LOOP ALGEBRAS OF ZASSENHAUS ALGEBRAS IN CHARACTERISTIC THREE

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1. Introduction

In [Car97] we studied the graded Lie algebra L associated to the Nottingham group in characteristic $p > 3$, from the point of view of providing a (finite) presentation of it. L is a loop algebra, with respect to a suitable grading, of the Witt algebra W_1. The latter has a one-dimensional Schur multiplier. Therefore the corresponding loop algebra M of the universal covering $\widehat{W_1}$ of W_1 has an infinite-dimensional centre. As M is finitely generated, and $M/Z(M) \cong L$, by a result of B. H. Neumann ([Neu37] or [Rob82, 2.2.3]) the algebra L is not finitely presented. We proved in [Car97], however, that M is finitely presented.

When p is 3, the Witt algebra W_1 is isomorphic to $sl(2)$, and has trivial Schur multiplier. Still, the loop algebra L has a central extension M by an infinite-dimensional centre which is finitely presented. In this case, the cocycles needed to construct $Z(M)$ are the products of the Killing form on $sl(2)$ and the cyclic cohomology (see below) on the polynomial part, in a transposition of the theory of [Gar80] to the case of positive characteristic. This is dealt with in [Car].

In [Car97] we studied more generally loop algebras of the Zassenhaus algebras W_n in characteristic $p > 3$. The theory is exactly the same as in the case $n = 1$.

It is the purpose of this paper to study finite presentations of the analogous loop algebras of the Zassenhaus algebras W_n in characteristic 3, for $n > 1$. As we will see, these algebras combine the features of the two cases described above. The cohomology of these loop algebras could be calculated with the methods developed by Zusmanovich in [Zus92] (see also [Zus94]); however, Zusmanovich considers loop algebras of Zassenhaus algebras only in characteristic greater than 3. We will therefore give a brief self-contained, elementary account of the matter here, stressing in particular the role played by the invariant form on W_n.

It is well known (see [Far87] and [Dzh85]) that the Schur multiplier of W_n has dimension $n - 1$ in characteristic 3; as above, we obtain infinitely many cocycles for the loop algebra L. We call these the *finite cohomology cocycles* (which are *nonclassical* in the terminology of Zusmanovich).

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The Killing form of W_n is zero, for $n > 1$; however, because of the characteristic 3, the algebra W_n has a nondegenerate associative form ([Far86]). This enables us, as in [Gar80] and [Car], to use the cyclic cohomology of the polynomial part of the loop algebra L to construct infinitely many more cocycles on L. We call these the *loop cocycles* (these are the *classical* ones according to Zusmanovich).

We prove that the central extension M of L obtained via the finite cohomology and loop cocycles is finitely presented. Note that by the result of B. H. Neumann mentioned above, this implies that all these cocycles generate a subspace of finite codimension in the second cohomology group of L (and also yields an alternative method for calculating the second cohomology group of W_n); one could see with the methods of Zusmanovich that these cocycles really exhaust the whole cohomology group.

Our proofs basically follow the pattern of [Car97], to which we refer. An argument similar to one used in [CMNS96] is employed to determine the occurrence of the loop cocycles.

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2. Preliminaries

We use [Jac79] and [SF88] as general references for Lie algebras. A good reference for the facts about cohomology, central extensions and universal coverings that we will be using is [Gar80].

At one point we will use Lucas' theorem ([Luc78]), which states that if the two nonnegative integers a and b are written p-adically

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n, \quad b = b_0 + b_1 p + b_2 p^2 + \cdots + b_n p^n,$$

so that $0 \le a_i, b_i < p$, then

$$\binom{a}{b} \equiv \prod_{i=0}^{n} \binom{a_i}{b_i} \pmod{p}.$$

For a prime-power q, we write $\mathbf{F}_q$ for the field with q elements. We will use Kronecker's delta

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

We will use several times without explicit mention the generalized Jacobi identity

$$[v[y\underbrace{x\dots x}_{n}]] = \sum_{i=0}^{n} (-1)^i \binom{n}{i} [v\underbrace{x\dots x}_{i}\, y\, \underbrace{x\dots x}_{n-i}].$$

3. The Zassenhaus algebras and their loop algebras

The Zassenhaus algebras are defined as algebras of special derivations of algebras of divided powers in one variable ([SF88]), and can be regarded abstractly as vector spaces

$$W_n = W(1;n) = \{y_i : -1 \le i \le q-2\},$$

of dimension q over the field $\mathbf{F}_p$ with p elements, p an odd prime, and $q = p^n$, with the product defined by

$$[y_i y_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i} \right) y_{i+j} \ .$$

These algebras are graded over the integers, weighting the y_i by their indices.

From now on, we will take $p = 3$, and $n > 1$. The corresponding theory for $p > 3$ is dealt with in [Car97], and that for $p = 3$ and $n = 1$ in [Car].

The original definition by Zassenhaus regarded W_n as an algebra over $\mathbf{F}_q$, with basis e_α, for $\alpha \in \mathbf{F}_q$, and multiplication defined by

$$[e_\alpha, e_\beta] = (\beta - \alpha) \cdot e_{\alpha+\beta}.$$

A connection between the two presentations, which we will be handy later, is given by the formulas

$$\begin{cases} y_{-1} = e_0 + \sum_\alpha e_\alpha, \\ y_i = -\sum_\alpha \alpha^{q-2-i} e_\alpha \quad \text{for } i \ne -1. \end{cases} \tag{1}$$

We will consider the loop algebra of W_n, with respect to a particular grading, as in [Car97]. This is a grading over a cyclic group of order $q-1$, and it is derived from the one over the integers by declaring y_{-1} and y_{q-2} to be of weight 1. In other words, we are viewing modulo $q-1$ the opposite grading to the above grading over the integers.

Thus y_{q-i} acquires weight $i-1$ in this grading, for $2 \le i \le q$, and y_{-1} acquires weight 1. Therefore all homogeneous component have dimension 1, except that the i-th one, for $i \equiv 1 \pmod{q-1}$, has dimension 2, and it is spanned by y_{-1} and y_{q-2}.

Now consider the loop algebra of W_n with respect to the latter grading. That is, take first the Lie algebra

$$W_n \otimes \mathbf{F}_p[t],$$

where $\mathbf{F}_p[t]$ is a polynomial algebra, and consider the subalgebra L generated by the two elements

$$x = y_{-1} \otimes t, \qquad y = y_{q-2} \otimes t.$$

One easily sees that the action of the two generators x and y on L is defined as follows. Define formally an element v_0 with the property

$$[v_0 x] = -x, \qquad [v_0 y] = y.$$

and elements v_i, for $i > 0$

$$v_i = [v_{i-1} x y \underbrace{x \dots x}_{q-3}].$$

Then we have, for $i \geq 0$,

$$
\begin{aligned}
&[v_i x y y] = [v_i x y x y] = \dots = [v_i x y \underbrace{x \dots x}_{q-4} y] = 0,\\
&[v_{i+1} y x] = [v_{i+1} x y],\\
&[v_{i+1} x x] = [v_{i+1} y y] = 0.
\end{aligned}
$$

4. The finite cohomology cocycles

We give here, for the convenience of the reader, a brief treatment of the second cohomology group of W_n in characteristic 3, in a form that is convenient for our purposes. This cohomology group is discussed in [Far87] and [Dzh85].

One sees readily that the following maps are 2-cocycles for W_n, in characteristic $p = 3$ and for $n > 1$, with respect to the presentation of Zassenhaus,

$$\phi_m(e_\alpha, e_\beta) = \alpha^{3^m} \delta(\alpha + \beta, 0),$$

where $1 \leq m < n$. Independence in cohomology follows from the linear independence of the distinct field automorphisms of $\mathbf{F}_{3^n}$.

We rewrite these cocycles in terms of the y_i, using formulas (1). We get, first for $i, j \neq -1$,

$$
\begin{aligned}
\phi_m(y_i, y_j) &= \sum_\alpha \alpha^{q-2-i}(-\alpha)^{q-2-j}\alpha^{3^m}\\
&= (-1)^{j+1} \sum_\alpha \alpha^{2q-i-j+3^m-4}\\
&= (-1)^{j+1} \sum_\alpha \alpha^{q-i-j+3^m-3}\\
&= (-1)^{j+1} \begin{cases} -1 & \text{if } q-1 \text{ divides } q-i-j+3^m-3\\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

Now $q - 1 \mid q - i - j + 3^m - 3$ when $i + j = 3^m - 2$, and when $i + j = q + 3^m - 3$. There are no other possibilities, as $i + j \leq 2q - 4$. Therefore

$$
\phi_m(y_i, y_j) = \begin{cases} (-1)^j & \text{if } i + j = 3^m - 2,\\ (-1)^j & \text{if } i + j = q + 3^m - 3,\\ 0 & \text{otherwise.} \end{cases}
$$

Now the map

$$
\theta_m(y_i, y_j) = \begin{cases} (-1)^j & \text{if } i + j = 3^m - 2,\\ 0 & \text{otherwise.} \end{cases}
$$

is a coboundary, as

$$\theta_m(u, v) = h([uv]),$$

where h is the linear map $h : L \to \mathbf{F}_3$ defined by $h(y_i) = -\delta(i, 3^m - 2)$. We obtain that $\psi_m = \phi_m - \theta_m$ is a cocycle, for $1 \leq m < n$, where

$$\psi_m(y_i, y_j) = \begin{cases} (-1)^j & \text{if } i + j = q + 3^m - 3, \\ 0 & \text{otherwise.} \end{cases}$$

The same formulas actually hold also when i or j take the value -1, as the extra e_0 does not play a role. We have thus recovered the cocycles of [Dzh85, Theorem 2].

Take Z to be an abelian Lie algebra with basis

$$z_1, z_2, \dots, z_{n-1},$$

and construct the universal covering $\widehat{W_n}$ of W_n as a central extension

$$0 \to Z \to \widehat{W_n} \to W_n \to 0,$$

with multiplication of (preimages of) y_i and y_j in $\widehat{W_n}$ defined by

$$[y_i y_j] + \sum_{m=1}^{n-1} \psi_m(y_i, y_j) \cdot z_m.$$

The grading over the integers of W_n extends to $\widehat{W_n}$, and z_m acquires weight $q + 3^m - 3$ in it. As these weights are distinct, and do not occur in W_n itself, we see that $\widehat{W_n}$ is perfect. (This shows once more that the cocycles ψ_m, for $1 \leq m < n$, are independent in cohomology.)

In the opposite grading modulo $q-1$ of the grading over the integers, z_m acquires weight $q - 3^m + 1 = q - (3^m - 1)$, that is, same as $y_{3^m - 2}$.

Each of the cocycles ψ will yield infinitely many cocycles of L, the loop algebra of W_n. We call these cocycles of L the *finite cohomology cocycles,* because of the way they arise. The extension M of L via all these cocycles can also be obtained as a loop algebra of $\widehat{W_n}$, as in the previous section. In the next section we will show that M has infinitely many more cocycles, arising from the loop process.

5. The loop cocycles for the loop algebra

It is shown in [Far86] that the following defines an associative form on W_n in characteristic 3:

$$\eta(y_i, y_j) = (-1)^{j+1} \delta(i + j, q - 3).$$

By this we mean that η is bilinear and symmetric, and that for $u, v, w \in W_n$ we have

$$\eta([uv], w) = \eta(u, [vw]).$$

Now consider the following bilinear maps, for $a \geq 0$,

$$\begin{aligned} \sigma_{ap} : \mathbf{F}_p[t] \times \mathbf{F}_p[t] &\to \mathbf{F}_p \\ (t^i, t^j) &\mapsto j \cdot \delta(i + j, ap). \end{aligned}$$

These are easily seen to be elements (actually a basis) of the first cyclic cohomology group of $\mathbf{F}_p[t]$, as they satisfy the identity

$$\sigma_{ap}(uv, w) + \sigma_{ap}(vw, u) + \sigma_{ap}(wu, v) = 0$$

(see for instance [Zus92]). As shown in [Kas84], one can generalize the arguments of [Gar80] by using the first cyclic cohomology group.

In fact, proceed as in [Gar80] and [Car], and combine η with the σ to get the following cocycles on the loop algebras L or M (remember that we are working in characteristic $p = 3$):

$$\begin{aligned}
\tau_{3a}(y_i \otimes t^{\eta(q-1)-i},\ y_j \otimes t^{\mu(q-1)-j}) =& \\
&= \eta(y_i, y_j) \cdot \sigma_{3a}(t^{\eta(q-1)-i}, t^{\mu(q-1)-j}) \\
&= (-1)^{j+1} \cdot \delta(i + j, q - 3) \cdot \\
&\quad (-j - \mu) \cdot \delta\left(\lambda(q - 1) - i + \mu(q - 1) - j, 3a\right) \\
&= (-1)^{i} \cdot (j + \mu) \cdot \delta(i + j, q - 3) \cdot \delta\left((\lambda + \mu)(q - 1) - (i + j), 3a\right),
\end{aligned}$$

where we have used the fact that $i+j = q-3$ is even. Note that $(\lambda+\mu)(q-1) = i + j + 3a = q - 3 + 3a$ is divisible by 3, so that $\lambda + \mu = 3b$ for some b. It follows that the values of a in the above for which the τ_{3a} do not obviously vanish on L are given by

$$3a = 3b(q - 1) - q + 3 = 2q + 3(q - 1)(b - 1).$$

The first few values of $3a$ are thus $2q, 5q - 3, 8q - 6, \dots$

We call the cocycles τ_{3a} corresponding to these values of a the *loop cocycles* of L. To show that they are linearly independent in the second cohomology group of M, note the calculation, for $i = (q - 3)/2$, any $b \geq 1$, and the corresponding value of a as above,

$$\begin{aligned}
\tau_{3a}(y_i \otimes t^{(q-1)-i},\ y_i \otimes t^{(3b-1)(q-1)-i}) =& \\
&= (-1)^{i}(i - 1) \\
&= (-1)^{i+1} \\
&\neq 0.
\end{aligned}$$

Now the elements $y_i \otimes t^{(q-1)-i}$ and $y_i \otimes t^{(3b-1)(q-1)-j}$ commute in M, and there is one value of a for which the cocycle τ_{3a} does not vanish on the corresponding pair. Independence follows.

6. The Presentation

Let now N be the central extension of L via the finite cohomology cocycles and the loop cocycles τ_{3a}. In the rest of the paper we will show that N is finitely presented. As remarked in the Introduction, a result of B. H. Neumann implies that the finite cohomology cocycles and the loop cocycles span a subspace of finite codimension in the second cohomology group of L.

It is not difficult now to write down the action of the generators x and y on N, starting from the presentation of L, and keeping into account the cocycles we have added. Keeping the notation we have already employed for L, and thus slightly abusing it, define formally an element v_0 with the property

$$[v_0 x] = -x, \qquad [v_0 y] = y.$$

and elements v_i, for $i > 0$

$$v_i = [v_{i-1} x y \underbrace{x \dots x}_{q-3}].$$

Write also $\theta_i = [v_i y x] - [v_i x y]$.

Then we have, for $i \geq 0$,

(2)
$$\begin{cases} [v_i x y \underbrace{x \dots x}_{j} y] = 0, \quad \text{for } 0 \leq j \leq q-4, \text{ and } j \neq q - 3^m - 2, \text{ for } 1 \leq m < n \\ [v_i x y \underbrace{x \dots x}_{q-3^m-2} y x] = [v_i x y \underbrace{x \dots x}_{q-3^m-2} y y] = 0, \quad \text{for } 1 \leq m < n \\ [v_{i+1} x x] = [v_{i+1} y y] = 0 \\ [\theta_i x] = [\theta_i y] = 0, \\ \theta_i = 0 \quad \text{for } i \not\equiv 2 \pmod 3. \end{cases}$$

Now consider the Lie algebra Q given by the presentation

$$(3) \qquad Q = \Big\langle\, x, y : \begin{aligned}[t] & [y x \underbrace{x \dots x}_{j} y], \\ & \text{for } 0 \leq j \leq q-4, \text{ and } j \neq q - 3^m - 2, \text{ for } 1 \leq m < n \\ & [y x \underbrace{x \dots x}_{q-3^m-2} y x], \ \text{for } 1 \leq m < n \\ & [y \underbrace{x \dots x}_{q}], \\ & [y \underbrace{x \dots x}_{q-1} y \underbrace{x \dots x}_{q-5} y x], \\ & [y \underbrace{x \dots x}_{q-1} y \underbrace{x \dots x}_{q-3} y x x] = [y \underbrace{x \dots x}_{q-1} y \underbrace{x \dots x}_{q-3} x y x] \Big\rangle. \end{aligned}$$

Note that the relations in (3) are a subset of those of (2). We want to show that Q is isomorphic to N, by showing that the defining relations (3) of Q imply all of (2).

We follow the pattern of [Car97], omitting the steps that are identical, and dealing only with the peculiarities for $p = 3$.

We have immediately

$$0 = [y x \underbrace{x \dots x}_{q-3^m-3} [y x y]] = -[y \underbrace{x \dots x}_{q-3^m-2} y y],$$

so that $[yx\underbrace{x\dots x}_{q-3^m-2}y]$ is central for all m. The expansion

$$0 = [[y\underbrace{x\dots x}_{(q-1)/2}][y\underbrace{x\dots x}_{(q-1)/2}]]$$

yields the relation $[v_1xy] = [v_1yx]$, or $\theta_1 = 0$.

So we can suppose (2) are verified for values less then some $i > 2$. Let $v = v_i \in Q_{i(q-1)}$. In [Car97] we proved by induction on j

$$[vxy\underbrace{x\dots x}_{j}y] = 0$$

for $0 \le j \le q-4$, and $j \ne q-3^m-2$, for $1 \le m < n$. For these special values we now show

$$[vxy\underbrace{x\dots x}_{q-3^m-2}yx] = [vxy\underbrace{x\dots x}_{q-3^m-2}xy] = [vxy\underbrace{x\dots x}_{q-3^m-2}yy] = 0.$$

Proving $[vxy\underbrace{x\dots x}_{q-3^m-2}yy] = 0$ is straightforward, as in [Car97]. For the rest, let $j = q-3^m-2$, for $1 \le m < n$.

We begin, as in [Car97], with

$$\begin{aligned}
0 &= [vx[y\underbrace{x\dots x}_{q-3^m-1}y]]\\
&= [vx[y\underbrace{x\dots x}_{q-3^m-1}]y] - [vxy[y\underbrace{x\dots x}_{q-3^m-1}]]\\
&= [vxy\underbrace{x\dots x}_{q-3^m-1}y]\\
&\quad - (-1)^{q-3^m-2}(q-3^m-1)[vxy\underbrace{x\dots x}_{q-3^m-2}yx] - (-1)^{q-3^m-1}[vxy\underbrace{x\dots x}_{q-3^m-1}y]\\
&= -[vxy\underbrace{x\dots x}_{q-3^m-2}xy] + [vxy\underbrace{x\dots x}_{q-3^m-2}yx].
\end{aligned}$$

Now we want to prove that the two commutators vanish separately. The argument of [Car97] for $m = 1$ works also in characteristic 3, so we have to consider the case $m > 1$, for which there is no counterpart in [Car97]. Here and in the following, write v^{-s}, for $0 \le s < q-1$, for a homogeneous element such that $[v^{-s}\underbrace{x\dots x}_{s}] = v$.

Let thus $m > 1$. Then

$$\begin{aligned}
0 &= [v^{-3^m+3^{m-1}+1}[y \underbrace{x \dots x}_{q-3^{m-1}-1} y]] \\
&= [v^{-3^m+3^{m-1}+1}[y \underbrace{x \dots x}_{q-3^{m-1}-1}]y] \\
&= (-1)^{3^m-3^{m-1}-1}\binom{q-3^{m-1}-1}{3^m-3^{m-1}-1}[vyx \underbrace{x \dots x}_{q-3^m-2} xy] \\
&\quad + (-1)^{3^m-3^{m-1}}\binom{q-3^{m-1}-1}{3^m-3^{m-1}}[vxy \underbrace{x \dots x}_{q-3^m-2} xy].
\end{aligned}$$

Here we apply Lucas' theorem to get

$$\begin{aligned}
\binom{q-3^{m-1}-1}{3^m-3^{m-1}-1} &= \binom{q-3^m+3^m-3^{m-1}-1}{3^m-3^{m-1}-1} \\
&= \binom{3^m(3^{n-m}-1)}{0} \cdot \binom{3^m-3^{m-1}-1}{3^m-3^{m-1}-1} \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\binom{q-3^{m-1}-1}{3^m-3^{m-1}} &= \binom{3^{n-1}(3-1)+\dots+3^m(3-1)+3^{m-1}(3-2)+\dots+(3-1)}{3^{m-1}(3-1)} \\
&= \binom{1}{2} = 0.
\end{aligned}$$

Therefore

$$0 = ((-1)\cdot 1)\cdot [vxy \underbrace{x \dots x}_{q-3^m-2} y] = -[vxy \underbrace{x \dots x}_{q-3^m-2} xy],$$

as requested.

We prove easily, as in [Car97]

$$[v_{i+1}xx] = [v_{i+1}yy] = 0.$$

We now have to prove that $\theta_{i+1} = [v_{i+1}yx] - [v_{i+1}xy]$ is central.

We first show that $[v_{i+1}xy]$ and $[v_{i+1}yx]$ are centralized by y. This follows from

$$0 = -[v_{i+1}[yxy]] = [v_{i+1}xyy] - 2[v_{i+1}yxy] + [v_{i+1}yyx] = [[v_{i+1}xy] + [v_{i+1}yx], y],$$

and

$$\begin{aligned}
0 &= [v_{i+1}^{-3}[yxxxxy]] = [v_{i+1}^{-3}[yxxxx]y] \\
&= (-1)^3\binom{4}{3}[v_{i+1}^{-3}xxxyxy] + (-1)^4\binom{4}{4}[v_{i+1}^{-3}xxxxyy] \\
&= [-[v_{i+1}yx] + [v_{i+1}xy], y],
\end{aligned}$$

To prove that θ_{i+1} commutes with x, we will expand, in a variation on the corresponding argument in [Car97], and proceeding by induction on i,

$$0 = [v_{i-1}x, [v_2yx] - [v_2xy]].$$

We will take a more conceptual approach than in [Car97]. One first proves by direct easy expansion, using only the extended Jacobi identity, the following formulas, for appropriate $j \le i$:

$$\begin{aligned}
&[v_jxv_1] = [v_{j+1}x], \qquad [v_jyv_1] = -[v_{j+1}y]\\
&[v_jxyv_1] = [v_{j+1}yx] - [v_{j+1}xy] = \theta_{j+1}\\
&[v_jv_1] = 0\\
&[v_jxyxv_1] = [v_jxyx].
\end{aligned}$$

We then use these to compute

$$\begin{aligned}
0 &= [v_{i-1}x, -[v_2yx] + [v_2xy]]\\
&= [v_{i-1}x[v_1yv_1x]] + [v_{i-1}x[v_1xv_1y]]\\
&= [v_{i-1}x[v_1yv_1]x] - [v_{i-1}xx[v_1yv_1]]\\
&\quad + [v_{i-1}x[v_1xv_1]y] - [v_{i-1}xy[v_1xv_1]]\\
&= [v_{i-1}x[v_1y]v_1x] - [v_{i-1}xv_1[v_1y]x]\\
&\quad + [v_{i-1}x[v_1x]v_1y] - [v_{i-1}xv_1[v_1x]y]\\
&\quad - [v_{i-1}xy[v_1x]v_1] + [v_{i-1}xyv_1[v_1x]]\\
&= [v_{i-1}x[v_1y]v_1x] - [v_ix[v_1y]x]\\
&\quad + [v_{i-1}x[v_1x]v_1y] - [v_ix[v_1x]y]\\
&\quad - [v_{i-1}xy[v_1x]v_1]\\
&= [v_{i-1}xv_1yv_1x] - [v_{i-1}xyv_1v_1x]\\
&\quad - [v_ixv_1yx] + [v_ixyv_1x]\\
&\quad + [v_{i-1}xv_1xv_1y] - [v_{i-1}xxv_1v_1y]\\
&\quad - [v_ixv_1xy] + [v_ixxv_1y]\\
&\quad - [v_{i-1}xyv_1xv_1] + [v_{i-1}xyxv_1v_1]\\
&= [\theta_{i+1}x] - 0 - [v_{i+1}xyx] + [\theta_{i+1}x] + 0 - 0 - 0 + 0 - 0 + [v_{i+1}xyx]\\
&= -[\theta_{i+1}x],
\end{aligned}$$

as desired.

We are left with proving that only one in three of the θ_i can be nonzero. We first prove, by induction on k, the formula $[[v_hx][v_ky]] = [v_{h+k}xy] - k\theta_{h+k}$. The basis of the induction is provided by

$$[[v_hx][v_1y]] = [v_hxv_1y] - [v_hv_1xy] = [v_{h+1}xy] - \theta_{h+1},$$

and then the induction step is

$$\begin{aligned}[][[v_h x][v_k y]] &= -[[v_h x][v_{k-1} y v_1]] \\ &= -[[v_h x][v_{k-1} y] v_1] + [[v_h x] v_1 [v_{k-1} y]] \\ &= -[[v_{h+k-1} x y] v_1] + [[v_{h+1} x][v_{k-1} y]] \\ &= -\theta_{h+k} + [v_{h+k} x y] - (k-1)\theta_{h+k} \\ &= [v_{h+k} x y] - k\theta_{h+k}. \end{aligned}$$

In particular, for $h = 1$ we have

$$[[v_1 x][v_i y]] = [v_{i+1} x y] - i\theta_{i+1},$$

and also

$$\begin{aligned}[][[v_1 x][v_i y]] &= -[[v_i y][v_1 x]] \\ &= -[v_i y v_1 x] + [v_i y x v_1] \\ &= [v_{i+1} y x] + \theta_{i+1}. \end{aligned}$$

Comparing the two expressions we obtain

$$(i+2)\theta_{i+1} = 0,$$

so that θ_j can indeed only be nonzero for

$$j \equiv 2 \pmod 3.$$

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