in \mathbb{Q}[X]$ such that

$$|M(G_k)| = f(p^k)$$

for large k.

Automorphism Group

Theorem 5 (Eick, 2006)

There is $e \in \mathbb{N}$ such that for large k

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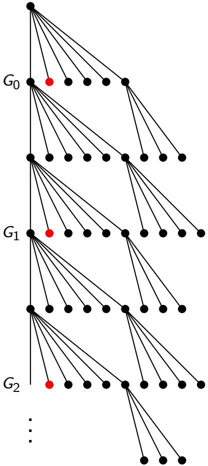
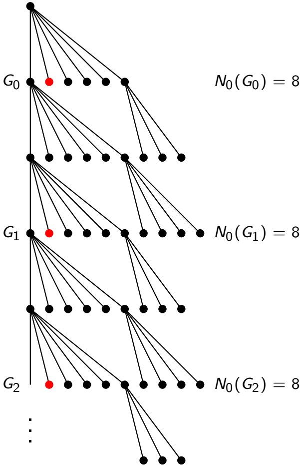
Coclass Family
Character Degrees



For a finite p-group G and $\ell \in \mathbb{N}_0$, denote

$$N_\ell(G) := |\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = p^\ell\}|.$$

Coclass Family
Character Degrees



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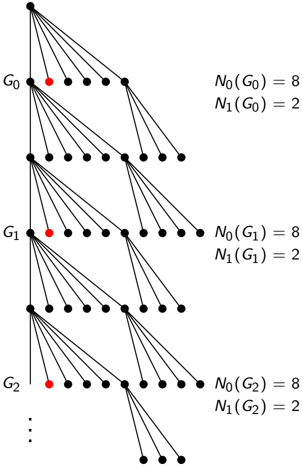

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Coclass Family Character Degrees

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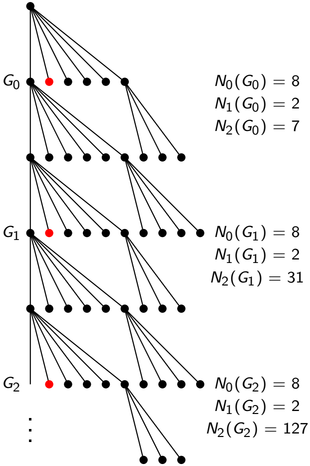

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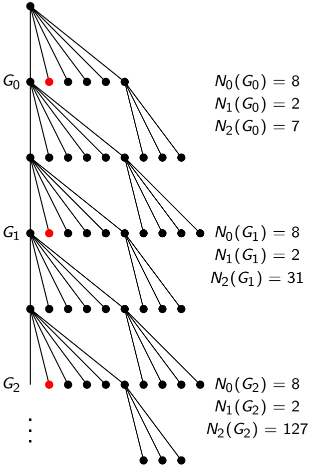

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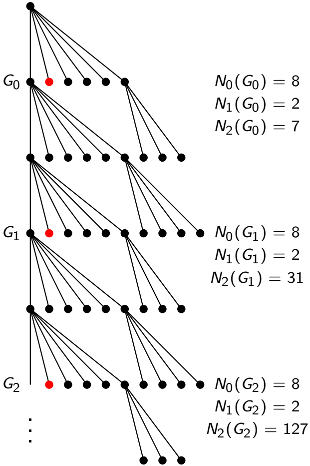

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$$f_2(X) := -1 + 8X^2$$

Theorem 7 (C. 2011)

Let $\mathcal{F} = (G_k \mid k \in \mathbb{N}_0)$ be a coclass family and d denote the dimension of the associated pro-p-group.

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- *There exists a bound b so that every irreducible character for every G_k has degree at most p^b. That is, $N_\ell(G_k) = 0$ if $\ell > b$.*
- *Let $\ell \in \{0, \dots, b\}$. Then there exists a polynomial $f_\ell(X) \in \mathbb{Q}[X]$ with $\deg(f_\ell) \leq d$ and a natural number w such that $N_\ell(G_k) = f_\ell(p^k)$ for every $k \geq w$.*

Overview

1. Coclass theory
2. Character theory of finite p-groups
3. Coclass families and character degrees
4. Coclass families and automorphism groups

Character theory of finite p-groups

Let G be a finite p-group.

Definition 8

Let $\chi \in \mathrm{Irr}(G)$ and $U \leq G$. We say, that χ is linearly induced from U, if there is a linear character μ of U such that $\chi = \mu{\uparrow}^G$.

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Theorem 9

G is monomial, that is, each irreducible character of G is linearly induced from some subgroup.

Let $U \leq G$ and let μ be a linear character of U. Denote $\chi := \mu\uparrow_U^G$.

1. Is χ irreducible?

If χ is irreducible:

2. How many linear characters of U induce to χ?
3. Given $V \leq G, V \neq U$, how many linear characters of V induce to χ?

Deciding irreducibility

Let μ be a linear character of U. Define

$$c_\mu : U \backslash G / U \to \{0, 1\}, UgU \mapsto \begin{cases} 1, & \text{if } [g, U] \cap U \subseteq \ker(\mu), \\ 0, & \text{otherwise.} \end{cases}$$

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Theorem 10 (Shoda)

$\mu{\uparrow}_U^G$ is irreducible if and only if $c_\mu = \mathbf{1}_{\{U\}}$.

Deciding equality of linearly induced characters

For $W \in \{U, V\}$ define the maps $a_{\mu,W} : W\backslash G/U \to \mathbb{N}_0$ and $a_U : U\backslash G/U \to \mathbb{N}_0$ by setting for $g \in G$:

$$a_{\mu,W}(WgU) = \begin{cases} 1, & \text{if } (W^g)' \cap U \subseteq \ker(\mu), \\ 0, & \text{otherwise,} \end{cases}$$

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Theorem 11

Assume $\chi = \mu\uparrow_U^G \in \mathrm{Irr}(G)$.

(i) χ *is linearly induced from* V *if and only if* $a_{\mu,V} \neq 0$.

(ii) $|\{\lambda \in \mathrm{Lin}(U) \mid \lambda\uparrow^G = \chi\}| = \sum_{d \in U\backslash G/U} a_{\mu,U}(d) \cdot a_U(d)$.

Overview

Applying results to coclass families

Let G be a finite p-group, acting on a $\mathbb{Z}_p$-module N of finite rank ($\mathbb{Z}_p = p$-adic numbers). Denote $N_0 := N$ and $N_{i+1} := [G, N_i]$ for $i \geq 0$.

Definition 12

G acts uniserially on N, if $[N_i : N_{i+1}] = p$ for every i with $N_i \neq 1$.

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Definition 12

G acts uniserially on N, if $[N_i : N_{i+1}] = p$ for every i with $N_i \neq 1$.

Let $d = \mathrm{rank}\ N$ and assume that G acts uniserially on N. Then $N_{i+d} = p \cdot N_i$ for all i.

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Let $\mathcal{F} = (G_k \mid k \in \mathbb{N}_0)$ be a coclass family. Then there are $n \in \mathbb{N}_0$, $d \in \mathbb{N}$ and a finite p-group P, acting uniserially on an abelian group T such that

- $T \cong \mathbb{Z}_p^d$ (where $\mathbb{Z}_p$ denotes the p-adic numbers),
- G_k is a group extension of T/T_{n+kd} by P.

(Eick & Leedham-Green, 2008)

Let

$$\tau_k : H^2(P, T) \to H^2(P, T/T_{n+kd})$$

be induced by the natural homomorphism $T \twoheadrightarrow T/T_{n+kd}$,

$$\rho_k : H^2(P, T/T_n) \to H^2(P, T/T_{n+kd})$$

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Theorem 13 (Eick & Leedham-Green, 2008)

If n is sufficiently large, then there is $K \leq H^2(P, T/T_n)$ such that for $k \in \mathbb{N}_0$

1. *τ_k is a monomorphism,*
2. *$K \cong H^3(P, T)$,*
3. *$H^2(P, T/T_{n+kd}) = \mathrm{Im}(\tau_k) \oplus \rho_k(K)$.*

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There are $\gamma \in Z^2(P,T)$ and $\delta \in Z^2(P,T/T_n)$ such that for all $k \in \mathbb{N}_0$ the group extension defined by the cocycle

$$\begin{aligned}\gamma + p^k\delta : P \times P &\to T/T_{n+kd},\\ (g,h) &\mapsto \gamma(g,h) + p^k\delta(g,h) + T_{n+kd}\end{aligned}$$

is isomorphic to G_k.

(Eick & Leedham-Green, 2008)

Applying results to coclass families

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Since $G_k/A_k = P$ for all k, this means that the character degrees of the irreducible characters of G_k are bounded globally.

Applying results to coclass families

- Let $\ell \in \mathbb{N}_0$ and let $\mathcal{U}_0 = \{U_{1,0}, \ldots, U_{s,0}\}$ be a transversal of the conjugacy classes of subgroups U in G_0 with $A_0 \leq U$ and $[G_0 : U] = p^\ell$.

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- Denote $\pi_k : G_0/A_0 \to G_k/A_k$, $gA_0 \mapsto gA_k$.
- For $1 \leq i \leq s$ let $U_{i,k} \leq G_k$ such that $A_k \leq U_{i,k}$ and

$$\pi_k(U_{i,0}/A_0) = U_{i,k}/A_k$$

and denote $\mathcal{U}_k = (U_{1,k}, \ldots, U_{s,k})$.

How to determine $N_\ell(G_k)$

- We can determine $N_\ell(G_k) = |\{\chi \in \mathrm{Irr}(G_k) \mid \chi(1) = p^\ell\}|$ similiar to $|\{\mu \in \mathrm{Lin}(U_{1,k}) \mid \mu\uparrow^G \in \mathrm{Irr}(G_k)\}|$.

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- Denote $U_k := U_{1,k}$ and recall that for $\mu \in \mathrm{Lin}(U_k)$

$$
\begin{aligned}
c_\mu : & U_k \backslash G_k / U_k \to \{0, 1\}, \\
& U_k g U_k \mapsto \begin{cases} 1, & \text{if } [g, U_k] \cap U_k \subseteq \ker(\mu), \\ 0, & \text{otherwise}, \end{cases}
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- $\mathrm{Lin}(U_k)$ is a group with respect to pointwise multiplication. But $\{\mu \in \mathrm{Lin}(U_k) \mid c_\mu = \mathbf{1}_{\{U_k\}}\}$ is **not** a subgroup.

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Let $c \in \mathrm{DCM}^*(k)$. Then

- $\{\mu \in \mathrm{Lin}(U_k) \mid c_\mu \preceq c\}$ is a group,
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Thus

$$\{\mu \in \mathrm{Lin}(U_k) \mid c_\mu = \mathbf{1}_{\{U_k\}}\} \\ = \{\mu \in \mathrm{Lin}(U_k) \mid c_\mu \preceq \mathbf{1}_{\{U_k\}}\} \setminus \bigcup_{c \prec \mathbf{1}_{\{U_k\}}} \{\mu \in \mathrm{Lin}(U_k) \mid c_\mu \preceq c\}$$

can be determined simultaneously for all k.

This enables us to prove Theorem 7:

Theorem 14 (C. 2011)

Let $\mathcal{F} = (G_k \mid k \in \mathbb{N}_0)$ be a coclass family and d denote the dimension of the associated pro-p-group.

- *There exists a bound b so that every irreducible character for every G_k has degree at most p^b. That is, $N_\ell(G_k) = 0$ if $\ell > b$.*
- *Let $\ell \in \{0, \ldots, b\}$. Then there exists a polynomial $f_\ell(X) \in \mathbb{Q}[X]$ with $\deg(f_\ell) \leq d$ and a natural number w such that $N_\ell(G_k) = f_\ell(p^k)$ for every $k \geq w$.*

Overview

1. Coclass theory
2. Character theory of finite p-groups
3. Coclass families and character degrees
4. Coclass families and automorphism groups

Automorphism groups $\mathrm{Aut}(G_k)$

$\mathcal{F} = (G_k \mid k \in \mathbb{N}_0)$ ...coclass family with associated pro-p-group S.
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Theorem 15 (C. 2012)

There are subgroups $B \trianglelefteq A \leq \mathrm{Aut}(S)$, a finite abelian group M and $e, v \in \mathbb{N}_0$ such for all large k

- $\mathrm{Aut}(G_k)$ *is a group extension of M by $A/B^{p^{k-v}}$,*
- $B \cong (1 + p^{e+n'}\mathrm{End}_P(T)) \ltimes Z^1(P, p^{n'}T)$,

(Here we use the notation $B^m := \langle b^m \mid b \in B\rangle$ for $m \in \mathbb{N}$)

$B \trianglelefteq A \leq \mathrm{Aut}(S)$; M finite abelian group; $v \in \mathbb{N}_0$ as in Theorem 15; $\mathrm{Aut}(G_k) \cong$ group presentation of M by $A/B^{p^{k-v}}$

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$$\begin{array}{ccc} H^2(A/B^{p^v}, M) & \xrightarrow{\lambda_k} & H^2(A/B^{p^{k-v}}, M) \\ & & \uparrow{\scriptstyle \kappa_k} \\ & & H^2(A/B^{p^{k-v}}, B^{p^{k-v}}/B^{p^k}) \end{array}$$

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Theorem 16 (C. 2012)

There is $\tau \in H^2(A/B^{p^v}, M)$ such that for every sufficiently large k the group extension defined by $\kappa_k(\rho_k) + \lambda_k(\tau)$ is isomorphic to $\mathrm{Aut}(G_k)$.

$$\begin{array}{ccc} H^2(A/B^{p^v}, M) & \xrightarrow{\lambda_k} & H^2(A/B^{p^{k-v}}, M) \\ & & \uparrow \kappa_k \\ & & H^2(A/B^{p^{k-v}}, B^{p^{k-v}}/B^{p^k}) \end{array}$$

- λ_k is induced by $A/B^{p^{k-v}} \twoheadrightarrow A/B^{p^v}$

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- λ_k is induced by $A/B^{p^{k-v}} \twoheadrightarrow A/B^{p^v}$
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- The following diagram commutes:

$$\begin{array}{ccc} B^{p^{k-v}}/B^{p^k} & \xrightarrow{\mu_k} & M \\ \downarrow \eta_k & & \| \\ B^{p^{k+1-v}}/B^{p^{k+1}} & \xrightarrow{\mu_{k+1}} & M \end{array}$$

where η_k is the group isomorphism sending bB^{p^k} to $b^p B^{p^{k+1}}$.

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- $N_l(G_k)$, the number of irreducible characters of degree p^l, can be described by a rational polynomial
- $\mathrm{Aut}(G_k)$ can be described by a sequence of cocycles induced by one cocycle and an infinite group