Pro-p Groups of Finite Virtual Length

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Finite p-groups versus simple groups

- A finite p-group G has normal subgroups of every possible order.
- The only simple finite p-group is the cyclic group of order p.
- Few connections between theory of finite p-groups and theory of finite simple groups. Well, almost.

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Simple pro-p groups?

- A pro-p group is the inverse limit of finite p-groups (each finite p-group is endowed with the discrete topology).
- The finite pro-p groups are just the finite p-groups.
- By a subgroup of a pro-p group we mean a closed one.
- A subgroup is open if and only if has finite index (and it is closed).
- Open subgroups have index a power of p.
- A pro-p group has (open) normal subgroups of every possible index p^n.
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Just infinite pro-p groups

- An infinite pro-p group G is just infinite if every non-trivial normal subgroup N is open in G (i.e. has finite index).
- Equivalently, G is just infinite if its proper quotients are finite.
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Hereditarily just infinite pro-p groups

Leedham-Green's point of view

- A simple group is a group whose only non-trivial subnormal subgroup is the group itself.
- The corresponding notion in pro-p group context is that of a hereditarily just infinite pro-p group: namely a pro-p group whose non-trivial subnormal subgroups have finite index.
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Composition series in pro-p groups?

- The Jordan–Hölder theorem motivated the study of finite simple groups.
- Let $\mathbf{S}$: $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ be a series of an infinite pro-p group G.
- At least one section G_{i+1}/G_i of $\mathbf{S}$ is infinite: it is therefore not simple.
- A new term can be inserted strictly between G_i and G_{i+1}.
- Every series of an infinite pro-p group can be properly refined.

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Elementary refinement of a series

- Insert a new term N strictly between two terms G_i and G_{i+1} of a series $\mathbf{S}$: let $\mathbf{S}'$ the new series obtained.
- The section G_{i+1}/G_i is replaced by two sections: G_{i+1}/N and N/G_i.

$\lvert G_{i+1} : G_i\rvert < \infty$	$G_i \vdash$ finite $\dashv N \vdash$ finite $\dashv G_{i+1}$	
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The length $\mathrm{L}_\infty(\mathbf{S})$ denotes the number of infinite sections of $\mathbf{S}$.

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Refining just infinite sections

- If G_{i+1}/G_i is just infinite we have only the case

$$G_i \vdash \text{infinite} \dashv N \vdash \text{finite} \dashv G_{i+1}$$

- However N/G_i is not necessarily just infinite. In this case a further refinement can yield a series of greater length.

$$G_i \vdash \text{infinite} \dashv M \vdash \text{infinite} \dashv N \vdash \text{finite} \dashv G_{i+1}$$

- Maybe more steps are needed to get a series of greater length.
- If G_{i+1}/G_i is hereditarily just infinite every iterated refinement (between G_{i+1} and G_i) replaces an infinite section with a finite section and an infinite one so the length is stable.

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Virtual composition series

- If every section of a series $\mathbf{S}$ is either hereditarily just infinite or finite then $L_\infty(\mathbf{S}') = L_\infty(\mathbf{S})$ for every refinement of $\mathbf{S}$.
- We say that a series $\mathbf{S}$ with this property is a virtual composition series (namely if $L_\infty(\mathbf{S}') = L_\infty(\mathbf{S})$ for every refinement of $\mathbf{S}$).

Questions

- When does a virtual composition series exist?
- Which is the relation between two virtual composition series?
- Is a virtual composition series with all sections either finite or hereditarily just infinite somehow typical?

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Virtual length

- Schreier Refinement Theorem holds in the context of pro-p groups: two series have isomorphic refinements (in particular they have refinements of the same length).
- If $\mathbf{S}'$ is a refinement of $\mathbf{S}$ then $\mathrm{L}_\infty(\mathbf{S}) \leq \mathrm{L}_\infty(\mathbf{S}')$.
- For a given pro-p group G define its virtual length $\mathrm{vl}(G)$ to be the supremum of $\mathrm{L}_\infty(\mathbf{S})$ taken over the series of G.
- If $\mathrm{vl}(G)$ is infinite then G has no virtual composition series.
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Properties of virtual length

- $\mathrm{vl}(G) = 0$ if and only if G is finite.
- If G is hereditarily just infinite then $\mathrm{vl}(G) = 1$.
- If N is a normal subgroup of G then $\mathrm{vl}(G) = \mathrm{vl}(G/N) + \mathrm{vl}(N)$ (we admit ∞ in this sum).
- If H is a subnormal subgroup then $\mathrm{vl}(H) \leq \mathrm{vl}(G)$: equality holds if and only if H is open.
- If $G = N_1 \oplus N_2 \oplus \cdots \oplus N_r$ then $\mathrm{vl}(G) = \mathrm{vl}(N_1) + \mathrm{vl}(N_2) + \cdots + \mathrm{vl}(N_r)$.

Theorem

If $\mathbf{S}$ and $\mathbf{S}'$ are virtual composition series of the same pro-p group then their infinite sections can be put in correspondence in such a way that corresponding sections are virtually isomorphic (i.e. they contain isomorphic open subgroups).

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- If N is a normal subgroup of G then $\mathrm{vl}(G) = \mathrm{vl}(G/N) + \mathrm{vl}(N)$ (we admit ∞ in this sum).
- If H is a subnormal subgroup then $\mathrm{vl}(H) \leq \mathrm{vl}(G)$: equality holds if and only if H is open.
- If $G = N_1 \oplus N_2 \oplus \cdots \oplus N_r$ then $\mathrm{vl}(G) = \mathrm{vl}(N_1) + \mathrm{vl}(N_2) + \cdots + \mathrm{vl}(N_r)$.

Theorem

If $\mathbf{S}$ and $\mathbf{S}'$ are virtual composition series of the same pro-p group then their infinite sections can be put in correspondence in such a way that corresponding sections are virtually isomorphic (i.e. they contain isomorphic open subgroups).

Pro-*p* groups of finite rank

They *have* finite virtual length

- A pro-*p* group has finite rank if there exists *r* such that every subgroup can be generated by *r* elements.
- A pro-*p* group of finite rank contains an open uniform subgroup.
- The dimension $\dim G$ of a pro-*p* group *G* of finite rank is the number of generators of an open uniform subgroup of *G*.
- Dimension is additive: if *N* is a normal subgroup of *G* then $\dim G = \dim(G/N) + \dim N$.

Theorem

A pro-p group G of finite rank has finite virtual length. More precisely $\mathrm{vl}(G) \leq \dim G$ *and the equality holds if and only if G is soluble.*

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Pro-*p* groups with Constant Normal Subgroup Growth

They *have* finite virtual length

- A pro-*p* group has Constant Normal Subgroup Growth if the number of open normal subgroups of the same index is bounded.
- Pro-*p* groups with Constant Normal Subgroup Growth have finite virtual length (this follows from structural results).
- Pro-*p* groups with Constant Normal Subgroup Growth include pro-*p* of finite coclass: so they have finite virtual length.
- Constant Normal Subgroup Group is not closed under taking extensions, while finite virtual length is.

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Branch groups

They *have not* finite virtual length

- There are two inequivalent definitions of branch groups: one involving a group action on a tree, and another purely algebraic.
- In both cases, a branch group is infinite and contains, for each natural number n, an open normal subgroup H_n which is the direct product of k_n copies of a subgroup L_n. Here $k_n > k_{n-1}$ for every n.
- $\mathrm{vl}(G) \geq \mathrm{vl}(H_n) = k_n \mathrm{vl}(L_n) \geq k_n$: therefore $\mathrm{vl}(G) = \infty$.
- Wilson's dichotomy: a just infinite pro-p group G is either a branch group ($\mathrm{vl}(G) = \infty$) or contains an open subgroup which is the direct sum of k copies of a hereditarily just infinite pro-p group ($\mathrm{vl}(G) = k$).

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Structural results

Theorem

Let G be a pro-p group with finite virtual length. Then:

- *G is finitely generated;*
- *every subnormal subgroup is finitely generated;*
- *every non-empty family of subnormal subgroups with bounded defect admits a maximal element.*

Corollary

Every non-empty family of normal subgroups of G closed under taking products (e.g. finite normal subgroups) admits a greatest element.

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For a pro-p group G the following are equivalent:

1. $\mathrm{vl}(G) = 1$;
2. *there exists a finite normal group N such that G/N is hereditarily just infinite;*
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Every virtual composition series of a pro-p group with finite virtual length can be refined to a virtual composition series whose sections are either hereditarily just infinite or finite cyclic of order p.

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A more general context

- Virtual composition series and virtual length can be defined also for profinite group.
- Even more: they can defined also for profinite groups with operators (once a suitable notion of profinite group with operators has been introduced).
- Basic results hold in this more general context.
- Stronger structural results do not hold however: e.g. there exist hereditarily just infinite profinite groups (hence with virtual length 1) which are not finitely generated (Wilson).
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How to characterize pro-p groups of finite virtual length?

- Pro-p groups of finite rank can be defined as pro-p groups in which the number of generators of subnormal subgroups has an upper bound.
- Pro-p groups of finite rank have finite virtual length.
- There exist groups of finite virtual length with infinite rank (e.g. the Nottingham group).
- In pro-p groups of finite virtual length every subnormal subgroup is finitely generated.

Question

Let G be a pro-p group whose subnormal subgroups are finitely generated. Is the virtual length of G finite?

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