QUOTIENTS OF MAXIMAL CLASS OF THIN LIE ALGEBRAS. THE CASE OF CHARACTERISTIC TWO

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ABSTRACT. In this paper we describe the infinite dimensional *thin* Lie algebras over a field of characteristic two which have non-metabelian factors of maximal class over terms of the lower central series.

1. INTRODUCTION

Thin infinite-dimensional graded Lie algebras can be considered as generalizations of algebras of maximal class (see [CMNS96]); the precise definition is given below. Among factors of thin algebras with respect to terms of the lower central series, there is a greatest one that is of maximal class. In odd characteristic one can prove that this largest quotient is always metabelian ([CJ97]). In characteristic two it turns out that there are thin algebras where this factor of maximal class is non-metabelian. A family of such examples is described here; we show that all thin Lie algebras with such a non-metabelian factor of maximal class are in this family.

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2. Preliminaries

As in [CJ97], all commutators are left-normed and the generalized Jacobi identity

$$[u[yx^n]] = \sum_{i=0}^{n} (-1)^i \binom{n}{i} [ux^i yx^{n-i}]$$

is used without mention in its characteristic two signless version:

$$[u[yx^n]] = \sum_{i=0}^{n} \binom{n}{i} [ux^i yx^{n-i}].$$

To compute binomial coefficients modulo two we will use Lucas' Theorem ([Luc78]): if the nonnegative integers a and b are written in base 2, that is, $a = \sum_{i=0}^{n} a_i 2^i$, $b = \sum_{i=0}^{n} b_i 2^i$, with $a_i, b_i \in \{0, 1\}$, then

$$\binom{a}{b} \equiv \prod_{i=0}^{n} \binom{a_i}{b_i} \pmod 2.$$

Note that if $a = 2^g - 1$ for some natural number g and $0 \le b \le a$, a_i is 1 for all i since

$$2^g - 1 = 2^0 + 2^1 + 2^2 + \cdots + 2^{g-1}, \tag{1}$$

and then the binomial coefficient $\binom{a}{b}$ is congruent to 1 modulo 2. We will use this fact in several instances without further mention.

Let $L = \bigoplus_{i=1}^{\infty} L_i$ be a Lie algebra over a field $\mathbf{F}$ of characteristic 2, which is graded over the positive integers. Suppose $\dim(L_1) = 2$ and $\dim(L_i) \le 2$ for all i.

We call L *thin* (see [CMNS96]) if the *covering property* holds: for each $i > 1$ and for each $0 \ne z \in L_i$ one has $[zL_1] = L_{i+1}$. The covering property implies that the algebra is generated by elements of weight one, and that $\dim(L_i) \le 2$, for each $i \ge 1$; a homogeneous component of dimension two of a thin Lie algebra is called a *diamond*.

We say that L is an *algebra of maximal class* (see [CMN97]) if $\dim(L_i) \le 1$, for $i > 1$ and it is generated by L_1 as an algebra, *i.e.* $L_i = [L_{i-1}L_1]$ for $i > 1$. If L has finite dimension d, then it has nilpotency class $d - 1$, otherwise if L is infinite-dimensional, then each finite-dimensional quotient L/L^d has this property, for $d > 1$. By convention, algebras of maximal class are not considered thin.

Finally, let L be infinite-dimensional of maximal class and define, for $i > 1$, the *two-step centralizer* C_i of the component L_i as the subspace $C_{L_1}(L_i)$ of L_1. (In the following, we refer to [CMN97] for the facts about algebras of maximal class mentioned without proof.) Choose $0 \neq y \in C_2$, so that $C_2 = \mathbf{F}y$. If all two-step centralizers coincide with $\mathbf{F}y$, then the algebra is metabelian; otherwise, let $\mathbf{F}x$ be the first two-step centralizer C_m distinct from $\mathbf{F}y$: then it can proved that $m = 2^{h+1}$, for some positive integer h: we call $q = 2^h$ the *parameter* of L. If, for some i and l, we have

$$C_i \neq \mathbf{F}y, \qquad C_{i+1} = C_{i+2} = \cdots = C_{i+l-1} = \mathbf{F}y, \qquad C_{i+l} \neq \mathbf{F}y,$$

then l can only assume the values $2q$ and $2q - 2^s$, for $0 \leq s \leq h$: we call the sequence $\{C_r\}_{r=i+1}^{i+l-1}$ a *constituent* of L. In particular, we will call a constituent *short* if it has the minimal possible length $q - 1$, *long* a maximal length one $2q-1$ and *intermediate* a constituent of any other length. If L has exactly two distinct two-step centralizers, the sequence of its constituent lengths, called *constituent sequence*, determines the algebra up to isomorphisms.

We will use an exponential notation to indicate repeating occurrences of a pattern: thus, we define

$$[\ldots (yx^{\alpha})^{\beta} \ldots] := [\ldots yx^{\alpha\beta} \ldots].$$

We will use the notation $C_i = C_{L_1}(L_i)$ also for the two-step centralizers of a thin Lie algebra: C_i will be one-dimensional, or zero when L_i or L_{i+1} is a diamond (see [CJ97]).

3. The example

For each pair of positive integers $(h, s) \in \mathbb{N} \times \mathbb{N}$, it is possible to construct a thin graded Lie algebra over the field $\mathbf{F}$ which has a non-metabelian factor of maximal class over a suitable term of the lower central series.

Let $k = 2^{h+s+2}$ and consider the two $(k+1) \times (k+1)$ matrices x and y over $\mathbf{F}[t, \frac{1}{t}]^{k+1}$ defined by:

$$x_{(i,j)} = \begin{cases} 1 & (i,j) = (2^{h+1} \cdot e + f, 2^{h+1} \cdot e + f + 1) \quad \begin{cases} e = 0, \ldots, 2^{s+1} - 1 \\ f = 1, \ldots, 2^{h+1} - 1 \end{cases} \\ \frac{1}{t^2} & (i,j) = (k+1, k) \\ 0 & \text{otherwise} \end{cases}$$

$$y_{(i,j)} = \begin{cases} 1 & (i,j) = (2^{h+1} \cdot e, 2^{h+1} \cdot e + 1) \quad e = 1, \ldots, 2^{s+1} - 1 \\ t & (i,j) = (k-1, 1) \text{ and } (i,j) = (k, 2) \\ 0 & \text{otherwise.} \end{cases}$$

A blockwise description of x and y can be given as follows: let A and B be the square matrices of dimension $2^{h+1} \times 2^{h+1}$ (where blank entries mean zero)

$$A = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \qquad B = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ 1 & & & & \end{pmatrix}.$$

Then x and y have the form

$$x = \begin{pmatrix} A & & & & & 0 \\ & \ddots & & & & \vdots \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & A & \vdots \\ & & & & \frac{1}{t^2} & 0 \end{pmatrix} \qquad y = \begin{pmatrix} & & B & & & & & 0 \\ & & & \ddots & & & & \vdots \\ & & & & \ddots & & & \vdots \\ & & & & & B & & \vdots \\ t & & & & & & & \vdots \\ & t & & & & & & \\ 0 & 0 & \dots & \dots & \dots & \dots & & 0 \end{pmatrix},$$

where the blocks A and B appear 2^{s+1} times.

Consider the Lie algebra over $\mathbf{F}$ spanned by x and y. This turns out to be a thin graded infinite-dimension Lie algebra. It has exactly the two distinct two-step centralizers y and x and its constituent sequence is

$$\left(2q-1, 2q-2, (2q-1)^{4n+1}, 2q-2, \diamond, \left(2q-2, (2q-1)^{2n}, 2q-2, \diamond\right)^{\infty}\right),$$

where $q = 2^h$, $n = 2^s - 1$ and the symbol $\diamond$ denotes a diamond of the algebra.

Every commutator between elements of weight bigger than k is zero. Therefore the algebra is solvable. This construction is analogous to the one of [CMN97, §8].

In this case, the indeterminate t plays the role of the generator of L_k, and if M is the free $k-$dimensional irreducible module $\mathbf{F}[t]^k$, the quotient algebra $A = L/L^k$ (where $L^k = \bigoplus_{i>k} L_i$ is the abelian ideal) embeds in $\mathrm{End}_{\mathbf{F}[t]}(M)$ and L can be viewed as a two-generated non-split extension of M by A. Note that the top right hand $k \times k$ minors of the above matrices x and y are just the generators of A, whilst the choice of $\frac{1}{t^2}$ is needed to have a non-trivial action of A over the copies of M contained in L. This can be verified looking at the element

$$\xi = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{2n}yx]$$

in class $k+1$, *i.e.* the first element of the first occurrence of M: the only non zero entry of its matrix is the element $\xi_{(k+1,2)} = 1$ (the top right hand $k \times k$ minor vanishes because it is actually in the abelian ideal), and so the action is non-trivial.

The representation of the above family can also be obtained, from another point of view, using Theorem 2 of [Cur54]: since the elements $h_1 = y, h_2 = x, h_3 = [yx], h_4 = [yxx], \dots, h_{k+1} = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{2n}y]$ form a regular basis for the nilpotent Lie algebra of maximal class L/L^k, if we take the irreducible polynomials $f_1 = f_2 = \cdots = f_k = Y, f_{k+1} = Y - t$ (with t as above) in $\mathbf{F}[Y]$, then there exists an irreducible representation of L such that the minimum polynomial of every matrix attached to h_i is a power of f_i; furthermore, this representation is unique up to equivalence since the f_i are linear. Now, if we write down the matrices $\alpha, \beta, [\alpha, \beta], \dots$ of h_i as elements of $\mathrm{End}_{\mathbf{F}[t]}(M)$ and we calculate their minimum polynomial, we find that these polynomials are just powers of the f_i as required, giving us another way to see the representation.

In the remaining part of the paper we will prove the following

Theorem. *Let L be an infinite-dimensional thin Lie algebra over a field of characteristic two.*

Suppose the second diamond occurs in weight k.

If L/L^k is not metabelian, then L is isomorphic to an algebra in the family constructed in this section.

4. The setup

Let L be a thin Lie algebra with the second diamond in class k. Thus L/L^k is an algebras of maximal class of dimension k. Let $q = 2^h$ be its parameter and let x and y be its generators. We suppose that this quotient is non-metabelian, that is it has at least 2 two-step centralizers and that its last constituent has length $t = 2q - 2^n - 1$ for some $0 \le n \le h$. Call v the element of L just before the diamond, u a non-zero element of weight $k - t - 3$ and write v^{-j} as a shortcut for $[uxyx^{t-j}]$, so that $[v^{-j}x^j] = v$.

Then the following relations hold:

$$[uy] = 0, \quad v^{-t} = [uxy] \neq 0, \quad [uxyx^s y] = 0 \text{ for } 0 \le s < t.$$

We want to show that almost all values of t force L to be finite dimensional, but the case shown in the previous section.

5. The easiest cases

As explained in the odd characteristic case [CJ97], the values of t which we have to deal with are essentially three:

1. $t = 2q - 1$;
2. $t = 2q - p^s - 1$ with $s > 0$;
3. $t = 2q - 2$.

Some of the computations performed in [CJ97] turn out to be independent of the characteristic of the field of definition of the algebra, so they can be used here, too: this happens when the last constituent before the second diamond has length $2q - 1$, where the arguments of the odd characteristic case can be integrally applied, while when the same constituent has length $2q - p^s - 1$ we have to check that the inequality $2q - p^s + 1 < 2q - 1$ still holds. As $q = 2^h$ and $p = 2$, this happens when $s > 1$: this implies that all cases are covered but the one with $s = 1$. As regards the last case $t = 2q - 2$, although the lemma still holds, the entire argument cannot be applied, since the coefficient that appears in it is even and thus vanishes in characteristic two.

This leaves us with two cases to consider:

1. $t = 2q - 2$ and
2. $t = 2q - 3$.

These ones need to be studied from a slightly different point of view, not looking at the last constituent only, but considering the entire structure of the involved Lie algebras.

6. The top structure

Before discussing the remaining cases, we state and prove some results about the classification of Lie algebras of maximal class that we need later. For the complete classification, see [CN98] and [Jur98].

Proposition. *Let $L = \bigoplus_{i=0}^{\infty} L_i$ be a graded Lie algebra of maximal class over a field $\mathbf{F}$ of characteristic 2. Let $q = 2^h$ be the parameter of L, and let its second constituent length be b.*

Then b cannot be $2q - 1$. If b is not $q - 1$, then all other constituents of the algebra have length b or $2q - 1$.

Furthermore, if the third constituent has also length b, then the fourth cannot have length $2q - 1$.

Proof. The proof has various steps that deal with the different possible values of the constituents: in every step we suppose the lengths are not the ones specified in the thesis and we prove that the algebra has finite dimension. Note that, for the lemma in [CJ97], we must consider the possibility of the occurrence of a third (or higher) centralizer only when short constituents other that the second are involved in the proof.

- *b cannot be $2q - 1$*

To prove the assumption, apply the generalized Jacobi identity to the element

$[yx^{2q-1}[yx^{2q-1}]]$:

$$\begin{aligned}0 &= [yx^{2q-1}[yx^{2q-1}]]\\ &= \sum_{i=0}^{2q-1}[yx^{2q-1+i}yx^{2q-1-i}]\\ &= [yx^{2q-1}yx^{2q-1}],\end{aligned}$$

so b cannot assume the value $2q-1$.

- *The length of the third constituent can only be b or $2q-1$*

Write b as $2q-2^{s_b}-1$ and call $c=2q-2^{s_c}-1$ the length of the third constituent.

Now let n be a natural number less or equal than b, and apply the generalized Jacobi identity to the element $[yx^{2q-1}yx^n[yx^{2q-1}yx^n]]$:

$$\begin{aligned}0 &= [yx^{2q-1}yx^n[yx^{2q-1}yx^n]]\\ &= \sum_{i=0}^{n}\binom{n}{i}[yx^{2q-1}yx^{n+i}[yx^{2q-1}y]x^{n-i}]\\ &= \sum_{i=0}^{n}\binom{n}{i}\{[yx^{2q-1}yx^{n+i}[yx^{2q-1}]yx^{n-i}]+\\ &+[yx^{2q-1}yx^{n+i}y[yx^{2q-1}]x^{n-i}]\}\\ &= \sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{2q-1}\{[yx^{2q-1}yx^{n+i+j}yx^{2q-1-j}yx^{n-i}]+\\ &+[yx^{2q-1}yx^{n+i}yx^{j}yx^{2q-1-j+n-i}]\}\\ &= \sum_{i=0}^{n}\sum_{j=0}^{2q-1}\binom{n}{i}[yx^{2q-1}yx^{n+i+j}yx^{2q-1-j}yx^{n-i}]+\\ &+\sum_{i=0}^{n}\sum_{j=0}^{2q-1}\binom{n}{i}[yx^{2q-1}yx^{n+i}yx^{j}yx^{2q-1-j+n-i}].\end{aligned}\tag{2}$$

The elements of the first sum are always zero except for $n+i+j=b$ and $2q-1-j=c$, while the only non-zero term of the second sum has $n+i=b$ and $j=c$, so that equation (2) reduces to

$$0=\left\{\binom{n}{2n+2q-(b+c+1)}+\binom{n}{2n-b}\right\}\cdot\\ \cdot[yx^{2q-1}yx^{b}yx^{c}yx^{2n+2q-(b+c+1)}].\tag{3}$$

To show that the algebra has finite dimension under the initial hypothesis we have to find a natural number n such that the coefficient of the element in (3) is odd (*i.e.* it is non-zero as an element of $\mathbf{F}$) and such that $2n+2q-(b+c+1)$ cannot be an admissible length for the fourth constituent (it will be sufficient that $2n+2q-(b+c+1)\leq q-1$).

The first case occurs when $b=2q-2$ (*i.e.* $s_b=0$), that is the only even possible value for a constituent. Since by hypothesis $c\neq 2q-1, 2q-2$, the sum $b+c$ is odd, so the number $n=\frac{b+c+1}{2}-q$ is a positive integer less than b; furthermore $2n-b<0$ because $c<2q-2$ and $2n+2q-(b+c+1)=b+c+1-2q+2q-(b+c+1)=0$, so that (3) becomes

$$0=\binom{n}{0}[yx^{2q-1}yx^{2q-2}yx^{c}y],$$

and the algebra must be finite-dimensional by definition of constituent.

Simmetrically, the situation turns out to be identical when $c=2q-2$, because, in such a case, b (which for the previous proof cannot be $2q-1$) cannot even be $2q-2$, so if we take n defined as above the same argument applies.

The general case occurs when both b and c are less than $2q-2$ or, equivalently, s_b and s_c are both bigger than 0 and then $b+c$ is even. Consider the quantity

$$\frac{b+c}{2}-q=\frac{2q-2^{s_b}-1+2q-2^{s_c}-1}{2}-q=q-1-2^{s_b-1}-2^{s_c-1}.$$

By (1), $\frac{b+c}{2}-q$ is the sum of all powers of 2 from 2^0 to $q=2^{h-1}$ except the two different powers 2^{s_b-1} and 2^{s_c-1}; so, if we put $\xi=\min\{s_b-1, s_c-1\}$, the number $n=\frac{b+c}{2}-q+2^{\xi}$ will be the right choice in (3), because

$$2n-b=b+c-2q+2^{\xi+1}-b=2^{\xi+1}-2^{s_c}-1<0, \tag{4}$$

as $\xi\leq s_c-1$, and

$$\begin{aligned}2n+2q-(b+c+1)&=b+c-2q+2^{\xi+1}+2q-b-c-1\\&=2^{\xi+1}-1\leq q-1,\end{aligned} \tag{5}$$

as $\xi\leq h-1$.

Equation (4) implies that the first binomial coefficient in (3) is zero, while (5) implies that the second one is congruent to 1 modulo 2, in view of (1) and Lucas' Theorem:

$$\begin{aligned}\binom{n}{2q+2n-(b+c+1)}&=\binom{\frac{b+c}{2}-q+2^{\xi}}{2^{\xi+1}-1}\\&=\binom{2^0+\cdots+2^{\xi}+\cdots+2^{s_b-2}+\cdots+2^{h-1}}{2^0+\cdots+2^{\xi}}\\&\equiv\underbrace{\binom{1}{1}\cdots\binom{1}{1}}_{\xi+1}\binom{1}{0}\cdots\binom{1}{0}\equiv 1 \pmod 2.\end{aligned}$$

So, equation (3) turns out to be

$$0 = [yx^{2q-1}yx^byx^cyx^{2^{\xi+1}-1}],$$

and the algebra has finite dimension also in this case.

Now we generalize the above calculations to any constituent: supposing that the second constituent is not short, we prove by induction that the possible constituent lengths are b or $2q-1$, assuming that the previous constituent has such a length. If we call c the length of the constituent we are studying and f the length of the previous one, we consider four different cases and we show the algebra could be infinite-dimensional only if $c = 2q-1$ or $c = b$.

- $f = b$, $c > b$

Let η the element before the constituent of length $f = b$: then η is of the form

$$\eta = [yx^{2q-1}yx^by\cdots y].$$

Apply the generalized Jacobi identity to the element

$$[\eta x^{2^{s_b}-1}[yx^{2q-1}yx^{b+1}]];$$

calculation are similar to those in (2), so we report here only the last step:

$$\begin{aligned}
0 &= [\eta x^{2^{s_b}-1}[yx^{2q-1}yx^{b+1}]] \\
&= \left\{\binom{b+1}{b+c-(2q+2^{s_b}-2)} + \binom{b+1}{b-2^{s_b}+1}\right\}\cdot \\
&\cdot [\eta x^byx^cyx^{2q+2^{s_b}-1-c}].
\end{aligned} \tag{6}$$

Using the definition of b, c, the properties of the binomial coefficients, the fact that $2^r - 2^s = 2^{r-1} + 2^{r-2} + \cdots + 2^s$ (a generalization of (1)), the hypothesis $c > b$ (that implies $s_c < s_b$) and Lucas' Theorem, (6) becomes

$$\begin{aligned}
0 &= \left\{\binom{2q-2^{s_b}}{2^{s_b}+2^{s_c}} + \binom{2q-2^{s_b}}{2^{s_b}}\right\}[\eta x^byx^cyx^{2^{s_b}+2^{s_c}}] \\
&= \left\{\binom{2^h+\cdots+2^{s_b}}{2^{s_b}+2^{s_c}} + \binom{2^h+2^{h-1}+\cdots+2^{s_b}}{2^{s_b}}\right\}[\eta x^byx^cyx^{2^{s_b}+2^{s_c}}] \\
&= \{0+1\}[\eta x^byx^cyx^{2^{s_b}+2^{s_c}}],
\end{aligned}$$

if $2^{s_c} \neq 0$, that is $c \neq 2q-1$; moreover, as $b \neq q-1$ we have that $s_c < s_b < h$, and hence $2^{s_b} + 2^{s_c} \leq 2^{h-1} + 2^{h-2} + \cdots + 2^0 = q-1$: therefore the thesis.

- $f = b$, $c < b$

This is one of the two cases where we may consider the possibility that after c may appear a different two-step centralizer.

First of all, suppose this is not the case, consider the element μ such that $[\mu y] = \eta$ and apply the Jacobi identity to $[\mu[yx^{2q-1}yx^c y]]$:

$$
\begin{aligned}
0 &= [\mu[yx^{2q-1}yx^c y]] \\
&= [\mu[yx^{2q-1}yx^c]y] + [\mu y[yx^{2q-1}yx^c]] \\
&= \sum_{i=0}^{c} \binom{c}{i} [\mu x^i [yx^{2q-1}y] x^{c-i} y] + \\
&+ \sum_{i=0}^{c} \binom{c}{i} [\mu y x^i [yx^{2q-1}y] x^{c-i}] \\
&= \sum_{i=0}^{c} \sum_{j=0}^{2q-1} \binom{c}{i} \left\{ [\mu x^{i+j} yx^{2q-1-j} yx^{c-i} y] + \right. \\
&+ [\mu x^i yx^j yx^{2q-1-j+c-i} y] + \\
&+ [\mu y x^{i+j} yx^{2q-1-j} yx^{c-i}] + \\
&+ \left. [\mu y x^i yx^j yx^{2q-1-j+c-i}] \right\}.
\end{aligned}
\tag{7}
$$

By the usual calculations and the hypothesis $c < b$, we obtain that the only non zero term occurs for $i + j = b$ and $2q - 1 - j = c$ in the first and the third elements of the above sum, and the corresponding associated binomial coefficient can be evaluated as an element of $\mathbf{F}$ in this way:

$$
\begin{aligned}
\binom{c}{b + c - 2q + 1} &= \binom{2q - 1 - 2^{s_c}}{2q - 1 - 2^{s_c} - 2^{s_b}} \\
&= \binom{2q - 1 - 2^{s_c}}{2^{s_b}} \\
&= \binom{2^h + 2^{h-1} + \cdots + 2^{s_c+1} + 2^{s_c-1} + \cdots + 2^0}{2^{s_b}} \\
&\equiv 1 \pmod{2}.
\end{aligned}
$$

So, equation (7) reads

$$0 = [\mu y x^b y x^c y x^{2q-1-b}],$$

that implies that the algebra is finite-dimensional, as $b > q$; this ends the first case.

Now suppose c short and centralized by an element $\varphi = x - \alpha y$ with $\alpha \in \mathbf{F} - \{0\}$. Using the same formula as in (7) we obtain:

$$
\begin{aligned}
0 &= [\mu y x^b y x^{3q-b-2} y] + [\mu y x^b y x^{q-1} y x^{2q-b-1}] \\
&= [\mu y x^b y x^{q-1} x x^{2q-b-2} y] + [\mu y x^b y x^{q-1} y x^{2q-b-1}] \\
&\simeq [\mu y x^b y x^{q-1} y x^{2q-b-2} y] + [\mu y x^b y x^{q-1} y x^{2q-b-1}].
\end{aligned}
$$

But $b > q - 1$, so $2q - b - 2 < q - 1$, that is not a possible constituent length, and thus the first element of the last equation is zero. This implies that the second element must be zero, too, obtaining the same conclusion of the first case.

- $f = 2q-1$, $c > b$

 Expanding the Jacobi relation as in the first case, we obtain

$$\begin{aligned} 0 &= [\eta x^{2^{s_b}-1}[yx^{2q-1}yx^{b+1}]] \\ &= \left\{ \binom{b+1}{c-2^{s_b}+1} + \binom{b+1}{2q-2^{s_b}} \right\} [\eta x^{2q-1}yx^c yx^{b-c+2^{s_b}}]. \end{aligned}$$

 By usual arguments, the two binomial coefficients can be written:

$$\begin{aligned} \binom{b+1}{c-2^{s_b}+1} &= \binom{2q-2^{s_b}}{2q-2^{s_c}-2^{s_b}} \\ &= \binom{2q-2^{s_b}}{2^{s_c}} \\ &= \binom{2^h+2^{h-1}+\cdots+2^{s_b}}{2^{s_c}} \\ &\equiv 0 \pmod 2, \end{aligned}$$

if $c \neq 2q-1$, and

$$\binom{b+1}{2q-2^{s_b}} = \binom{2q-1-2^{s_b}+1}{2q-2^{s_b}} = 1,$$

so obtaining

$$[\eta x^{2q-1}yx^c yx^{b-c+2^{s_b}}] = 0,$$

that implies that the algebra has finite dimension again, as $b - c + 2^{s_b} = 2q-1-2^{s_b}-2q+1+2^{s_c}+2^{s_b} = 2^{s_c} \leq q-1$.

- $f = 2q-1$, $c < b$

 We must again consider the possibility that, when c is short, it could be centralized by something different from x, but now it is easier: using the same calculation (7) of the second case, we obtain four non-zero terms, those with couples $(i,j) = (0,0), (0,2q-1), (c, 2q-1-c), (2q-1, c)$, whose associated binomial coefficients are

$$\binom{c}{0} = \binom{c}{c} = 1 \qquad \text{and} \qquad \binom{c}{2q-1} \equiv 0 \pmod 2,$$

so that (7) becomes

$$[\mu y x^{2q-1} y x^c y] = 0,$$

and the algebra is always finite-dimensional whatever the centralizer of c is.

Now the last statement of the thesis:

• *If the third constituent has length b, the fourth can not have length $2q-1$*

Apply the Jacobi identity to the element $[yx^{2q-1}yx^b[yx^{2q-1}yx^b]]$:

$$\begin{aligned}
0 &= [yx^{2q-1}yx^b[yx^{2q-1}yx^b]] \\
&= \left\{ \binom{b}{b} + \binom{b}{2q-1} \right\} [yx^{2q-1}yx^b yx^b yx^{2q-1}] \\
&= [yx^{2q-1}yx^b yx^b yx^{2q-1}].
\end{aligned}$$

□

To conclude, an observation: if the second constituent is short, the above limitations to the other constituents lengths do not hold anymore and so the structure of the algebra cannot be determined in a similar way and we cannot say anything about it.

7. The case $t = 2q-3$

First of all, let us suppose $[uxx]$ to be different from zero: this implies that $[uxx] = \mu[uxy]$ for some $\mu \in \mathbf{F} - \{0\}$; then, since $\binom{2q}{i}$ is non zero only for $i=0$ and $i=2q$, we have:

$$\begin{aligned}
0 &= [u[yx^{2q}]] \\
&= \sum_{i=0}^{2q} \binom{2q}{i} [ux^i yx^{2q-i}] \\
&= [uyx^{2q}] + [ux^{2q}y] \\
&= \mu[uxyx^{2q-2}y] \\
&= \mu[vxy].
\end{aligned}$$

But a standard calculation, as in [CMNS96], yields

$$\begin{aligned}
0 &= [v^{-1}[yxy]] \\
&= [v^{-1}xyy] + 2[v^{-1}yxy] + [v^{-1}yyx] \\
&= [vyy],
\end{aligned}$$

and the algebra cannot be infinite-dimensional.

So it remains to study the case $[uxx] = 0$. Suppose first $[vxx]$ to be non-zero; then take ϑ the element such that $[\vartheta yx^{2q-3}] = v$ and compute:

$$0 = [\vartheta[yx^{2q}]] = [\vartheta yx^{2q}] = [vxxx]$$
$$0 = [\vartheta y[yx^{2q-2}y]] = [\vartheta yx^{2q-2}yy] = [vxyy].$$

Now, $[vxx]$ and $[vxy]$ must be linearly dependent, so $[vxx] = \lambda[vxy]$, with $\lambda \in \mathbf{F} - \{0\}$: but then the second equation reads $[vxxy] = 0$ and thus $[vxx]$ is central, against the covering property.

So we can consider $[vxx] = 0$, too.

At this point we have to consider some subcases, depending on the length of the second constituent and on the length of the constituent just before the one of length t, that from now on we will denote by f. In fact, the results of Section (6) show that it is necessary to distinguish the cases with b short and non short: in this last case, we know that b can only be equal to t.

7.1. $\mathbf{b = 2q - 3}$. Now, always by the results of Section (6), we have that f can only be long or equal to b.

7.1.1. $f = 2q - 1$. Take s the element just before the constituent f:

$$\begin{aligned} 0 &= [s[yx^{2q-1}yx^{2q-2}]] \\ &= \binom{2q-2}{i}[sx^i yx^j yx^{4q-3-(i+j)}] + [sx^{i+j}yx^{2q-1-j}yx^{2q-2-i}] \\ &= [vxy], \end{aligned}$$

and the algebra has finite dimension again (remember that $[vyy] = 0$).

7.1.2. $f = 2q - 3$. Here we must suppose $q \neq 2$, so we will have to study this case separately. Using the same s as above, compute:

$$\begin{aligned} 0 &= [s[yx^{2q-1}yx^{2q-4}y]] \\ &= \binom{2q-4}{i}[sx^i yx^j yx^{4q-5-(i+j)}y] + [sx^{i+j}yx^{2q-1-j}yx^{2q-4-i}y] + \\ &+ [syx^i yx^j yx^{4q-5-(i+j)}] + [syx^{i+j}yx^{2q-1-j}yx^{2q-4-i}] \\ &= [vxyy]. \end{aligned}$$

Now either $[vxy] = 0$, and then we are done, or not. If $[vxy]$ does not vanish, but $[vxyx] = 0$ we finish again, otherwise

$$\begin{aligned} 0 &= [s[yx^{2q-1}yx^{2q-2}]] \\ &= \binom{2q-2}{i}[sx^i yx^j yx^{4q-3-(i+j)}] \\ &\quad + [sx^{i+j}yx^{2q-1-j}yx^{2q-2-i}] \\ &= [vxyxx] \end{aligned}$$

and

$$\begin{aligned} 0 &= [v^{-3}[yx^5y]] \quad \text{(remember } q > 2) \\ &= [vxyxy], \end{aligned}$$

show that $[vxyx]$ is central, against the covering property again.

7.2. $\mathbf{b = q - 1}$. Here f has no limitation on its length, so we will consider the cases f not short and f short.

7.2.1. $f \neq q-1$. Take an element k exactly $q+1$ places before the last constituent (this is possible if its length $2q-2^g-1$ is at least $q+1$, *i.e.* if $q \neq 2$) and compute:

$$\begin{aligned}
0 &= [k[yx^{2q-1}yx^q]] \\
&= \sum_{i=0}^{q} \binom{q}{i} [kx^i[yx^{2q-1}y]x^{q-i}] \\
&= \sum_{i=0}^{q} \sum_{j=0}^{2q-1} \binom{q}{i} \{[kx^{i+j}yx^{2q-1-j}yx^{q-i}]+ \\
&+ [kx^iyx^jyx^{3q-1-(j+i)}]\}.
\end{aligned}$$

The only couple (i,j) which gives a nonzero term in the previous sum is $(q,1)$, and for such a value the above equation reduces to

$$0 = [kx^{q+1}yx^{2q-2}y] = [vxy],$$

and so this subcase is completed, too.

7.2.2. $f = q-1$. Suppose $q > 2$ (for $q = 2$ it is just $b = f = 2q-3$) and take k three elements before the last constituent:

$$\begin{aligned}
0 &= [k[yx^{2q-1}yx^q]] \\
&= \sum_{i=1}^{q} \binom{q}{i} [kx^i[yx^{2q-1}y]x^{q-i}] \\
&= [k[yx^{2q-1}y]x^q] + [kx^q[yx^{2q-1}y]] \\
&= [kxxyx^{2q-3}yx^q] \\
&= [vyx^q] \\
&\simeq [vxyx^{q-2}x].
\end{aligned}$$

Note that $[kx^q[yx^{2q-1}y]]$ is zero because $[kxx]$ must be centralized by x, as the next constituent is not short.

Furthermore

$$\begin{aligned}
0 &= [v^{-(q-1)}[yx^{2q-2}y]] \\
&= [v^{-(q-1)}[yx^{2q-2}]y] \\
&= \sum_{i=1}^{2q-2} \binom{2q-2}{i} [v^{-(q-1)}x^iyx^{2q-2-i}y] \\
&= [vxyx^{q-2}y],
\end{aligned}$$

and so $[vxyx^{q-2}]$ is central and the algebra has finite dimension.

7.3. **The $\mathbf{q = 2}$ remaining cases.** Here the algebra starts out with constituents 3 and 1, has $t = 1$ and f may assume the values 3, 2 or 1.

If f is 3, take d as the element before the constituent of length f and compute:

$$
\begin{aligned}
0 &= [d[yxxxyxx]] \\
&= \sum_{i=0}^{2}\sum_{j=0}^{3}\binom{2}{i}[wx^iyx^jyx^{5-(i+j)}] + [wx^{i+j}yx^{3-j}yx^{2-i}] \\
&= [dxxxyxxy] \\
&= [vxy],
\end{aligned}
\tag{8}
$$

and so every element of weight bigger than $k+2$ is zero.

If f is 2, the above equation gives $[vxyx] = 0$, that, together with

$$
\begin{aligned}
0 &= [v^{-1}[yxxy]] \\
&= [v^{-1}[yxx]y] + [v^{-1}y[yxx]] \\
&= \sum_{i=1}^{2}\binom{2}{i}[v^{-1}x^iyx^{2-i}y] \\
&= [v^{-1}xxyy] \\
&= [vxyy],
\end{aligned}
$$

shows the centrality of $[vxy]$ and so the assumption.

If f is 1 we may also consider the possibility of the presence of a different centralizer φ after the centralizer f; but then take d as above and expand:

$$
\begin{aligned}
0 &= [d[yxxxx]] \\
&= [dxxxxy] \\
&= [dx(\varphi - \alpha y)xxy] \\
&\simeq [dxyxxy] \\
&= [vxy].
\end{aligned}
$$

So we may assume that f is centralized by x and use (8) to obtain $[vxyxx] = 0$ and $[v[yxy]] = 0$ to obtain $[vxyy] = 0$. Thus, if $[vxyx]$ is zero, we find a central element $[vxy]$, otherwise we have to study the whole structure of the algebra to conclude.

First of all, note that in the case of two short constituents before the diamonds, if we consider the maximal class branch of the thin algebra raising from the element $[vxy]$, we see that this algebra has really a constituent of length 2 and if this branch ends, the entire thin Lie algebra has to be finite dimensional. So we start studying the behaviour of a length 2 constituent in a Lie algebra of maximal class that begins with constituent sequence $(3, 1, \dots)$.

It can be shown that, if the first non short constituent after the top one is long, then 2 cannot appear as an admissible constituent length, so we can never have the situation we are studying, whilst if such a constituent is short the Lie algebra of maximal class that rises is periodic, so we cannot have a second diamond in this case, too. In fact, when the algebra starts out with constituent length sequence $(3, 1^g, 3)$, first of all we can see using just one deflation that $g = 2^s - 3$ for some power $s \geq 2$: in fact if we write explicitly down the top part of the constituent sequences of the original and the deflated algebra we get

$$\begin{array}{llll}
C_2 = \mathbf{F}y & C_3 = \mathbf{F}x & & \\
C_{2i+2} = \mathbf{F}x & C_{2i+3} = \ Fy & \rightsquigarrow & C^{\downarrow}_{i+1} = \mathbf{F}x' \quad \text{for } i = 1 \dots g \\
C_{2g+4} = \mathbf{F}x & C_{2g+5} = \mathbf{F}y & \rightsquigarrow & C^{\downarrow}_{g+2} = \mathbf{F}x' \\
C_{2g+6} = \mathbf{F}y & C_{2g+7} = \mathbf{F}y & \rightsquigarrow & C^{\downarrow}_{g+3} = \mathbf{F}y' \\
C_{2g+8} = \mathbf{F}x & C_{2g+9} = \mathbf{F}y & \rightsquigarrow & C^{\downarrow}_{g+4} = \mathbf{F}x'
\end{array}$$

and so $g + 3 - 2$ has to be of the kind $2q - 2$, and so the thesis. Then, if we suppose there is a length 2 constituent somewhere in the algebra we can consider its first occurrence and call m the element just before the beginning of the constituent immediately before the length 2 one; but now

$$\begin{aligned}
0 &= [m[yxxxyxx]] \\
&= \sum_{i=0}^{2}\sum_{j=0}^{3}\binom{2}{i}[mx^i yx^j yx^{5-i-j}] + [mx^{i+j}yx^{3-j}yx^{2-i}] \\
&= [mxxxyxxy],
\end{aligned}$$

and this implies that before the length 2 constituent there must be at least a short one. So we are in the situation $(3, 1^g, 3, \dots, 3, 1^k, 2 \dots)$. Now call r the element just before the branch of sequence $(\dots 3, 1^k, 2, \dots)$; then there are two possibilities, depending on k: if $k \geq g$ expanding

$$0 = [r[yxxx(yx)^g yxy]],$$

we obtain that the algebra has finite dimension (if $g > 1$), whilst if $k < g$ we obtain the same result applying

$$0 = [r[yxxx(yx)^k yxx]]$$

for k odd and

$$0 = [c[yxxx(yx)^{k+1} yxx]]$$

for k even, where $r = [cyx]$.

When $g = 1$ we can see that there are two possibilities, depending on the constituent after the second long one. First of all, expanding

$$0 = [yxxxyxyxy[yxxxyxx]]$$

we note that this constituent cannot be intermediate. If this is a short one, the algebra goes on periodic with constituents sequence $(3, 1)^\infty$ (this facts can be seen using the two zero relations $[yxxxyxyxy]$ and $[yxxyxx]$). Finally, suppose it is long and suppose further that there is a constituent of length 2 somewhere in an upper class of the grading; then take the first occurrence of such a constituent: from above, we know that it follows a short constituent and, using the relations just seen for the previous cases, we show that there can be only one short constituent between the intermediate one we are considering and the previous non short one (that in view of our assumption is long). So we are in the situation $(3, 1, 3, 3, \dots, 3, 1, 2)$, but also this can be shown to be a finite-dimensional Lie algebra: call $\varkappa$ the element just before the sequence $(3, 1, 2, \dots)$ and expand $0 = [\varkappa xx[yxxxyxyxy]]$ and $0 = [\varkappa xxxy[yxxxyxx]]$ to see that, if we call a the weight of the elements at the end of the sequence $(\dots 3, 1, 2)$, every commutator of weight $a + 2$ is zero.

When the first non short constituent is an intermediate one (with g short constituents before it) we can prove, using essentially relations similar to those used for the previous case, that when the algebra is infinite dimensional it is periodic with period (after the top 3) of kind $(1^g, a_1, \dots, 1^g, a_n)$ where a_i can be 2 or 3: also this result implies that every element of the grading is one-dimensional, so there is no diamonds other than the top one.

8. The case $t = 2q - 2$

Also in this case, first of all suppose $[uxx] \neq 0$ and compute as above

$$\begin{aligned} 0 &= [u[yx^{2q}]] \\ &= \sum_{i=0}^{2q} \binom{2q}{i} [ux^i yx^{2q-i}] \\ &= [uyx^{2q}] + [ux^{2q}y] \\ &= \mu[uxyx^{2q-2}y] \\ &= \mu[vy], \end{aligned}$$

so there is not a diamond after the element v, against the hypothesis.

So we can assume $[uxx] = 0$ and compute

$$\begin{aligned} 0 &= [[ux][yx^{2q}]] \\ &= \sum_{i=0}^{2q} \binom{2q}{i} [ux^{i+1} yx^{2q-i}] \\ &= [uxyx^{2q}] + [ux^{2q+1}y] \\ &= [vxx]. \end{aligned}$$

Then $[vxx] = 0$ and (by the covering property) $[vxy]$ and $[vyx]$ are linearly dependent. Furthermore, we can apply the same calculation used in [CJ97], obtaining $[vxyx^j y] = 0$ for $0 \leq j \leq q - 2$.

As in the previous section, we have two main subcases, depending on the two possible lengths of the second constituent.

8.1. $\mathbf{b} = \mathbf{q} - \mathbf{1}$. If, in the same notation as above, f is not short, we can find an element w such that $[wx^q y] \neq 0$ and $[wy] = 0$. But then

$$\begin{aligned} 0 &= [w[yx^{2q-1}yx^q]] \\ &= \sum_{i=0}^{q} \sum_{j=0}^{2q-1} \binom{q}{i} [wx^i yx^j yx^{3q-1-(i+j)}] + [wx^{i+j} yx^{2q-1-j} yx^{q-i}] \\ &= [vyx]. \end{aligned}$$

Otherwise, if f is short (note that even here the centralizer after this constituent must be x, because there is not another short constituent after it) we can take the element r such that $[rxyx^t] = v$ and compute

$$\begin{aligned} 0 &= [r[yx^{2q-1}yx^q]] \\ &= \sum_{i=0}^{q} \binom{q}{i} [kx^i[yx^{2q-1}y]x^{q-i}] \\ &= [k[yx^{2q-1}y]x^q] + [kx^q[yx^{2q-1}y]] \\ &= [kxyx^{2q-2}yx^q] \\ &= [vyx^q]. \end{aligned}$$

But this last term can be written as $[vyxx^{q-2}x] \simeq [vxyx^{q-2}x] = 0$, that implies, together with the above identity $[vxyx^j y] = 0$ for $0 \leq j \leq q-2$, that $[vxyx^{q-2}]$ is a central element.

8.2. $\mathbf{b} = \mathbf{2q} - \mathbf{2}$. This is the hardest case: in fact, the length of the third and the fourth constituent (that will be called respectively b_3 and b_4) determines a different behaviour of the algebra that must be studied separately. In view of the results of Section (6), they both can be long or equal to the second one b, but if the third constituent is not long, the fourth cannot be long, too.

Thus there three subcases to consider:

- $b_3 = b_4 = 2q - 2$.
- $b_3 = 2q - 1, b_4 = 2q - 2$.
- $b_3 = b_4 = 2q - 1$.

Before analyzing the single cases, we introduce here a new technique to expand the generalized Jacobi identities involved in the sequel, similar to that used in the case $t = 2q-2$ of [CJ97]. Since from now on, for what we have seen in Section (6), we will not meet anymore short constituents, we can without loss of generality assume that the Lie algebras we are studying have only the two centralizers y and x. Then consider the element $z = x + y$ and take any element v: then $[vz]$ turns out to be $[vx]$ if v is centralized by y and $[vy]$ if v is centralized by x. This means that a non zero commutator $[yx_1x_2 \cdots x_n]$ can be written in the form $[yz^n]$.

Furthermore note what happens when the generalized Jacobi involves the element v: the result of the expansion contains the term $[vyx\cdots]$ and $[vxy\cdots]$ (we already known $[vxx]=[vyy]=0$); each one is multiplied for an appropriate sum, that we call S_1 and S_2. Since we known nothing about the structure of the algebra for the elements in class bigger than $k+1$, the sums have as their last term the one involving the constituent of length t just before v: this implies that S_1 and S_2 coincide up to the last term, so that these two last terms turns out to be the only relevant ones for the final calculation.

8.2.1. $b_3=b_4=2q-2$. First of all, for what we have seen in Section (6), we have no short constituents in all the maximal class top part of the algebra, so we can always find an element η q places before the beginning of the f constituent.

If, in particular, f is long, we can expand

$$\begin{aligned}
0 &= [\eta x[yx^{2q-1}yx^{2q-2}yx^{2q-2}x]] \\
&= [\eta x[yz^{6q-3}]x]+[\eta xx[yz^{6q-3}]] \\
&= \left\{S+\binom{6q-3}{q-2}+S+\binom{6q-3}{q-1}\right\}[vxyz^{q-2}x]+ \\
&+\left\{T+\binom{6q-3}{q-1}+T+\binom{6q-3}{q}\right\}[vxyz^{q-2}z] \\
&= [vxyx^{q-2}x],
\end{aligned} \tag{9}$$

as $[vxy]$ and $[vyx]$ are linearly dependent and we already know that

$$[vxyx^{q-2}y]=0.$$

This shows that $[vxyx^{q-2}]$ is a central element.

If, otherwise, f is not long, we know that f must be $2q-2$ and we can use a slightly different version of (9) to obtain

$$\begin{aligned}
0 &= [\eta[yx^{2q-1}yx^{2q-2}yx^{2q-2}x]] \\
&= [\eta[yz^{6q-3}]x]+[\eta x[yz^{6q-3}]] \\
&= \left\{S+\binom{6q-3}{q-2}+S+\binom{6q-3}{q-1}\right\}[vxyz^{q-2}x]+ \\
&+\left\{T+\binom{6q-3}{q-1}+T+\binom{6q-3}{q}\right\}[vxyz^{q-2}z] \\
&= [vxyx^{q-2}x],
\end{aligned} \tag{10}$$

and so the same conclusion.

8.2.2. $b_3=2q-1, b_4=2q-2$. Here we can expand:

$$0=[\psi[yx^{2q-1}yx^{2q-2}yx^{2q-1}yx^{2q-2}x]], \tag{11}$$

changing the element ψ owing to different values of the constituent f and of the one just before, that we denote f_3; in fact we must choose ψ such that $[\psi x^a y x^{f_3} y x^f y x^t] = v$ where

$$\begin{cases} \text{a=q+1} & \text{if } f = f_3 = 2q-2 \text{ (note that this choice is possible only if } q > 2) \\ \text{a=q} & \text{if } f_3 = 2q-1 \text{ and } f = 2q-2 \text{ or if } f = 2q-1 \text{ and } f_3 = 2q-2 \\ \text{a=q-1} & \text{if } f = f_3 = 2q-1. \end{cases}$$

So, expanding equation (11), we have in any case

$$\begin{aligned} 0 &= [\psi[yx^{2q-1}yx^{2q-2}yx^{2q-1}yx^{2q-2}x]] \\ &= [\psi[yx^{2q-1}yx^{2q-2}yx^{2q-1}yx^{2q-2}]x] + \\ &+ [\psi x[yx^{2q-1}yx^{2q-2}yx^{2q-1}yx^{2q-2}x]] \\ &= \left\{S + \binom{8q-3}{q-2} + S + \binom{8q-3}{q-1}\right\}[vxyz^{q-2}x] + \\ &+ \left\{T + \binom{8q-3}{q-1} + T + \binom{8q-3}{q}\right\}[vxyz^{q-2}z] \\ &= [vxyx^{q-2}x], \end{aligned}$$

so we can conclude as in the previous cases.

(Note that the generalized Jacobi identity can also be expanded directly using the definition as previously: clearly the results are the same, but they need a bigger computational work.)

The case we have left occurs when $q = 2$ and $b = 2$: here our claim is that such an algebra is periodic, that is, its constituent sequence is $(3, 2)^\infty$. We will prove this by induction on the exponent of the period, where the case $n = 1$ is just our hypothesis. So suppose that the constituent sequence is periodic for n times, that is $(3, 2)^n$: we may prove that it goes on with another $(3, 2)$. So, if we call e the element $[yxxxyxx^{n \text{ times}}]$, we can prove that

$[ey]$ is zero: take r such that $[rxy] = e$ and expand

$$0 = [r[yxy]] = [rxyy] = [ey].$$

$[exy]$ is central:

$$0 = [e[yxy]] = [exyy],$$

and, taking η such that $[\eta yxxy] = e$,

$$0 = [\eta[yxxxyxy]] = [\eta yxxyxyx] = [exyx].$$

$[exxy]$ lies in the second centre of the algebra:

$$0 = [ex[yxy]] = [exxyy],$$

then take ν such that $[\nu xxyxxxyxxy] = e$:

$$0 = [\nu x[yxxxyxxyxxxyxy]] = [exxyxy];$$

and

$$0 = [\nu[yxxxyxxyxxxyxxx]] = [exxyxx].$$

$[exxxx]$ is zero:

$$0 = [rx[yxxxx]] = [rxyxxxx] = [exxxx].$$

$[exxxyy]$ is zero:

$$0 = [exx[yxy]] = [exxy[yx]] + [exx[yx]y] = [exxyyx] + [exxxyy].$$

$[exxxyxy]$ is zero:

$$0 = [rx[yxxxyxy]] = [rxyxxxyxy] = [exxxyxy].$$

$[exxxyxxx]$ is a central element:

$$0 = [exxx[yxxxx]] = [exxxyxxxx];$$

and

$$\begin{aligned} 0 &= [\nu xxy[yxxxyxxyxxxyxxx]] \\ &= [\nu xxyxxxyxxyxxxyxxxy] \\ &= [exxxyxxxy]. \end{aligned}$$

So the algebra is periodic, thus there cannot be another diamond other than the first one, because one of its element would lie in the first or in the second centre of the algebra.

8.2.3. $b_3 = b_4 = 2q-1$. In this case, the methods used so far do not yield any results, so we do something different. The aim of this last section is to show that an algebra of maximal class with the above initial characteristic is periodic and so cannot turn out to be thin.

By hypothesis, the algebra starts out with constituent sequence

$$\left(2q-1, 2q-2, (2q-1)^2\right).$$

For what we have seen in Section (6) the next constituents can only be of length $2q-1$ or $2q-2$, so we can describe the constituent sequence as

$$\begin{aligned} \bigl(2q-1, 2q-2, (2q-1)^k, (2q-2)^l, (2q-1)^{k_1}, (2q-2)^{l_1}, \ldots \\ \ldots, (2q-1)^{k_i}, (2q-2)^{l_i} \ldots\bigr). \end{aligned}$$

If there are no more intermediate constituent, the algebra goes on of maximal class with sequence

$$\left(2q-1, 2q-2, (2q-1)^\infty\right),$$

so suppose there is at least another intermediate one.

First of all, we prove that $l_i = 1$ or 2, that is, there cannot be 3 (or more) successive non long constituents. In fact, take an element g one class before the three consecutive non long constituents, and expand, in the same notation as above,

$$
\begin{aligned}
0 &= [g[yx^{2q-1}yx^{2q-2}yx^{2q-2}y]] \\
&= [g[yx^{2q-1}yx^{2q-2}yx^{2q-2}]y] \\
&= \left\{ \binom{6q-3}{1} + \binom{6q-3}{2q} + \binom{6q-3}{4q-1} \right\} \cdot \\
&\cdot [gxyx^{2q-2}yx^{2q-2}yx^{2q-2}y] \\
&= [gxyx^{2q-2}yx^{2q-2}yx^{2q-2}y].
\end{aligned} \tag{12}
$$

Now we want to show that $k_i = k$ and $l_i = l$ for all i, and that the pair (k, l) can only assume the values $(2^g - 1, 1)$ or $(2^g - 2, 2)$ for some g, otherwise the algebra has finite dimension. To prove this, consider separately the cases k even and k odd and suppose we only know that the algebra has $\big(2q-1, 2q-2, (2q-1)^k, 2q-2\big)$ as its initial constituents sequence.

k odd. If k is odd, we show that, if we call e the element

$$e = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}y],$$

then the elements $[e(yx^{2q-1})^a yx^{2q-2}y]$ are zero or central for all values $0 \le a < k$: this would imply that the algebra is finite-dimensional in view of the previous observation on the possible constituents lengths.

If $a = 0$, consider $\beta = [yx^{2q-1}yx^{2q-2}y]$ and expand:

$$
\begin{aligned}
0 &= [\beta[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}x]] \\
&= [\beta[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}]x] + \\
&+ [\beta x[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}]] \\
&= \left\{ \binom{2q(k+3)-3}{2q-2} + \sum_{\xi=0}^{k-1} \binom{2q(k+3)-3}{2q(\xi+2)-2} + \binom{2q(k+3)-3}{2q(k+2)-3} + \right. \\
&\left. + \binom{2q(k+3)-3}{2q(k+3)-4} \right\} \cdot [\beta z^{2q(k+3)-2}x] + \\
&+ \left\{ \binom{2q(k+3)-3}{2q-3} + \sum_{\xi=0}^{k-1} \binom{2q(k+3)-3}{2q(\xi+2)-3} + \binom{2q(k+3)-3}{2q(k+2)-4} + \right. \\
&\left. + \binom{2q(k+3)-3}{2q(k+3)-5} \right\} \cdot [\beta x z^{2q(k+3)-3} z] \\
&= [\beta x^{2q-2}(yx^{2q-1})^k yx^{2q-2}yx^{2q-2}yxx],
\end{aligned} \tag{13}
$$

where we use the equivalences

$$\begin{aligned}
\sum_{\xi=0}^{k-1}\binom{2q(k+2)}{2q(\xi+1)} &\equiv \sum_{\xi=0}^{k-1}\binom{k+2}{\xi+1}\\
&\equiv \sum_{i=1}^{k}\binom{k+2}{i}\\
&\equiv \sum_{i=0}^{k+2}\binom{k+2}{i}-\binom{k+2}{0}-\binom{k+2}{k+1}-\binom{k+2}{k+2}\\
&\equiv (1+1)^{k+2}-1-(k+2)-1\\
&\equiv k\\
&\equiv 1 \pmod 2.
\end{aligned}$$

From now on, we will not show the entire expansion of the generalized Jacobi identities, but only their most relevant points.

Now we have to show that, in case k odd, are zero or central the elements $[yx^{2q-1}x^{2q-2}(yx^{2q-1})^k yx^{2q-2}(yx^{2q-1})^a yx^{2q-2}y]$ for any $0 < a < k$.

Take $h, t < k$ such that $h = a + R$ and $h + t = k + a - 1$ and expand

$$
\begin{aligned}
0 &= [yz^{2q-1}z^{2q-2}(yz^{2q-1})^t[yz^{2q-1}z^{2q-2}(yz^{2q-1})^h yz^{2q-2}y]] \\
&= [yz^{2q-1}z^{2q-2}(yz^{2q-1})^t[yz^{2q-1}z^{2q-2}(yz^{2q-1})^h yz^{2q-2}]y] + \\
&+ [yz^{2q-1}z^{2q-2}(yz^{2q-1})^t y[yz^{2q-1}z^{2q-2}(yz^{2q-1})^h yz^{2q-2}]] \\
&= \left\{ \binom{2q(h+3)-3}{0} + \sum_{\eta=0}^{k-t-1} \binom{2q(h+3)-3}{2q(\eta+1)} + \right. \\
&\left. + \binom{2q(h+3)-3}{2q(k-t+1)-1} + \sum_{\zeta=0}^{a-1} \binom{2q(h+3)-3}{2q(k-t+2+\zeta)-1} \right\} \cdot \\
&\cdot [yz^{2q-1}z^{2q-2}(yz^{2q-1})^t z^{2q(h+3)-2}y] + \\
&+ [yz^{2q-1}z^{2q-2}(yz^{2q-1})^t z^{2q(h+3)-2}y] + \\
&+ \left\{ + \sum_{\eta=0}^{k-t-1} \binom{2q(h+3)-3}{2q(\eta+1)-1} + \binom{2q(h+3)-3}{2q(k-t+1)-2} + \right. \\
&\left. + \sum_{\zeta=0}^{a-1} \binom{2q(h+3)-3}{2q(k-t+2+\zeta)-2} + \right\} \cdot \\
&\cdot [yz^{2q-1}z^{2q-2}(yz^{2q-1})^t yz^{2q(h+3)-3}z] \\
&= \left\{ 1 + \sum_{i=1}^{R+1} \binom{a+R+2}{i} + 0 + 0 + 1 \right\} \cdot \\
&\cdot [yx^{2q-1}x^{2q-2}(yx^{2q-1})^k yx^{2q-2}(yx^{2q-1})^a yx^{2q-2}y].
\end{aligned} \tag{14}
$$

At this point we have to check when $\sum_{i=1}^{R+1} \binom{a+R+2}{i}$ does not vanish $\pmod 2$, *i.e.* when it is 1: this means that we have to choose the right R depending on the different values of a. So write a 2–adically as $a_0 + a_1 \cdot 2 + a_2 \cdot 4 + \ldots$ and take $R = 2^e - 1$ where e is the index of the first non-zero a_j: so, if we write a as $2^e + \boxed{\textit{terms of order} > 2^e}$, the sum becomes

$$
\begin{aligned}
&\sum_{i=1}^{2^e+1} \binom{2^e + \boxed{\textit{terms of order} > 2^e} + 2^e - 1 + 2}{i} = \\
&\sum_{i=1}^{2^e+1} \binom{1 + \boxed{\textit{terms of order} > 2^e}}{i} = \\
&1.
\end{aligned}
$$

Note that this works only if $h = a + R < k$, that is $k - a > R = 2^e - 1$: this is not really a reduction, because in the following we will show that for the wrong (in the sense of *not of the type* 2^g-1) values of k all the commutators of the algebra vanish in a weight that satisfies the above condition.

Now we know that after the second non long constituent, there are at least other k long constituents. Suppose $k=2s-1$ but not of the form 2^g-1 for any positive integer g. Take $\vartheta=[yx^{2q-1}yx^{2q-2}y(x^{2q-1}y)^{2r-2}x^{2q-1}]$ for some $r<s$ and compute:

$$\begin{aligned}
0 &= [\vartheta\vartheta] \\
&= \sum_{j=0}^{2q(2r+1)-2}\binom{2q(2r+1)-2}{j}[\vartheta z^j yz^{2q(2r+1)-2-j}] \\
&= \left\{\binom{2q(2r+1)-2}{0}+\sum_{\eta=0}^{k-2r}\binom{2q(2r+1)-2}{2q(\eta+1)}+\right. \\
&\left.+\binom{2q(2r+1)-2}{2q(k-2r+2)-1}+\sum_{\xi=0}^{4r-k-3}\binom{2q(2r+1)-2}{2q(k-2r+3+\xi)-1}\right\}\cdot \\
&\cdot[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}(yx^{2q-1})^{4r-2s}] \\
&= \{1+S+0+0\}\cdot[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}(yx^{2q-1})^{4r-2s}].
\end{aligned}$$

To conclude that the algebra has finite dimension, we must find an r such that $0<4r-2s<k$ and $S\equiv 0 \pmod 2$. If s is odd, we can take $r=\frac{s+1}{2}$; if s is even, as $k\neq 2^g-1$, we have that $2s-1\neq 2^g-1$, so s is not a power of 2. So write s 2−adically as $\sum_{w=0}^{d}s_w\cdot 2^w$ with $s_d=1$ and take $d-\beta$ to be the highest exponent w such that $s_w=0$ if there exists $v<d-\beta$ such that $s_v=1$ and to be the lowest exponent w such that $s_w=1$ otherwise and put $r=s-2^{d-\beta}$.

In the first case, as $\beta>1$,

$$r=\boxed{\textit{terms of order} > 2^{d-\beta}}+2^{d-\beta}+\boxed{\textit{terms of order} < 2^{d-\beta}}.$$

Then $s-r=2^{d-\beta}$ and S becomes

$$\begin{aligned}
S &= \sum_{\eta=0}^{k-2r}\binom{2q(2r+1)-2}{2q(\eta+1)} \\
&= \sum_{j=1}^{s-r}\binom{2r}{2j} \\
&= \sum_{c=1}^{2^{d-\beta+1}}\binom{\boxed{\textit{terms of order} > 2^{d-\beta+1}}+2^{d-\beta+1}+\boxed{\textit{terms of order} < 2^{d-\beta+1}}}{c}.
\end{aligned}$$

The elements of this sum are non-zero only for $c = 2^{d-\beta}+1$ and for all values of c whose $2-$adic summands are $2-$adic summands of $2r$ (*i.e.* they appears also as summands in the numerator of the last binomial coefficient above): they are therefore $1+(2^n-1)$, where n is the number of summands of such a numerator; this number is always even (and so the sum is zero) as we are in the case $n \neq 0$.

In the second case (remember that s is not a power of 2, so $\beta > 1$) $r = \sum_{w=d-\beta+1}^{d} s_w \cdot 2^w$ and the sum becomes

$$
\begin{aligned}
S &= \sum_{\eta=0}^{k-2r} \binom{2q(2r+1)-2}{2q(\eta+1)} \\
&= \sum_{c=1}^{2^{d-\beta}+1} \binom{2^{d+1}+\cdots+2^{d-\beta+2}}{c}.
\end{aligned}
$$

But the $2-$adic summands of c are less than $2^{d-\beta+2}$, so every term of the above sum is zero, and so $S=0$. Note that in every case the difference $k-a$ turns out to be bigger than $2^{d-\beta}+1$ and so bigger than 2^e-1 satisfying the condition on R required at page 26.

So the algebra can be infinite-dimensional only if k is of the form 2^g-1 for some exponent g. Now we want to prove that in this case the resulting algebra is periodic, so it cannot evolve as thin. To prove the periodicity, we have to show that every elements of type

$$\psi_1 = [yx^{2q-1}yx^{2q-2}((yx^{2q-1})^k yx^{2q-2})^n (yx^{2q-1})^k yx^{2q-1}]$$

and

$$\psi_2 = [yx^{2q-1}yx^{2q-2}((yx^{2q-1})^k yx^{2q-2})^n (yx^{2q-1})^a yx^{2q-2}]$$

are zero or central for all values of $n \in \mathbb{N}$ and $0 \leq a < k$. (Note that equations (13) and (14) solve the case $n=1$ for the element of the second type).

First of all, take

$$\epsilon = [yx^{2q-1}yx^{2q-2}((yx^{2q-1})^k yx^{2q-2})^{n-1} (yx^{2q-1})^{k-1} y]$$

and compute:

$$
\begin{aligned}
0 &= [\epsilon[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}x]]\\
&= [\epsilon x[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}x]] +\\
&+ [\epsilon[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}]x]\\
&= [\psi_1 y] +\\
&+ \left\{\binom{2q(k+3)-3}{2q-2} + \binom{2q(k+3)-3}{4q-3} +\right.\\
&+ \left.\sum_{\gamma=0}^{k-1}\binom{2q(k+3)-3}{2q(\gamma+3)-3} +\right\}[\psi_1 z]\\
&+ \left\{\binom{2q(k+3)-3}{2q-1} + \binom{2q(k+3)-3}{4q-2} +\right.\\
&+ \left.\sum_{\gamma=0}^{k-1}\binom{2q(k+3)-3}{2q(\gamma+3)-2} +\right\}[\psi_1 x]\\
&= \{0+1+S+1\}\,[\psi_1 y]\\
&= [\psi_1 y],
\end{aligned}
$$

expanding S as in the following:

$$
\begin{aligned}
\sum_{\gamma=0}^{k-1}\binom{2q(k+3)-2}{2q(\gamma+3)-2} &\equiv \sum_{\gamma=0}^{k-1}\binom{2q(k+2)}{2q(\gamma+2)}\binom{2q-2}{2q-2} \pmod 2\\
&\equiv \sum_{\gamma=0}^{k-1}\binom{k+2}{\gamma+2} \pmod 2\\
&= \sum_{i=2}^{k+1}\binom{k+2}{i}\\
&= \sum_{i=0}^{k+2}\binom{k+2}{i} - \binom{k+2}{0} - \binom{k+2}{1} - \binom{k+2}{k+2}\\
&= (1+1)^{k+2} - 1 - (k+2) - 1\\
&\equiv 0+1+1+1 = 1 \pmod 2.
\end{aligned}
$$

Now we consider the elements of the form ψ_2, proving that they are central if $0 < a < k$ and that they lie in the second centre of the algebra if $a = 0$ (note that in both cases $[\psi_2 yy]$ and $[\psi_2 yxy]$ vanish because they have wrong constituents length).

If $a \neq 0$ take

$$\zeta = [yx^{2q-1}yx^{2q-2}((yx^{2q-1})^k yx^{2q-2})^{n-1}(yx^{2q-1})^{a-1}]$$

(note that $n \geq 2$, because we have already seen the case $n = 1$) and expand:

$$
\begin{aligned}
0 &= [\zeta[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}x]] \\
&= [\zeta x[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}]] \\
&+ [\zeta[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}x]x] \\
&= \left\{\sum_{\gamma=0}^{k-a} \binom{2q(k+3)-3}{2q(\gamma+1)-2} + \binom{2q(k+3)-3}{2q(k-a+2)-3} + \right. \\
&+ \left. \sum_{\delta=0}^{a-1} \binom{2q(k+3)-3}{2q(k-a+3+\delta)-3} + \binom{2q(k+3)-3}{2q(k+3)-4} \right\} [\psi_2 yz] + \\
&+ \left\{\sum_{\gamma=0}^{k-a} \binom{2q(k+3)-3}{2q(\gamma+1)-1} + \binom{2q(k+3)-3}{2q(k-a+2)-2} + \right. \\
&+ \left. \sum_{\delta=0}^{a-1} \binom{2q(k+3)-3}{2q(k-a+3+\delta)-2} + \binom{2q(k+3)-3}{2q(k+3)-3} \right\} [\psi_2 yx] = \\
&= 0 + A + S + 1[\psi_2 yz] + [\psi_2 yx] \\
&= [\psi_2 yx],
\end{aligned}
$$

where

$$
\begin{aligned}
A + S &= \binom{2q(k+3)-3}{2q(k-a+2)-3} + \sum_{\delta=0}^{a-1} \binom{2q(k+3)-3}{2q(k-a+3+\delta)-3} \\
&= \sum_{\vartheta=k-a+2}^{k+2} \binom{2q(k+3)-3}{2q\vartheta-3} \\
&\equiv \sum_{\vartheta=k-a+2}^{k+2} \binom{k+2}{\vartheta-1} \pmod 2 \\
&= \sum_{i=k-a+1}^{k+1} \binom{k+2}{i} \\
&= \sum_{i=2^g-a}^{2^g} \binom{2^g+1}{i} \\
&\equiv 1 \pmod 2,
\end{aligned}
$$

as $k = 2^g - 1$, $a < k$ and $\binom{2^g+1}{i}$ is non-zero $\pmod 2$ only for $i = 2^g$.
Otherwise, if $a = 0$:

$$\begin{aligned}
0 &= [[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2})^{n-2}(yx^{2q-1})^k y],\\
&\quad [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}x]]\\
&= [[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2})^{n-2}(yx^{2q-1})^k yx],\\
&\quad [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}]] +\\
&+ [[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2})^{n-2}(yx^{2q-1})^k y],\\
&\quad [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}]]x]\\
&= \left\{\binom{2q(k+3)-3}{2q-3} + \sum_{\gamma=0}^{k-1}\binom{2q(k+3)-3}{2q(\gamma+2)-3} + \binom{2q(k+3)-3}{2q(k+2)-4} + \right.\\
&+ \left.\binom{2q(k+3)-3}{2q(k+3)-5}\right\}[\psi_2 yxz] +\\
&+ \left\{\binom{2q(k+3)-3}{2q-2} + \sum_{\gamma=0}^{k-1}\binom{2q(k+3)-3}{2q(\gamma+2)-2} + \binom{2q(k+3)-3}{2q(k+2)-3} + \right.\\
&+ \left.\binom{2q(k+3)-3}{2q(k+3)-4}\right\}[\psi_2 yxz]\\
&= \left\{1 + \left((1+1)^{k+2} - \binom{k+2}{0} - \binom{k+2}{k+1} - \binom{k+2}{k+2}\right) + 1 + 0\right\}\cdot\\
&\cdot [\psi_2 yxz] + \{0+0+1+1\}[\psi_2 yxx]\\
&= [\psi_2 yxx],
\end{aligned}$$

evaluating the sum as in the previous cases.

This shows that the algebra is periodic with constituent sequence

$$\left(2q-1, 2q-2, \left((2q-1)^{2^g-1}, (2q-2)\right)^\infty\right).$$

k even. If k is even, as in the previous case, first of all we have to prove that after the second non long constituent there is another non long one and then at least k long ones. This means that we have to show that are zero the elements

$$[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}yx^{2q-1}]$$

and

$$[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}yx^{2q-2}(yx^{2q-1})^a yx^{2q-2}y]$$

for $0 < a < k$ (note that the case $a = 0$ is just (12)).

For the first identity, take $\nu = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{\frac{k}{2}}]$ and expand

$$
\begin{aligned}
0 &= [\nu\nu] \\
&= \sum_{j=0}^{2q(\frac{k}{2}+2)-2} \binom{2q(\frac{k}{2}+2)-2}{j} [\nu z^j y z^{2q(\frac{k}{2}+2)-2-j}] \\
&= \left\{ \binom{2q(\frac{k}{2}+2)-2}{0} + \sum_{\gamma=0}^{\frac{k}{2}-1} \binom{2q(\frac{k}{2}+2)-2}{2q(\gamma+1)} + \binom{2q(\frac{k}{2}+2)-2}{2q(\frac{k}{2}+1)-1} \right\} \cdot \\
&\cdot [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}yx^{2q-1}] \\
&= \left\{ 1 + \left((1+1)^{\frac{k}{2}+1} - \binom{\frac{k}{2}+1}{0} - \binom{\frac{k}{2}+1}{\frac{k}{2}+1} \right) + 0 \right\} \cdot \\
&\cdot [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}yx^{2q-1}] \\
&= [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}yx^{2q-1}].
\end{aligned}
$$

For the elements of the second kind, we introduce a notation that will be useful in the sequel: we call $\rho_{n,a}$ the element

$$\rho_{(n,a)} = [yx^{2q-1}yx^{2q-2}((yx^{2q-1})^k yx^{2q-2}yx^{2q-2})^n (yx^{2q-1})^a y].$$

So we have to show that the element $[\rho_{(1,a)}x^{2q-2}y]$ vanishes: take $b, c < k$ such that $b + c = k + a - 1$ and expand

$$
\begin{aligned}
0 &= [\rho_{(0,c)}x^{2q-1}[yx^{2q-1}yx^{2q-2}y(yx^{2q-1})^b yx^{2q-2}y]] \\
&= [\rho_{(0,c)}x^{2q-1}[yx^{2q-1}yx^{2q-2}y(yx^{2q-1})^b yx^{2q-2}]y] \\
&= \left\{ \binom{2q(b+2)+2q-3}{1} + \sum_{\eta=0}^{k-(c+1)-1} \binom{2q(b+2)+2q-3}{2q(\eta+1)} + \right. \\
&+ \binom{2q(b+2)+2q-3}{2q(k-c)} + \binom{2q(b+2)+2q-3}{2q(k-c)+2q-1} + \\
&+ \left. \sum_{\eta=0}^{a-1} \binom{2q(b+2)+2q-3}{2q(k-c+2+\eta)+2q-1} \right\} \cdot [\rho_{(1,a)}yx^{2q-2}y] \qquad (15) \\
&= \left\{ 1 + \sum_{\eta=0}^{k-c-2} \binom{b+2}{\eta+1} + \binom{b+2}{k-c} + 0 + 0 \right\} \cdot [\rho_{(1,a)}yx^{2q-2}y] \\
&= \sum_{i=0}^{b+1-a} \binom{b+2}{i} \cdot [\rho_{(1,a)}yx^{2q-2}y] \\
&= [\rho_{(1,a)}yx^{2q-2}y],
\end{aligned}
$$

evaluating the sum in the same way we did in page 26.

Now we proceed as in the k odd case, showing that the algebra has finite dimension if k is even but not of the kind 2^g-2 for some exponent g: we need the above calculation because in the sequel we use the facts that after the second non long constituent there is another one and then there are k long ones. So put $k=2s$ and take an a such that $\frac{s}{2}<a<s$.

Let ϑ be $[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{2a}yx^{q-1}]$.

Then

$$
\begin{aligned}
0 &= [\vartheta\vartheta] \\
&= \sum_{j=0}^{2q(2a+2)+q-2} \binom{2q(2a+2)+q-2}{j} [\vartheta z^j y z^{2q(2a+2)+q-2-j}] \\
&= \Bigg\{ \binom{2q(2a+2)+q-2}{q} + \sum_{\eta=0}^{k-2a-2} \binom{2q(2a+2)+q-2}{q(2\eta+3)} + \\
&+ \binom{2q(2a+2)+q-2}{2q(k-2a)+q-1} + \binom{2q(2a+2)+q-2}{2q(k-2a+1)+q-2} + \\
&+ \sum_{\zeta=0}^{4a-k} \binom{2q(2a+2)+q-2}{2q(k-2a+2+\zeta)+q-2} \Bigg\} [\rho_{(1,4a-k)} yx^{2q-1}] \\
&= \{0+0+0+0+S\} [\rho_{(1,4a-k)} yx^{2q-1}] \\
&= [\rho_{(1,4a-k)} yx^{2q-1}],
\end{aligned}
\tag{16}
$$

where the indices of the summations are positive and less than k for the bounds on a and where the sum S is congruent to $1 \pmod 2$ if s is even and we take

$a = s-1$: in fact, under these hypothesis, we have

$$\begin{aligned}
S &= \sum_{\zeta=0}^{4a-k} \binom{2q(2a+2)+q-2}{2q(k-2a+2+\zeta)+q-2} \\
&= \sum_{\zeta=0}^{4a-k} \binom{2a+2}{k-2a+2+\zeta}\binom{q-2}{q-2} \\
&= \sum_{j=0}^{2a-s} \binom{2(a+1)}{2(s-a+1+j)} \\
&= \sum_{j=0}^{s-2} \binom{2s}{2j+4} \\
&= \sum_{i=4}^{2s} \binom{2s}{i} \\
&= (1+1)^{2s} - \binom{2s}{0} - \binom{2s}{1} - \binom{2s}{2} - \binom{2s}{3} \\
&\equiv 1,
\end{aligned}$$

as s is even.

If s is odd, call $s = 2w-1$; as $k \neq 2^g - 2$ for any g, we have that $2(2n-1) \neq 2^g - 2$, *i.e.* n is not a power of 2. We can rewrite S in (16) as

$$S = \sum_{i=4n-2a}^{2a+2} \binom{2a+2}{i} = (1+1)^{2a+2} - \sum_{i=0}^{4n-2a-1} \binom{2a+2}{i}, \tag{17}$$

and use the decomposition of n as sum of 2-powers: as n is not a power of 2, there exists 2 different exponents α and β such that $n = 2^\alpha + 2^\beta + \boxed{\textit{terms of higher order}}$; so take $a = s - 2^{\alpha+1}$ (note that this choice of a satisfies the bounds cited above) and look at every binomial coefficient in the sum in the right member of (17):

$$\binom{2a+2}{i} = \binom{2^{\beta+2} + \boxed{\textit{terms of higher order}}}{i} \equiv 0 \pmod 2,$$

as $i \leq 4n - 2a - 1 = 2^{\alpha+2} - 1$.

This implies that the only values of k for which the algebra can have infinite dimension on are of the kind $2^g - 2$ for any g. Now we finish showing, as in the previous case, that for such a value of k, the algebra is periodic: to show this, we have to prove that the elements of the form $[\rho_{(n,a)} x^{2q-2} y]$ for all values $0 \leq a \leq k-1$, $[\rho_{(n,k)} x^{2q-1}]$ and $[\rho_{(n,k)} x^{2q-2} y x^{2q-1}]$ for every $n > 0$ vanish or are central, while we have already seen the case $n = 0$.

As regards $[\rho_{(n,0)} x^{2q-2} y]$, there is nothing to prove, because it follows immediately from (12); furthermore, also the centrality of $[\rho_{(n,a)} x^{2q-2} y]$ when

$1 \leq a \leq k-1$ follows from equation (15) above substituting $[\rho_{(n,k-1)}x^{2q-1}y]$ to $[\rho_{(n,k-1)}x^{2q-1}y]$.
Both $[\rho_{(n,k)}x^{2q-1}]$ and $[\rho_{(n,k)}x^{2q-2}yx^{2q-1}]$ can be shown to lie in the second center expanding a Jacobi relation with the same zero element

$$[\chi_k x] = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^k yx^{2q-2}x]$$

$$\begin{aligned}
0 &= [\rho_{(n-1,k)}[\chi_k x]] \\
&= [\rho_{(n-1,k)}x\chi_k] + [\rho_{(n-1,k)}\chi_k x] \\
&= \left\{ \binom{2q(k+3)-3}{2q-3} + \binom{2q(k+3)-3}{4q-4} + \sum_{\delta=0}^{k-1} \binom{2q(k+3)-3}{2q(\delta+3)-3} + \right. \\
&+ \left. \binom{2q(k+3)-3}{2q(k+3)-4} \right\} [\rho_{(n,k)}xx^{2q-1}yz] + \\
&+ \left\{ \binom{2q(k+3)-3}{2q-2} + \binom{2q(k+3)-3}{4q-3} + \sum_{\delta=0}^{k-1} \binom{2q(k+3)-3}{2q(\delta+3)-2} + \right. \\
&+ \left. \binom{2q(k+3)-3}{2q(k+3)-3} \right\} [\rho_{(n,k)}xx^{2q-1}yx] + \\
&= [\rho_{(n,k)}x^{2q-1}yx],
\end{aligned}$$

where

$$\begin{aligned}
\sum_{\delta=0}^{k-1} \binom{2q(k+3)-3}{2q(\delta+3)-3} &\equiv \sum_{\delta=0}^{k-1} \binom{k+2}{\delta+2} \binom{2q-3}{2q-3} \\
&\equiv \sum_{i=2}^{k+1} \binom{k+2}{i} \\
&\equiv (1+1)^{k+2} - \binom{k+2}{0} - \binom{k+2}{1} - \binom{k+2}{k+2} \\
&\equiv 0 \pmod 2,
\end{aligned}$$

and

$$
\begin{aligned}
0 &= [\rho_{(n-1,k)} x^{2q-2} y [\chi_k x]] \\
&= [\rho_{(n-1,k)} x^{2q-2} y x \chi_k] \\
&+ [\rho_{(n-1,k)} x^{2q-2} y \chi_k x] \\
&= \left\{ \binom{2q(k+3)-3}{2q-3} + \sum_{\delta=0}^{k-1} \binom{2q(k+3)-3}{2q(\delta+2)-3} + \binom{2q(k+3)-3}{2q(k+2)-4} + \right. \\
&+ \left. \binom{2q(k+3)-3}{2q(k+3)-4} \right\} [\rho_{(n,k)} x^{2q-2} y x^{2q-1} y z] \\
&+ \left\{ \binom{2q(k+3)-3}{2q-2} + \sum_{\delta=0}^{k-1} \binom{2q(k+3)-3}{2q(\delta+2)-2} + \binom{2q(k+3)-3}{2q(k+2)-3} + \right. \\
&+ \left. \binom{2q(k+3)-3}{2q(k+3)-3} \right\} [\rho_{(n,k)} x^{2q-2} y x^{2q-1} y x] \\
&+ \left\{ 1 + \left((1+1)^{k+2} - \binom{k+2}{0} - \binom{k+2}{k+1} - \binom{k+2}{k+2} \right) + 0 + 1 \right\} \cdot \\
&\cdot [\rho_{(n,k)} x^{2q-2} y x^{2q-1} y z] + \\
&+ \{0+0+0+1\} \cdot [\rho_{(n,k)} x^{2q-2} y x^{2q-1} y x] \\
&= [\rho_{(n,k)} x^{2q-2} y x^{2q-1} y x],
\end{aligned}
$$

as k is even.

Further, previous general considerations about constituents lengths show that the elements $[\rho_{(n,k)} x^{2q-1} x]$, $[\rho_{(n,k)} x^{2q-1} yy]$, $[\rho_{(n,k)} x^{2q-2} y x^{2q-1} x]$ and $[\rho_{(n,k)} x^{2q-2} y x^{2q-1} yy]$ are zero.

So we have proved that every algebra that starts out with constituents lengths

$$\big(2q-1, 2q-2, (2q-1)^k, 2q-2\big)$$

goes on as

$$\Big(2q-1, 2q-2, \big((2q-1)^{2^g-1}, (2q-2)\big)^{\infty}\Big)$$

or as

$$\Big(2q-1, 2q-2, \big((2q-1)^{2^g-2}, (2q-2)^2\big)^{\infty}\Big)$$

for any exponent g.

Now we want to discuss the possibility of having another diamond starting after a $2q-2$ constituent, with element v. So, first of all we must show when such a situation cannot take place.

First of all, it can be noted that the diamond must lie before the end of the first period, otherwise if at least the sequence

$$\left(2q-1, 2q-2, (2q-1)^k, (2q-2)^j\right)$$

(with k, j as above) takes place in the algebra before the diamond, the algebra can go on only as a periodic one of maximal class, as the above calculations show. So we can reduce to consider the element v before the diamond to be

$$[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^d(yx^{2q-2})^i]$$

where i can be 1 or 2, studying separately what happens if d is even or odd.

$i = 1, d$ even. Here we show that

$$[vxyx^{2q-3}] = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^d yx^{2q-1}yx^{2q-3}]$$

is central. In fact, if we call ϑ the element $[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{\frac{d}{2}}]$ and μ the element $[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^d]$, we obtain

$$\begin{aligned}
0 &= [\vartheta\vartheta] \\
&= \sum_{i=0}^{2q\left(\frac{d}{2}+1\right)+2q-2} \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{i} [\vartheta z^i y z^{2q\left(\frac{d}{2}+1\right)+2q-2-i}] \\
&= \left\{ \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{0} + \sum_{\xi=0}^{\frac{d}{2}-1} \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{2q(\xi+1)} + \right. \\
&+ \left. \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{2q(\frac{d}{2}+1)-1} + \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{2q(\frac{d}{2}+2)-2} \right\} \cdot \\
&\cdot [vyxx^{2q-3}x] + \\
&+ \left\{ \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{0} + \sum_{\xi=0}^{\frac{d}{2}-1} \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{2q(\xi+1)} + \right. \\
&+ \left. \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{2q(\frac{d}{2}+1)} + \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{2q(\frac{d}{2}+2)-2} \right\} \cdot \\
&\cdot [vxyx^{2q-3}x] \\
&= [vxyx^{2q-3}x],
\end{aligned}$$

and

$$\begin{aligned}
0 &= [\mu[yx^{2q-1}yx^{2q-3}y]] \\
&= [vxyx^{2q-3}y].
\end{aligned}$$

$i=1, d$ odd. As above, we will show that there exists a central element, against the covering property: here this element is $[vxyx^{2q-2}yx^{2q-2}]$: in fact, using

$$[v[yx^{2q}]] = 0$$

we show that after the diamond must be an intermediate constituent and, if we write d as $2n-1$ and put

$$\vartheta = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{\frac{d+1}{2}}]$$

and

$$\mu = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{d-1}],$$

we can show that:

$$\begin{aligned}
0 &= [\mu[yx^{2q-1}yx^{2q-2}yx^{2q-1}yx^{2q-2}y]]\\
&= [\mu y[yz^{8q-3}]] + [\mu[yz^{8q-3}]y]\\
&= \sum_{i=0}^{8q-3}\binom{8q-3}{i}\{[\mu yz^i yz^{8q-3-i}] + [\mu z^i yz^{8q-3-i}y]\}\\
&= \left\{\binom{8q-3}{2q-1} + \binom{8q-3}{4q-1} + \binom{8q-3}{6q-2} + \binom{8q-3}{8q-3}\right\}\cdot\\
&\cdot [vxyx^{2q-2}yx^{2q-2}y] +\\
&+ \left\{\binom{8q-3}{2q-1} + \binom{8q-3}{4q-2} + \binom{8q-3}{6q-2} + \binom{8q-3}{8q-3}\right\}\cdot\\
&\cdot [vyxx^{2q-2}yx^{2q-2}y] +\\
&+ \left\{\binom{8q-3}{0} + \binom{8q-3}{2q} + \binom{8q-3}{4q} + \binom{8q-3}{6q-1}\right\}\cdot\\
&\cdot [vxyx^{2q-2}yx^{2q-2}y] +\\
&+ \left\{\binom{8q-3}{0} + \binom{8q-3}{2q} + \binom{8q-3}{4q-2} + \binom{8q-3}{6q-1}\right\}\cdot\\
&\cdot [vyxx^{2q-2}yx^{2q-2}y]\\
&= [vxyx^{2q-2}yx^{2q-2}y],
\end{aligned} \tag{18}$$

and

$$
\begin{aligned}
0 &= [\vartheta\vartheta] \\
&= \sum_{i=0}^{2q\left(\frac{d+3}{2}\right)+2q-2} \binom{2q\left(\frac{d+3}{2}\right)+2q-2}{i} [\vartheta z^i y z^{2q\left(\frac{d+3}{2}\right)+2q-2-i}] \\
&= \left\{ \binom{2q\left(\frac{d+3}{2}\right)+2q-2}{0} + \sum_{\xi=0}^{\frac{d-3}{2}} \binom{2q\left(\frac{d+3}{2}\right)+2q-2}{2q(\xi+1)} + \right. \\
&+ \left. \binom{2q\left(\frac{d+3}{2}\right)+2q-2}{2q(\frac{d+1}{2})} + \binom{2q\left(\frac{d+3}{2}\right)+2q-2}{2q(\frac{d+3}{2})-1} \right\} \cdot \\
&\cdot [vxyx^{2q-2}yx^{2q-1}] + \\
&+ \left\{ \binom{2q\left(\frac{d+3}{2}\right)+2q-2}{0} + \sum_{\xi=0}^{\frac{d-3}{2}} \binom{2q\left(\frac{d+3}{2}\right)+2q-2}{2q(\xi+1)} + \right. \\
&+ \left. \binom{2q\left(\frac{d+3}{2}\right)+2q-2}{2q(\frac{d+1}{2})-1} + \binom{2q\left(\frac{d+3}{2}\right)+2q-2}{2q(\frac{d+3}{2})-1} \right\} \cdot \\
&\cdot [vyxx^{2q-2}yx^{2q-1}] \\
&= \binom{\frac{d+3}{2}}{\frac{d+1}{2}} [vxyx^{2q-2}yx^{2q-1}] \\
&= \binom{n+1}{n} [vxyx^{2q-2}yx^{2q-1}] \\
&= (n+1) \cdot [vxyx^{2q-2}yx^{2q-2}x].
\end{aligned}
$$

So there cannot be a diamond in such a situation but in the cases n odd, *i.e.* $d \equiv 1 \pmod 4$. Repeating the computation (18) increasing the number of constituents of the involved elements) it can be shown that the algebra has at least three more long constituents (and, in general, exactly $\lfloor \frac{d}{2} \rfloor$). Using this fact, we can improve the previous bound showing that n can assume only some very special kind of values: call as usual v the element just before the diamond

$$v = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{4n+1}yx^{2q-2}],$$

and let

$$m = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{\eta}],$$

with $2n+1 < \eta < 3n+1$ (bounds required to obtain a non vanishing part up to the last element) and

$$\iota = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{4n+1}yx^{2q-2}(yx^{2q-1})^{2\eta-4n-1}yx^{2q-2}].$$

Then expand

$$
\begin{aligned}
0 &= [mm] \\
&= \sum_{i=0}^{2q(\eta+1)+2q-2} \binom{2q\cdot(\eta+1)+2q-2}{i} [mz^i y z^{2q\cdot(\eta+1)+2q-2-i}] \\
&= \Bigg\{ \sum_{a=0}^{4n-\eta} \binom{2q\cdot(\eta+1)+2q-2}{2qa+2q-1} + \binom{2q\cdot(\eta+1)+2q-2}{2q\cdot(4n-\eta)+2q-1} + \\
&+ \binom{2q\cdot(\eta+1)+2q-2}{2q\cdot(4n-\eta+1)+2q-2} + \\
&+ \sum_{a=0}^{2\eta-4n-1} \binom{2q\cdot(\eta+1)+2q-2}{2q\cdot(4n-\eta+1)+2q-1+2qa+2q-1} \Bigg\} \cdot \\
&\cdot [vxyx^{2q-2}(yx^{2q-1})^{2\eta-4n-1}] + \\
&+ \Bigg\{ \sum_{a=0}^{4n-\eta} \binom{2q\cdot(\eta+1)+2q-2}{2qa+2q-1} + \binom{2q\cdot(\eta+1)+2q-2}{2q\cdot(4n-\eta)+2q-2} + \\
&+ \binom{2q\cdot(\eta+1)+2q-2}{2q\cdot(4n-\eta+1)+2q-2} + \\
&+ \sum_{a=0}^{2\eta-4n-1} \binom{2q\cdot(\eta+1)+2q-2}{2q\cdot(4n-\eta+1)+2q-1+2qa+2q-1} \Bigg\} \cdot \\
&\cdot [vyxx^{2q-2}(yx^{2q-1})^{2\eta-4n-1}] \\
&= \binom{2q\cdot(\eta+1)+2q-2}{2q\cdot(4n-\eta)+2q-2} \cdot [vxyx^{2q-2}(yx^{2q-1})^{2\eta-4n-1}] \\
&= \binom{2q\cdot(\eta+1)+2q-2}{2q\cdot(4n-\eta)+2q-2} \cdot [\iota x],
\end{aligned}
\tag{19}
$$

so the algebra is finite-dimensional if in the bounds above there is at least a value of η that makes the last binomial coefficient to be odd. This coefficient is congruent to $\binom{\eta+1}{4n-\eta}$ modulo 2, so the problem reduces to check the existence of at least a value of η in the range $2n+1 < \eta < 3n+1$.

Now we claim the such an η always exists but in the case n of the kind 2^k-1 for any k strictly bigger than 1. This can be proved using the 2-adic expansion together with Lucas' theorem.

First of all, we look at what happens when n is of the kind just cited above: we have that the above coefficient vanishes for every value of η. It can be easily seen that if η is odd, the binomial coefficient is always zero (this really happens for any value of n), so assume $\eta = 2s$; you obtain the following equivalences:

$$\begin{aligned}\binom{\eta+1}{4n-\eta} &\equiv \binom{2s+1}{4(2^k-1)-2s}\\ &\equiv \binom{2s+1}{2^{k+1}+2^k+\cdots+4-2s}\\ &\equiv \binom{s}{2^k+\cdots+2-s} \pmod 2.\end{aligned}$$

If s is even, say $s = 2r$ (with r 2-adically written as $\sum_{i=0}^{k-1} a_i 2^i$) we obtain

$$\binom{2r}{2^k+\cdots+2-2r} \equiv \binom{r}{2^{k-1}+\cdots+1-r} \equiv \prod_{i=0}^{k-1}\binom{a_i}{1-a_i}$$

that is congruent to zero $\pmod 2$ unless $a_i = 1$ for all i, *i.e.* $\eta = 4n$, but that is impossible in view of the above bounds.

If s is odd, say $s = 2r+1$ we have

$$\begin{aligned}\binom{2r+1}{2^k+\cdots+2-2r-1} &\equiv \binom{2r}{2^k+\cdots+4-2r}\\ &\equiv \binom{r}{2^{k-1}+\cdots+2-r} \pmod 2,\end{aligned}$$

and again distinguishing the cases r even and r odd we fall in the above solved cases, and so we have our thesis.

This means that in case $n+1$ is a power of 2, the Jacobi expansion (19) is just an identity and does not give us information about the structure of the algebra.

On the contrary, if n is not of such a kind, writing 2-adically $n = \sum_{i=2}^{k+2} b_i 2^{i-2}$ (letting $b_0 = b_1 = 0$) and $\eta = \sum_{i=1}^{k+2} a_i 2^i$ (letting $a_0 = 0$) we obtain

$$\binom{\eta+1}{4n-\eta} \equiv \binom{\sum_{i=1}^{k+2} a_i 2^i + 1}{\sum_{i=0}^{k+2}(b_i-a_i)2^i} \equiv \prod_{i=1}^{k+2}\binom{a_i}{b_i-a_i} \pmod 2,$$

and it can be seen that, given the sequence of the a_i (which cannot be all equal to 1 as n is different from 2^k-1), it can always be found at least a valid (in the sense of the bounds) sequence of b_i that makes the above product to be non-zero, and thus a value of η for which equation (19) is meaningful, forcing the algebra to be finite dimensional for every value of n not of the kind $2^k - 1$, because it can always be shown that $[\iota y]$ vanishes. In fact, take the element

$$o = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{2\eta-1-k}]$$

with $1 \le k \le 4n$ to be specified, and expand:

$$\begin{aligned}
0 &= [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{2\eta-1-k}[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{k}yx^{2q-2}y]] \\
&= [oy[yz^{2q\cdot(k+2)+2q-3}]] + [o[yz^{2q\cdot(k+2)+2q-3}]y] \\
&= \sum_{i=0}^{2q\cdot(k+2)+2q-3} \binom{2q\cdot(k+2)+2q-3}{i} \cdot \\
&\cdot \left\{[oyz^{i}yz^{2q\cdot(k+2)+2q-3-i}] + [oz^{i}yz^{2q\cdot(k+2)+2q-3-i}y]\right\} \\
&= \Bigg(\Bigg\{ \sum_{m=0}^{4n+1-2\eta+k} \binom{2q\cdot(k+2)+2q-3}{2q\cdot m+2q-1} + \\
&+ \binom{2q\cdot(k+2)+2q-3}{2q\cdot(4n-2\eta+k+2)+2q-2} + \\
&+ \sum_{m=0}^{2\eta-4n-2} \binom{2q\cdot(k+2)+2q-3}{2q\cdot(4n-2\eta+k+3+m)+2q-2} + \\
&+ \binom{2q\cdot(k+2)+2q-3}{2q\cdot(k+2)+2q-3} \Bigg\} + \\
&+ \Bigg\{ \sum_{m=0}^{4n+2-2\eta+k} \binom{2q\cdot(k+2)+2q-3}{2q\cdot m+2q-1} + \\
&+ \binom{2q\cdot(k+2)+2q-3}{2q\cdot(4n-2\eta+k+3)+2q-2} + \\
&+ \sum_{m=0}^{2\eta-4n-3} \binom{2q\cdot(k+2)+2q-3}{2q\cdot(4n-2\eta+k+4+m)+2q-2} + \\
&+ \binom{2q\cdot(k+2)+2q-3}{2q\cdot(k+2)+2q-3} \Bigg\} +
\end{aligned}$$

$$
\begin{aligned}
&+\left\{\binom{2q\cdot(k+2)+2q-3}{0}+\sum_{m=0}^{4n+1-2\eta+k}\binom{2q\cdot(k+2)+2q-3}{2q\cdot(m+1)}+\right.\\
&+\binom{2q\cdot(k+2)+2q-3}{2q\cdot(4n-2\eta+k+2)+2q-1}+\\
&\left.+\sum_{m=0}^{2\eta-4n-2}\binom{2q\cdot(k+2)+2q-3}{2q\cdot(4n-2\eta+k+3+m)+2q-1}\right\}+\\
&+\left\{\binom{2q\cdot(k+2)+2q-3}{0}+\sum_{m=0}^{4n+2-2\eta+k}\binom{2q\cdot(k+2)+2q-3}{2q\cdot(m+1)}+\right.\\
&+\binom{2q\cdot(k+2)+2q-3}{2q\cdot(4n-2\eta+k+3)+2q-1}+\\
&\left.\left.+\sum_{m=0}^{2\eta-4n-3}\binom{2q\cdot(k+2)+2q-3}{2q\cdot(4n-2\eta+k+4+m)+2q-1}\right\}\right)\cdot[\iota y]\\
&=\binom{k+2}{4n-2\eta+k+3}[\iota y]\\
&=[\iota y],
\end{aligned}
$$

choosing a suitable k in the above range (for example, $k=2^w-3$ with the right w).

$i=2$. In this subcase, we can show that after v there cannot be a diamond, *i.e.* L_k has dimension one: in fact, if d is even then $[vx]=0$, while if d is odd then $[vy]=0$. To show this we use two balanced Jacobi identities $[\vartheta\vartheta]$, with

$$
\vartheta=\begin{cases}[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{\frac{d}{2}}] & \text{for } d \text{ even}\\ [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{\frac{d-1}{2}}yx^{q-1}] & \text{for } d \text{ odd.}\end{cases}
$$

Expanding, we get

$$\begin{aligned}
0 &= [\vartheta\vartheta] \\
&= \sum_{i=0}^{2q\left(\frac{d}{2}+1\right)+2q-2} \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{i} [\vartheta z^i y z^{2q\left(\frac{d}{2}+1\right)+2q-2-i}] \\
&= \left\{ \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{0} + \sum_{\xi=0}^{\frac{d}{2}-1} \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{2q(\xi+1)} + \right. \\
&+ \left. \binom{2q\left(\frac{d}{2}+1\right)+2q-2}{2q(\frac{d}{2}+1)-1} \right\} [vx] \\
&= \left\{ 1 + \sum_{i=1}^{\frac{d}{2}} \binom{\frac{d}{2}+1}{i} + 0 \right\} [vx] \\
&= \left\{ 1 + (1+1)^{\frac{d}{2}+1} - \binom{\frac{d}{2}+1}{0} - \binom{\frac{d}{2}+1}{\frac{d}{2}+1} \right\} [vx] \\
&= [vx],
\end{aligned}$$

for even values of d and

$$\begin{aligned}
0 &= [\vartheta\vartheta] \\
&= \sum_{i=0}^{q(d+3)+q-2} \binom{q(d+3)+q-2}{i} [\vartheta z^i y z^{q(d+3)+q-2-i}] \\
&= \left\{ \binom{q(d+3)+q-2}{q} + \sum_{\xi=0}^{\frac{d-3}{2}} \binom{q(d+3)+q-2}{q(2\xi+3)} + \binom{q(d+3)+q-2}{q(d+1)+q-1} + \right. \\
&+ \left. \binom{q(d+3)+q-2}{q(d+3)+q-2} \right\} [vy] \\
&= \left\{ \binom{d+3}{1} + \sum_{i=1}^{d} \binom{d+3}{i} + 0 + 1 \right\} [vy] \\
&= \left\{ d+3 + (1+1)^{d+3} - \binom{d+3}{0} - \binom{d+3}{1} - \binom{d+3}{2} + \right. \\
&+ \left. \binom{d+3}{d+1} - \binom{d+3}{d+2} - \binom{d+3}{d+3} + 1 \right\} [vy] \\
&= [vy],
\end{aligned}$$

in case d odd.

This ends the proof: we have shown that if a thin Lie algebra has a non-metabelian largest maximal class quotient, its only possible constituent sequence is

$$\left(2q-1, 2q-2, (2q-1)^{4n+1}, 2q-2, \diamond, \left(2q-2, (2q-1)^{2n}, 2q-2, \diamond\right)^{\infty}\right),$$

i.e. it is just the algebra builded up in Section (3).

9. Computations

As in the first part [CJ97] of the paper, the ANU $p-$Quotient Program ([HNO97]) and GAP ([S$^+$97]) were very useful to discover the structures of the involved algebras and to see how to perform computations in full generality.

References

[Bon97] C. Bonmassar, *Normally constrained p-groups*, to appear, 1997.

[Car97a] A. Caranti, *Presenting the graded Nottingham Lie algebra*, J. Algebra **198** (1997), 266–289.

[Car97b] A. Caranti, *Thin groups of prime-power order and thin Lie algebras: an addendum*, accepted for publication, 1997.

[CJ97] A. Caranti and G. Jurman, *Quotients of maximal class of thin Lie algebras. The odd characteristic case.*, in preparation, 1997.

[CM97] A. Caranti and S. Mattarei, *Thin Lie algebras related to Hamiltonian algebras and algebras of maximal class*, preliminary version, 1997.

[CMN97] A. Caranti, S. Mattarei, and M. F. Newman, *Graded Lie algebras of maximal class*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 4021–4051.

[CMNS96] A. Caranti, S. Mattarei, M. F. Newman, and C. M. Scoppola, *Thin groups of prime-power order and thin Lie algebras*, Quart. J. Math. Oxford Ser. (2) **47** (1996), no. 187, 279–296.

[CN98] A. Caranti and M. F. Newman, *Graded Lie algebras of maximal class. II*, in preparation, 1998.

[Cur54] C.W. Curtis, *A note on the representations of nilpotent Lie algebras.*, Proc. Amer. Mat. Soc. **5** (1954), 813–824;1001.

[Fia83] A. Fialowski, *Classification of graded Lie algebras with two generators*, Moscow Univ. Math. Bull. **38** (1983), no. 2, 76–79.

[HNO97] G. Havas, M.F. Newman, and E.A. O'Brien, *ANU p-Quotient Program (version 1.4), written in C, available as a share library with GAP and as part of Magma, or from http://wwwmaths.anu.edu.au/services/ftp.html*, 1997.

[Jur98] G. Jurman, *Graded Lie algebras of maximal class. III*, in preparation, 1998.

[LG94] C. R. Leedham-Green, *The structure of finite p-groups*, J. London Math. Soc. (2) **50** (1994), no. 1, 49–67.

[LGN80] C. R. Leedham-Green and M. F. Newman, *Space groups and groups of prime-power order. I.*, Arch. Math. (Basel) **35** (1980), no. 3, 193–202.

[LGPK97] C. R. Leedham-Green, W. Plesken, and G. Klaas, *Pro-p-groups of finite width*, Lecture Notes in Mathematics, vol. 1674, Springer-Verlag, Berlin, 1997.

[LM78] J. Lepowsky and S. Milne, *Lie algebraic approaches to classical partition identities.*, Adv. in Math. **29** (1978), no. 1, 15–59.

[Luc78] E. Lucas, *Sur les congruences des nombres eulériens et des coefficients différentiels des fonctions trigonométriques, suivant un module premier*, Bull. Soc. Math. France **6** (1878), 49–74.
[S+97] M. Schönert et al., *GAP - Groups, Algorithms and Programming, Release 3.4, available from* ***http://www-history.mcs.st-and.ac.uk/~gap/***, School of Mathematical and Computational Sciences, University of St Andrews, Scotland, 1997.
[Sha94] Aner Shalev, *The structure of finite p-groups: effective proof of the coclass conjectures*, Invent. Math. **115** (1994), no. 2, 315–345.
[SZ97] Aner Shalev and Efim I. Zelmanov, *Narrow Lie algebras I: a coclass theory and a characterization of the Witt algebra*, J. Algebra **189** (1997), no. 2, 294–331.

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