QUOTIENTS OF MAXIMAL CLASS OF THIN LIE ALGEBRAS. THE ODD CHARACTERISTIC CASE

A. CARANTI AND G. JURMAN

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DEGLI STUDI DI TRENTO
I-38050 POVO (TRENTO)
ITALY

ABSTRACT. Among thin graded Lie algebras, which are particular instances of Lie algebras of finite width, there are many interesting objects, such as the graded Lie algebra associated to the Nottingham group. Among the factors of a thin algebra with respect to the terms of the lower central series, there is a greatest factor which is of maximal class. In thin Lie algebras associated to groups, this factor is metabelian.

In this paper we show that the same holds in general, provided the characteristic of the underlying field is odd. In another paper by the second author it is shown that this is not the case for characteristic two.

1. Introduction

C. R. Leedham-Green [LG94] and A. Shalev [Sha94] have proved the conjectures about pro-p-groups of finite coclass proposed by C. R. Leedham-Green and M. F. Newman in [LGN80], thereby establishing what could be considered the first classification theorem for p-groups.

A. Shalev and E. Zelmanov [SZ97] have shown that a satisfactory theory also holds for graded Lie algebras of finite coclass in characteristic zero, provided they are generated by elements of weight one.

There are other *narrowness* conditions, more general that than of being of finite coclass, that one may consider for groups and algebras. A rather general one is that of having *finite width*, that is to say, there exists a constant that bounds the orders (resp. dimensions) of the lower central factors. A finer

Date: 5 March 1998.

1991 *Mathematics Subject Classification.* 17B70 17B65 17B30 20F40.

Key words and phrases. Graded Lie algebras of maximal class, thin Lie algebras.

First author partially supported by MURST-Italy, and by CNR-GNSAGA under *Sottoprogetto "Calcolo Simbolico" del Progetto Strategico "Applicazioni della Matematica per la Tecnologia e la Società"*. Both authors are members of GNSAGA, Italy.

measure of narrowness is provided by the so called *obliquity* introduced in [LGPK97]. (Groups of obliquity zero have been considered in [BS97] where they are called normally constrained.) Groups of obliquity zero with $|G/G_2| = p^2$ have been called thin (see [Bra88] and [BCS92]), and this definition can be carried over to graded Lie algebras. Here and in the following, we refer to the next section for the precise definitions.

The class of thin pro-p-groups, and of graded Lie algebras over fields of characteristic $p > 0$, contains several interesting objects, such as certain groups and algebras arising from simple Lie algebras of type A_1 and A_2 (see [CMNS96], [Car99], [LM78], [Fia83]), the Nottingham group and its associated Lie algebra (see [Car97]), and certain Lie algebras related to Hamiltonian algebras (see [CM98]). The homogeneous components of an infinite-dimensional thin Lie algebra $L = \bigoplus_{i=1}^{\infty} L_i$ have dimension 1 or 2. When $\dim(L_i) = 2$, we say L_i is a diamond. Now L_1 is always a diamond. If there are no other diamonds, then L is of maximal class, and so by convention is not considered thin. If L is thin proper, then there is a second diamond L_k, and

$$S = L/L^k = L/\bigoplus_{i \geq k} L_i$$

has coclass one, and it is called a *graded Lie algebra of maximal class.* Such an algebra S is determined up to isomorphism by its two-step centralizers [CMN97]. In thin Lie algebras associated to groups, these two-step centralizers are all the same. Equivalently, S is metabelian (see [CMN97, Lemma 3.1]). This is not obviously the case for Lie algebras not coming from groups. It is the goal of this paper and its sequel [Jur98] to investigate this situation.

In this paper we investigate the situation over fields of odd characteristic. It turns out that if a factor of maximal class of a thin Lie algebra L with respect to a term of the lower central series has at least two distinct two-step centralizers, that is, it is not metabelian, then L is finite-dimensional. In [Jur98] the situation over fields of characteristic 2 is investigated. This turns out to be more complicated. In particular, there are infinite-dimensional thin Lie algebras L with a non-metabelian factor of maximal class. However, these algebras can be satisfactorily classified.

We are grateful to Mike Newman for reading an earlier version of the manuscript and suggesting several improvements.

2. Preliminaries

Our iterated commutators are left-normed. We use without further mention the generalized Jacobi identity

$$[u[yx^n]] = \sum_{i=0}^{n} (-1)^i \binom{n}{i} [ux^i yx^{n-i}].$$

Here we use shorthands like

$$[yx^n] = [y\underbrace{x \dots x}_{n}].$$

To compute binomial coefficients modulo a prime p we will use several times Lucas' Theorem [Luc78]: if the nonnegative integers a and b are written in base p, that is, $a = \sum_{i=0}^{n} a_i p^i$, $b = \sum_{i=0}^{n} b_i p^i$, with $0 \le a_i, b_i < p$ for all i, then

$$\binom{a}{b} \equiv \prod_{i=0}^{n} \binom{a_i}{b_i} \pmod{p}.$$

Let $L = \bigoplus_{i=1}^{\infty} L_i$ be a Lie algebra over a field $\mathbf{F}$ of positive characteristic p, graded over the positive integers. If such an algebra is generated by L_1, then we have $L^m = \bigoplus_{i \ge m} L_i$ for the Lie powers of L.

L is said to be *of maximal class* (that is, of coclass one, see [CMN97]) if $\dim(L_1) = 2$, while $\dim(L_i) \le 1$, for $i > 1$, and $L_i = [L_{i-1}, L_1]$ for $i > 1$. The last condition amounts to saying that L is generated by L_1 as an algebra. If L is of finite dimension d, then it has nilpotency class $d - 1$, that is, the largest nilpotency class compatible with its dimension. If L is infinite-dimensional, then each finite-dimensional quotient L/L^d has this property, for $d > 1$.

L is said to be *thin* (see [CMNS96]) if $\dim L_1 = 2$, and the following *covering property* holds: for each $i \ge 1$ and for each $0 \ne z \in L_i$ one has $[zL_1] = L_{i+1}$. The covering property implies that the algebra is generated by elements of weight 1, and that $\dim(L_i) \le 2$, for each $i \ge 1$. Algebras of maximal class are not considered thin by convention. If L is thin, a homogeneous component L_i of dimension 2 is called a *diamond*.

An algebra $L = \bigoplus_{i=1}^{\infty} L_i$ of maximal class over the field $\mathbf{F}$ is determined up to isomorphism by its sequence of *two-step centralizers* (here and in the following, see [CMN97] for details). For simplicity, suppose L to be infinite-dimensional; then a two-step centralizer in L is defined to be

$$C_i = C_{L_1}(L_i),$$

for $i > 1$; this is a subspace of dimension 1 of the two-dimensional space L_1. We fix the notation $\mathbf{F}y = C_2$. One can see that if all two-step centralizers coincide with $\mathbf{F}y$, then L is metabelian. Otherwise, write $\mathbf{F}x$ for the first two-step centralizer C_m distinct from $\mathbf{F}y$. One can see that if m is not of the form $2p^h$, for some $h \ge 1$, then L is finite-dimensional, against our assumption. If $m = 2p^h$, we call $q = p^h$ the *parameter* of L. One sees that if for some i and l

$$C_i \ne \mathbf{F}y, \qquad C_{i+1} = C_{i+2} = \dots = C_{i+l-1} = \mathbf{F}y, \qquad C_{i+l} \ne \mathbf{F}y,$$

then, provided L is infinite-dimensional, l can only assume the values $2q$ and $2q - p^s$, for $0 \leq s \leq h$. We call the sequence

$$C_{i+1}, C_{i+2}, \dots, C_{i+l-1}$$

a *constituent* of L.

3. The problem

We prove

Theorem. *Let $L = \bigoplus_{i=1}^{\infty} L_i$ be an infinite-dimensional thin algebra over the field $\mathbf{F}$ of characteristic $p > 2$. Then its quotients of maximal class are metabelian.*

According to [CMN97, Lemma 3.1], an algebra of maximal class is metabelian if and only if it has only one two-step centralizer. We will prove that if the biggest quotient of maximal class of L has at least two distinct two-step centralizers, then L is finite-dimensional.

We use the notation

$$C_i = C_{L_1}(L_i)$$

for the two-step centralizers in L, as in algebras of maximal class. These will be one-dimensional, except that $C_i = \{0\}$ if L_i or L_{i+1} is a diamond. The second case is immediate. The first one follows from the covering property: let L_i be a diamond, and $0 \neq v \in L_{i-1}$. If $C_i = L_1$, then $L_{i+1} = [L_i L_1] = \{0\}$, a contradiction. If $C_i = \mathbf{F}z \neq \{0\}$, choose $z' \in L_1$ so that

$$L_1 = \mathbf{F}z + \mathbf{F}z', \qquad \text{and thus} \qquad L_i = \mathbf{F}[vz] + \mathbf{F}[vz'].$$

We have $[vzz] = [vz'z] = 0$, so that by the covering property

$$L_{i+1} = \mathbf{F}[vzz'] = \mathbf{F}[vz'z']$$

is one-dimensional. But if $[vzz'] = \lambda[vz'z']$ for some λ, we obtain $[v, z - \lambda z', z'] = 0$, so that $[[v, z - \lambda z'], L_1] = \{0\}$, thereby violating the covering property.

Suppose L_k is the second diamond after L_1. Therefore L/L^k is an algebra of maximal class of dimension k. Let $\mathbf{F}y = C_2$. We assume that L/L^k has at least two distinct two-step centralizers, and want to show that this leads to a contradiction. As L is infinite-dimensional, the arguments from [CMN97, Theorem 5.5] we have just mentioned imply that the second two-step centralizer of L/L^k occurs as $\mathbf{F}x = C_{2q}$, where $q = p^h$ for some $h \geq 1$. This is because these arguments show that if an algebra M has a quotient of maximal class in which the second two-step centralizer does not occur in such a position, then M is finite-dimensional. Similarly, the constituents of L/L^k have the form we have just described. Note that we have the identities $[yx^j y] = 0$, for $0 \leq j < 2q - 1$, and $[yx^{2q}] = 0$.

Suppose we have

$$C_m \neq \mathbf{F}y, \qquad C_{m+1} = C_{m+2} = \cdots = C_{k-2} = \mathbf{F}y, \qquad C_{k-1} = \{0\}\,.$$

The proof of [CMN97, Proposition 5.6] applies here to show that if $t = k-m-2$, then $t+1$ has to obey the restriction on the length of a constituent mentioned above.

We fix some notation. Let $0 \neq u \in L_{m-1}$, so that $0 \neq v = [uxyx^t] \in L_{k-1}$. Therefore L_k is spanned by the independent elements $[vx]$ and $[vy]$. Write $v^{-j} = [uxyx^{t-j}]$, for $0 < j \leq t$, so that $[v^{-j}x^j] = v$. We have

$$[uy] = 0, \quad v^{-t} = [uxy] \neq 0, \quad [uxyx^s y] = 0 \text{ for } 0 \leq s < t.$$

We distinguish three cases:

- $t = 2q-1$;
- $t = 2q - p^s - 1$, for $s > 0$;
- $t = 2q-2$.

3.1. **The case $t = 2q-1$.** As $L_{m+1} = \mathbf{F}[uxy]$, we have $[uxx] = \lambda[uxy]$ for some $\lambda \in \mathbf{F}$. (Thus $C_m = \mathbf{F}\cdot(x-\lambda y)$.) We compute

$$\begin{aligned} 0 &= [[ux][yx^{2q}]] \\ &= [uxyx^{2q}] + [uxx^{2q}y] \\ &= [uxyx^{t+1}] + \lambda[uxyx^t y] \\ &= [vx] + \lambda[vy] \\ &= [v, x+\lambda y]. \end{aligned}$$

It follows that $\dim(L_k) < 2$, a contradiction.

3.2. **The case $t = 2q - p^s - 1,\ s > 0$.** First of all, a standard calculation, as in [CMNS96], yields

$$\begin{aligned} 0 &= [v^{-1}[yxy]] \\ &= -[v^{-1}xyy] + 2[v^{-1}yxy] - [v^{-1}yyx] \\ &= -[vyy] + 2[v^{-1}yxy] - [v^{-1}yyx] \\ &= -[vyy]. \end{aligned}$$

We have $2q - p^s + 1 < 2q-1$, as $p^s \geq p > 2$, so that $[yx^{2q-p^s+1}y] = 0$. This yields

$$\begin{aligned} 0 &= [u[yx^{2q-p^s+1}y]] = [u[yx^{2q-p^s+1}]y] \\ &= \sum_{i=0}^{2q-p^s+1} (-1)^i \binom{2q-p^s+1}{i} [ux^i yx^{2q-p^s+1-i}y] \\ &= -(2q-p^s+1)[uxyx^{2q-p^s}y] \\ &= -[vxy]. \end{aligned}$$

We obtain $C_k \supseteq \mathbf{F}y$, a contradiction.

3.3. **The case $t = 2q - 2$.** We need the following Lemma. This is part of a much more detailed theory of graded Lie algebras of maximal class, developed in [CN99], which includes an enumeration of isomorphisms classes.

Lemma. *Let L be an algebra of maximal class over the field $\mathbf{F}$ of characteristic $p > 2$. We assume that L is infinite-dimensional, or that it is a factor of an infinite-dimensional thin algebra. In either case the parameter $q = p^h$ is defined, for some $h \geq 1$. Let $C_2 = \mathbf{F}y$, and let $\mathbf{F}x$ be the second two-step centralizer in order of occurrence.*

1. *If $C_i \neq \mathbf{F}y$, then $C_{i+1} = \mathbf{F}y$.*
2. *Suppose the sequence of two-step centralizers contains the pattern*

$$\mathbf{F}z, \mathbf{F}y, \dots, \mathbf{F}y, \mathbf{F}t, \mathbf{F}y, \dots, \mathbf{F}y$$

   where $\mathbf{F}t \neq \mathbf{F}y$, and $\mathbf{F}z \neq \mathbf{F}y, \mathbf{F}x$. (Note that by the theory of constituent lengths there must be at least $q - 1$ trailing $\mathbf{F}y$ here.) Then $\mathbf{F}t = \mathbf{F}x$.
3. *A two-step centralizer other than $\mathbf{F}y$ and $\mathbf{F}x$ must occur at the end of a constituent of length $q - 1$.*
4. *A two-step centralizer other than $\mathbf{F}y$ and $\mathbf{F}x$ is followed by a constituent of length $q - 1$.*

Proof. The first statement is already in [CMN97]. Let $L_{i-1} = \mathbf{F}w$, and $x \in L_1 \setminus C_2$. We have by induction $C_{i-1} = \mathbf{F}y$, so that $[wy] = 0$ and

$$0 = [w[yxy]] = -[wxyy] + 2[wyxy] - [wyyx] = -[wxyy].$$

The result follows from the fact that $L_{i+1} = \mathbf{F}[wxy]$.

The second statement is a second level refinement of the first one, and can be conveniently proved using the *deflation* technique of [CMN97]. We refer to this paper for a justification of the arguments employed in the following. Deflating h times the algebra L we obtain an algebra of maximal class M whose sequence of two-step centralizers contains a pattern of the form

$$\mathbf{F}z, \mathbf{F}t \qquad \text{or} \qquad \mathbf{F}z, \mathbf{F}y, \mathbf{F}t.$$

Now $C_{M_1}(M_2) = \mathbf{F}x \neq \mathbf{F}z$. In the first case we apply the previous result to get $\mathbf{F}t = \mathbf{F}x$, while the second case leads to a contradiction.

For the third and fourth statement, suppose we have the pattern

$$\mathbf{F}x, \mathbf{F}y, \dots, \mathbf{F}y, \mathbf{F}z, \mathbf{F}y, \dots, \mathbf{F}y$$

in the sequence of two-step centralizer, with $l \geq q - 1$ intermediate $\mathbf{F}y$, and at least $q - 1$ trailing $\mathbf{F}y$. If the first centralizer in the pattern is C_i, choose $0 \neq w \in L_i$, and compute

$$0 = [w[yx^{2q}]] = [wyx^{2q}] = [wyx^l x x^{2q-l-2} x].$$

This shows that there is a trailing constituent of length $2q - l - 2 \leq q - 1$, as $l \geq q - 1$. It follows $2q - l - 2 = q - 1$, and $l = q - 1$. This proves both statements. ∎

Because of the last statement of the Lemma, we have $C_m = \mathbf{F}x$. (Note that we might have used this to simplify slightly the argument in the case $t = 2q-1$ above.) We obtain

$$0 = [[ux][yx^{2q}]] = [uxyx^{2q}] = [vxx].$$

By the covering property, $[vxy]$ and $[vyx]$ are nonzero and linearly dependent.

We have $[yx^{q+j}y] = 0$ for $0 \le j \le q-2$, so that we obtain, proceeding by induction on j,

$$\begin{aligned} 0 &= [v^{-(q-1)}[yx^{q+j}y]] \\ &= [v^{-(q-1)}[yx^{q+j}]y] - [v^{-(q-1)}y[yx^{q+j}]] \\ &= (-1)^{q-1}\binom{q+j}{q-1}[vyxx^j y] + (-1)^q \binom{q+j}{q}[vxyx^j y] \end{aligned}$$

for $0 \le j \le q-2$. Lucas' theorem yields

$$\binom{q+j}{q-1} = \binom{q}{0} \cdot \binom{j}{q-1} = 0,$$

as $j < q-1$, and

$$\binom{q+j}{q} = \binom{q}{q} \cdot \binom{j}{0} = 1,$$

so that

$$[vxyx^j y] = 0$$

for $0 \le j \le q-2$. Therefore

$$L_{k+i} = \mathbf{F}[vxyx^{i-1}]$$

for $1 \le i \le q$.

Now

$$\begin{aligned} 0 &= [v^{-q}[yx^{2q}]] \\ &= (-1)^q \binom{2q}{q}[vyxx^{q-1}] + (-1)^{q+1}\binom{2q}{q+1}[vxyx^{q-1}] \\ &= -2\ [vyxx^{q-1}], \end{aligned}$$

again by Lucas' theorem. We obtain $L_{k+q} = 0$, a final contradiction. This is the place where the peculiarities of the case $p = 2$ sneak in (see [Jur98]).

4. Computations

Extensive usage of the ANU p-Quotient Program ([HNO97]) and of GAP ([S$^+$97]) has been made to obtain computational evidence while working at this paper. We refer to [CMNS96] and [CMN97] for similar details.

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(A. Caranti) Dipartimento di Matematica, Università degli Studi di Trento, via Sommarive 14, I-38050 Povo (Trento), Italy
E-mail address: caranti@science.unitn.it
URL: http://www-math.science.unitn.it/~caranti/

(G. Jurman) Dipartimento di Matematica, Università degli Studi di Trento, via Sommarive 14, I-38050 Povo (Trento), Italy
E-mail address: jurman@science.unitn.it
URL: http://www-math.science.unitn.it/~jurman/