THIN GROUPS OF PRIME-POWER ORDER AND THIN LIE ALGEBRAS: AN ADDENDUM

A. CARANTI

Introduction

In [CMNS] we have shown that certain modular graded Lie algebras called thin Lie algebras T are uniquely determined by their factors of low dimension. This was achieved by constructing certain graded Lie algebras L which are finitely presented, and by showing that their central quotients $L/Z(L)$ are isomorphic to T. The question left open in [CMNS] was how big $Z(L)$ actually was. As we will see below, this is equivalent to asking whether T itself is finitely presented or not. This was not critical for our purposes in [CMNS], so we were content with giving an upper bound for $Z(L)$, and by providing computational evidence for the fact that $Z(L) \neq \{0\}$.

In this short note we determine exactly $Z(L)$, which turns out to be infinite-dimensional. In [CMNS] the Lie algebras T were determined by grading special linear Lie algebras of dimension one and two over the p-adic integers: this approach was used in [Mat] to construct the corresponding groups. The idea here is to rewrite first the algebras of [CMNS] in terms of loop algebras of special linear Lie algebras of dimension one and two over the prime field. Once this is done, one uses an adaptation to the modular case of an argument in [Gar]. As in [Car] the fact that $Z(L)$ is infinite-dimensional implies, by the Lie algebra analogue of a well-known result of B. H. Neumann ([Neu], or see [Rob, 2.2.3]), that the thin Lie algebras $T \cong L/Z(L)$ are not finitely presented although one might say that, just because of this, they come close.

We also extend to characteristic 3 a result of [CMNS] on the presentation of the algebras L as above. Together with the results mentioned in the previous paragraph, this covers the case of the graded Lie algebra associated to the Nottingham group in characteristic 3, which could not be dealt with using the techniques of [Car].

General notation is standard. We write $\langle x, y \rangle$ for the linear span of x and y in an appropriate vector space, and use Kronecker's delta

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

We refer to [CMNS] for several details.

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Loop algebras of Lie algebras of type A_1 and their cocycles

We deal first with the case of an algebra of type A_1 over the field $\mathbf{F}_p$ with p elements, for $p > 2$, which we write in the notation of [CMNS] as

$$sl(2, \mathbf{F}_p) = \langle\, x, c, y : [xy] = c, [cx] = y, [cy] = \lambda x \,\rangle\,,$$

for some $\lambda \in \mathbf{F}_p^{\star}$. Here $\langle\, c \,\rangle$ is a Cartan subalgebra; the algebra will be split (see [Jac]) if and only if λ is a square in $\mathbf{F}_p$. This algebra admits a cyclic grading over the group with two elements, assigning weight 1 to x and y, and weight 2 to c. Consider a polynomial algebra $\mathbf{F}_p[\xi]$, and in the tensor Lie algebra $sl(2, \mathbf{F}_p) \otimes \mathbf{F}_p[\xi]$ consider the subalgebra L generated by

$$x \otimes \xi \quad \text{and} \quad y \otimes \xi.$$

This is usually called the loop algebra with respect to the given grading, and inherits a grading over the integers

$$T = \bigoplus_{i=1}^{\infty} T_i,$$

where

$$T_i = \begin{cases} \langle\, x \otimes \xi^i, y \otimes \xi^i \,\rangle & \text{if } i \text{ is odd} \\ \langle\, c \otimes \xi^i \,\rangle & \text{if } i \text{ is even} \end{cases}$$

This algebra is visibly isomorphic to the thin Lie algebra T with first chain length 1 of [CMNS].

In [CMNS] we gave a finite presentation for a certain graded Lie algebra L, for $p > 3$, and showed that $L/Z(L) \cong T$. Moreover we showed that $Z(L)$ has at most one nonzero element z_{ap} of weight ap, for each even positive integer a.

We now show that all these z_{ap} are nonzero. We first exhibit countably many linearly independent 2-cocyles for T, mimicking the characteristic zero construction of [Gar]. Let κ be the Killing form on $sl(2, \mathbf{F}_p)$, and for each even integer $a \geq 2$ define a bilinear map by prescribing

$$\begin{aligned} \sigma_{ap} : \mathbf{F}_p[\xi] \times \mathbf{F}_p[\xi] &\to \mathbf{F}_p \\ (\xi^i, \xi^j) &\mapsto j \cdot \delta(i + j, ap). \end{aligned}$$

We claim that the bilinear maps defined by

$$\begin{aligned} \tau_{ap} : T \times T &\to \mathbf{F}_p \\ (u \otimes \xi^i, v \otimes \xi^j) &\mapsto \kappa(u, v) \cdot \sigma_{ap}(\xi^i, \xi^j) \end{aligned}$$

are 2-cocyles.

Note first that as κ is symmetric and associative we have

$$\kappa([vw], u) = \kappa(u, [vw]) = \kappa([uv], w) \quad \text{and} \quad \kappa([wu], v) = \kappa(w, [uv]) = \kappa([uv], w).$$

Moreover

$$\begin{aligned} \tau_{ap}([u \otimes \xi^i, v \otimes \xi^j], w \otimes \xi^k) &= \tau_{ap}([uv] \otimes \xi^{i+j}, w \otimes \xi^k) \\ &= \kappa([uv], w) \cdot \sigma_{ap}(\xi^{i+j}, \xi^k), \end{aligned}$$

so that one has to verify the cocycle condition for σ_{ap}, for it to hold for τ_{ap}:

$$\begin{aligned}&\sigma_{ap}(\xi^{i+j}, \xi^k) + \sigma_{ap}(\xi^{j+k}, \xi^i) + \sigma_{ap}(\xi^{k+i}, \xi^j)\\ &= (k + j + i) \cdot \delta(i + j + k, ap)\\ &= 0.\end{aligned}$$

On a given pair $u \otimes \xi^i, v \otimes \xi^j$ of homogeneous elements the only cocyle τ that can possibly take a nonzero value is τ_{ap}, for $i + j = ap$, provided $i + j$ is an even multiple of p. As

$$\tau_{ap}(c \otimes \xi^{ap-2}, c \otimes \xi^2) = \kappa(c, c) \cdot 2 = 4\lambda \neq 0,$$

it follows that the τ_{ap} are linearly independent in the cohomology group, and in particular none of them vanishes. Now take an abelian Lie algebra

$$Z = \bigoplus_{a \in 2\mathbb{N}^\star} \langle\, z_{ap} \,\rangle\,,$$

that is, Z has a basis z_{ap} indexed over the positive even integers a, and construct a central extension

$$0 \to Z \to L \to T \to 0$$

by defining the Lie product of $u \otimes \xi^i$ and $v \otimes \xi^j$ in L as

$$[uv] \otimes \xi^{i+j} + \sum_{a \in 2\mathbb{N}^\star} \tau_{ap}(u \otimes \xi^i, v \otimes \xi^j) \cdot z_{ap}\,.$$

The discussion above implies that in the infinite summation all terms but possibly one vanish. Clearly the z_{ap} are compatible with the given grading, as they acquire weight ap. The resulting algebra L is thus the appropriate algebra L of [CMNS].

The results of [CMNS] show that no z_{ap} appear for a odd. From the point of view we are taking here this depends on the fact that

$$\kappa(x, c) = \kappa(y, c) = 0,$$

so that for a odd the τ_{ap} vanish. See the end of the next section for a more general comment.

It might be noted that the graded Lie algebra associated to the Nottingham group in characteristic 3 is the split form of T here. We comment on this in the last section.

The Lie algebra of type A_2

Consider the Lie algebra of type A_2 over the algebraic closure E of $\mathbf{F}_p$, for $p > 3$, given in terms of a Chevalley basis, with the choice of signs which is implicit in the following calculations (see [Hum, 25.2]):

$$sl(3, E) = \langle\, h_\alpha, h_\beta, e_\alpha, e_\beta, e_{\alpha+\beta}, e_{-\alpha}, e_{-\beta}, e_{-\alpha-\beta} \,\rangle\,.$$

Here α and β are the simple roots, $\langle\, h_\alpha, h_\beta \,\rangle$ is a Cartan subalgebra, and the e_γ are generators for the root spaces.

Consider the elements

$$x = e_\beta + \mu^{-1} e_{-\alpha-\beta}, \quad y = e_\alpha,$$

where $\mu \in E$, with $\mu^2 = 3$, and let S be the subalgebra of $sl(3, E)$ generated by x and y. We have, as in [Mat], and using notation compatible with [CMNS],

$$\begin{cases} c = [yx] = e_{\alpha+\beta} - \mu^{-1} e_{-\beta}, & \\ d = [cx] = \mu^{-1} \cdot (h_\alpha + 2h_\beta), & [cy] = 0 \\ v = [dx] = \mu e_\beta - e_{-\alpha-\beta} & [dy] = 0 \\ s = [vx] = 2e_{-\alpha}, & t = [vy] = -\mu e_{\alpha+\beta} - e_{-\beta}, \\ [sx] = 0, & w = [tx] = -h_\alpha, \\ [sy] = -2h_\alpha = 2w, & [ty] = 0, \\ [wx] = x, & [wy] = -2y. \end{cases}$$

Thus the Lie algebra S has dimension 8, and it is defined over $\mathbf{F}_p$. Moreover, S has a grading

$$S = \bigoplus_{i=1}^{6} S_i$$

over a cyclic group of order 6, obtained by assigning weight 1 to x and y, so that

$$S_1 = \langle\, x, y \,\rangle, S_2 = \langle\, c \,\rangle, S_3 = \langle\, d \,\rangle, S_4 = \langle\, v \,\rangle, S_5 = \langle\, s, t \,\rangle, S_6 = \langle\, w \,\rangle.$$

As before, we may construct the Lie subalgebra T of $S \otimes \mathbf{F}_p[\xi]$ generated by $x \otimes \xi$ and $y \otimes \xi$, which is graded over the positive integers. This is easily seen to be isomorphic to the infinite-dimensional thin algebra with first chain length 3 of [CMNS].

We can construct 2-cocycles τ_{ap} for T, for a positive, $a \equiv 0 \pmod 6$, as in the previous case. As before, these cocycles are linearly independent in the cohomology group, and thus nonzero, since

$$\kappa(h_\alpha, h_\alpha) = 12,$$

so that for $a \equiv 0 \pmod 6$

$$\tau_{ap}(w \otimes \xi^{ap-6}, w \otimes \xi^6) = \kappa(w, w) \cdot 6 = 72 \neq 0$$

as $p > 3$. As in the previous section, this allows us to construct the algebra L of [CMNS] for this case. The results of [CMNS] imply that for $a \not\equiv 0 \pmod 6$ the τ_{ap} vanish. Sandro Mattarei has remarked that this can be read off from the cyclic grading. In fact, if e and f are homogeneous with respect to the grading over a cyclic group of order 6, then the value of the Killing form

$$\kappa(e, f) = \mathrm{tr}(\mathrm{ad}(e)\mathrm{ad}(f))$$

can only be nonzero when the sum of the weights of the endomorphisms $\mathrm{ad}(e)$ and $\mathrm{ad}(f)$ (which coincide with the weights of the elements e and f) is a multiple of 6, the period of the grading.

The graded Lie algebra associated to the Nottingham group for the prime 3

It follows easily from the results of [Car] that the graded Lie algebra associated to the Nottingham group in characteristic 3 is the loop algebra T of the split form of $sl(2, \mathbf{F}_3)$ (that is, for $\lambda = 1$) that we have constructed here. We have shown earlier that $T \cong L/Z(L)$ for a suitable graded Lie algebra L, with $Z(L)$ infinite-dimensional. By the result of B. H. Neumann mentioned in the introduction, T is not finitely presented: this extends to this case a result of [Car].

In [CMNS] we had given finite presentations for the algebras L as above, for primes $p > 3$. We now show that the algebras L are finitely presented for $p = 3$ too; the argument is just slightly more complicated than that of [CMNS]. This extends another result of [Car] to this case.

In [CMNS] we had shown that the algebras L for $p > 3$ admit the finite presentation

$$L = \langle\, x, y : [yxxy], [yxyx], \lambda[yxxx] + [yxyy] \,\rangle .$$

We claim that for $p = 3$ a presentation is

$$\begin{aligned} L = \Big\langle\, x, y : [yxxy], [yxyx], \lambda[yxxx] + [yxyy] \\ [yxxxxyx], [yxxxxyy] \,\Big\rangle. \end{aligned}$$

All is needed to complete the proof already given in [CMNS, pp. 291–3] is to show that if one defines $c_0 = [yx]$, and $c_i = [c_{i-1}xx]$, for $i > 0$, then the elements $[c_i xy]$ are central. The calculations in [CMNS] that derived this from the other relations failed for $p = 3$, and computations show that extra relations are indeed necessary. Note that in this notation the relations we have added force $[c_1 xy]$ to be central.

We can exploit the results of [CMNS], and proceed by induction on i, so that we may assume

$$\begin{aligned} &[c_j yy] = -\lambda[c_j xx] \text{ for } j \leq i, \\ &[c_j xy], [c_j yx] \in Z(L) \text{ for } j < i, \\ &[c_i xy] + [c_i yx] \in Z(L). \end{aligned}$$

The result follows from the two calculations, for $i \geq 2$,

$$\begin{aligned} 0 &= [c_{i-2}x[yxxxxy]] \\ &= [c_{i-2}x[yxxxx]y] \\ &= -[c_{i-2}xxxxyxy] + [c_{i-2}xxxxxyy] \\ &= [-[c_i yx] + [c_i xy], y], \end{aligned}$$

which together with the above shows that $[c_i xy]$ and $[c_i yx]$ are centralized by y; and

$$\begin{aligned} 0 &= [c_{i-2}y[yxxxxy]] \\ &= [c_{i-2}y[yxxxx]y] - [c_{i-2}yy[yxxxx]] \\ &= [c_{i-2}yyxxxxy] + [c_{i-2}yyxxxyx] - [c_{i-2}yyxxxxy] \\ &= -\lambda[[c_i xy], x], \end{aligned}$$

which shows that $[c_i xy]$, and thus $[c_i yx]$, are centralized also by x.

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Dipartimento di Matematica, Università degli Studi di Trento, via Sommarive 14, I-38050 Povo (Trento), Italy

E-mail address: caranti@science.unitn.it