PRESENTING THE GRADED LIE ALGEBRA ASSOCIATED TO THE NOTTINGHAM GROUP

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Abstract. The graded Lie algebra L associated to the Nottingham group is a loop algebra of the Witt algebra W_1. The universal covering $\widehat{W_1}$ of W_1 has 1-dimensional centre, so that the corresponding loop algebra M of $\widehat{W_1}$ has an infinite-dimensional centre $Z(M)$. As $M/Z(M)$ is isomorphic to L, it follows from a result of B. H. Neumann that L is not finitely presented. However we are able to show that M itself is finitely presented.

We work more generally with the Zassenhaus algebras W_n. In the group context, examples of finitely presented groups whose centre is not finitely generated were given by V. N. Remeslennikov and H. Abels.

Introduction

The graded Lie algebra L associated to the Nottingham group over the field $\mathbf{F}_p$ with p elements, p an odd prime, is easily seen to be a loop algebra of the Witt algebra $W_1 = W(1;1)$ of dimension p over $\mathbf{F}_p$. In this paper we investigate presentations for this algebra.

It is well-known ([Blo]) that the Schur multiplier of W_1 has dimension 1, for $p > 3$. Consider the universal covering $\widehat{W_1}$ of W_1, and the corresponding loop algebra M. The latter algebra has thus an infinite-dimensional centre, is finitely generated, and its central quotient $M/Z(M)$ is isomorphic to L. Under these circumstances, the Lie algebra counterpart of a well-known result of B. H. Neumann for groups (see [Neu] or [Rob, 2.2.3]) yields that L is not finitely presented. However we are able to prove that M is finitely presented. This may be the indication of a more general phenomenon, as indicated by results of [CMNS] and [Carr]. In the group context, examples of finitely presented groups whose centre is not finitely generated were given by V. N. Remeslennikov ([Rem]), and by H. Abels ([Ab]) in the soluble case.

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We work, more generally, with the Zassenhaus algebra $W_n = W(1;n)$, which has dimension p^n over $\mathbf{F}_p$. The Schur multiplier of W_n has also dimension 1, for $p > 3$ ([Blo], [Far2]). We prove that a certain loop algebra of $\widehat{W}_n$, which is a natural generalization of the graded Lie algebra associated to the Nottingham group, is finitely presented.

Eamonn O'Brien has shown us strong computational evidence for the fact that the Nottingham group is finitely presented as a pro-p-group. Charles Leedham-Green has explained to us how it might be possible to start from this work to construct a proof of this fact.

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Preliminaries and Notation

We use [Hu] as a general reference for group theory, and [Jac] and [StrFar] for Lie algebras. A good reference for the facts about cohomology, central extensions and universal coverings that we will be using is [Gar].

We will use several times Lucas' theorem ([Lucas]) for computing binomial coefficients modulo a prime number p. This states that if the two nonnegative integers a and b are written p-adically

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n, \quad b = b_0 + b_1 p + b_2 p^2 + \cdots + b_n p^n,$$

so that $0 \le a_i, b_i < p$, then

$$\binom{a}{b} \equiv \prod_{i=0}^{n} \binom{a_i}{b_i} \pmod{p}.$$

This has among others the following useful consequences. For all m,

$$\binom{up^m + v}{sp^m + t} \equiv \binom{u}{s}\binom{v}{t} \pmod{p},$$

for $v, t < p^m$. Also, if $b < a$, and $b, a - b < p^m$, but $a \ge p^m$, then $\binom{a}{b} \equiv 0 \pmod{p}$. This is because a carry will occur in the p-adic addition of b and $a - b$, so that $a_i < b_i$ for some i.

For a topological group G, we will denote by G_i the i-th term of its lower central series, as defined by letting $G_1 = G$, and G_i to be the closure of $[G_{i-1}, G]$, for $i > 1$.

For a prime-power q, we write $\mathbf{F}_q$ for the field with q elements. We write $\langle\, u, v, \dots \rangle$ for the vector space generated by vectors $u, v, \dots$. We will use Kronecker's delta

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

The graded Lie algebra associated to the Nottingham group

The Nottingham group $J(p)$ over the field $\mathbf{F}_p$ with p elements, p an odd prime, is defined as the group of normalized automorphisms of the algebra $\mathbf{F}_p[[t]]$ of formal power series. In other words, $J(p)$ is the set of all formal power series

$$t + \sum_{i=2}^{\infty} a_i t^i,$$

and substitution is taken as a binary operation.

In the following, we refer to [John], [York], [LSW], [Cam1], [Cam2] for the details we omit. As far as we know, [York] is the only source where calculations in $J(p)$ are developed with an approximation better than the first-degree one of formula (N) below. As this source is not easily available, we will make our calculations independent of it.

$J = J(p)$ is a pro-p-group. The factors J_i/J_{i+1} of the lower central series are elementary abelian of order p^2, if $i \equiv 1 \pmod{p-1}$, or of order p otherwise. Define

$$g_i = t \cdot (1 + t^i) \in J.$$

Then J_i/J_{i+1} is generated by the coset of

$$g_{\lceil i/(p-1) \rceil + i}$$

for $i \not\equiv 1 \pmod{p-1}$, whereas for $i \equiv 1 \pmod{p-1}$ it is generated by the cosets of

$$g_{\lfloor i/(p-1) \rfloor + i} = g_{\lceil (i-1)/(p-1) \rceil + i}, \quad \text{and} \quad g_{\lfloor i/(p-1) \rfloor + i+1} = g_{\lceil i/(p-1) \rceil + i}.$$

Note that if $i = 1 + k(p-1)$, then

$$\lceil (i-1)/(p-1) \rceil + i = kp + 1, \quad \text{and} \quad \lceil i/(p-1) \rceil + i = kp + 2.$$

It follows

$$g_i \in J_j \setminus J_{j+1}, \qquad \text{for } j = \psi(i) = i - \left\lceil \frac{i-1}{p} \right\rceil.$$

We have also

$$\text{(N)} \qquad [g_i g_j] \equiv g_{i+j}^{i-j} \pmod{J_{\min\{\psi(i+2j), \psi(2i+j)\}}}.$$

Let now $L_i = J_i/J_{i+1}$, and consider the graded Lie ring

$$L = \bigoplus_{i=1}^{\infty} L_i$$

associated to J with respect to the lower central series. Write

$$f_i = g_i J_{\psi(i)+1} \in L_{\psi(i)}.$$

Because of the above, L will be a Lie algebra over $\mathbf{F}_p$. We call a homogeneous component of dimension two a *diamond*. Thus L_i is a diamond for $i \equiv 1 \pmod{p-1}$. Consider the generators for L

$$x = f_1 + f_2 = g_1 g_2 J_2, \qquad y = f_2 = g_2 J_2.$$

Note that x is the class of $t + t^2 + t^3$ modulo J_2.

We have

$$[g_i g_2] \equiv g_{i+2}^{i-2} \pmod{J_{\psi(i+4)}}.$$

Now we have

$$\psi(i) + \psi(2) = \psi(i) + 1 = i - \left\lceil \frac{i-1}{p} \right\rceil + 1 = i + 2 - \left\lceil \frac{i+1}{p} \right\rceil = \psi(i+2)$$

iff $i \equiv 0, 1 \pmod{p}$. Therefore y centralizes all f_i, except for the cases

$$[f_{kp} y] = -2 f_{kp+2}, \qquad [f_{kp+1} y] = -f_{kp+3}.$$

For $i \not\equiv 0, 1 \pmod{p}$ we have

$$[f_i x] = (i-1) f_{i+1}.$$

In fact,

$$[g_i, g_1 g_2] \equiv g_{i+1}^{i-1} \pmod{J_{\psi(i+1)+1}},$$

as $\psi(i+1) = \psi(i) + \psi(1) = \psi(i) + 1$ in this case. When $i \equiv 0 \pmod{p}$, so that $i = kp$ for some k, we will have

$$[f_{kp} x] = -f_{kp+1} + \alpha f_{kp+2}$$

for some α. This value might be calculated exactly with the formulas of [York] (and seen to be zero), and one could also see that we then get

$$[f_{kp} x x] = 0. \tag{xx}$$

However we are content with showing that the latter identity holds, possibly by redefining x. In fact

$$[f_p, x + \beta y, x + \beta y] = [f_p x x] + \beta([f_p x y] + [f_p y x]) = [f_p x x] - \beta [f_p x y],$$

as $[f_{kp} y x] = -2[f_{kp+2} x] = -2 f_{kp+3}$ and $[f_p x y] = [-f_{kp+1} + \alpha f_{kp+2}, y] = f_{kp+3}$. Therefore redefining x to be $x + \beta y$, for a suitable choice of β (which, as we said, may be seen to be zero in this case) will ensure that (xx) holds for $k = 1$, that is, $[y \underbrace{x \cdots x}_{p}] = 0$. Now we have, proceeding by induction on k,

$$0 = [f_{(k-1)p}[y \underbrace{x \cdots x}_{p}]] = [f_{(k-1)p} y \underbrace{x \cdots x}_{p}] - [f_{(k-1)p} \underbrace{x \cdots x}_{p} y] = \gamma [f_{kp} x x],$$

for some $\gamma \neq 0$.

Note thus that the centralizer of x in L consists of one element in each diamond. Note also, for later use, that we have in particular

$$[f_{kp} x] = -f_{kp+1}, \qquad [f_{kp+1} x] = 0, \qquad [f_{kp+2} x] = f_{kp+3}.$$

It is now easy to write down the action of the generators x and y on the algebra L. To write this in a uniform way, it is convenient to define formally an element v_0 such that

$$[v_0 x] = -x, \qquad [v_0 y] = -2y.$$

and elements v_i, for $i > 0$

$$v_i = [v_{i-1} x y \underbrace{x \cdots x}_{p-3}].$$

In the notation above, v_i is a nonzero multiple of f_{ip}, for $i > 0$.

Then we have, for $i \geq 0$,

$$\begin{aligned} &[v_i x y y] = [v_i x y x y] = \cdots = [v_i x y \underbrace{x \cdots x}_{p-4} y] = 0, \\ &[v_{i+1} y x] = -2[v_{i+1} x y], \\ &[v_{i+1} x x] = [v_{i+1} y y] = 0. \end{aligned}$$

One sees readily that all these relations actually define the Lie algebra L.

The Zassenhaus algebras and their Schur multipliers

The Zassenhaus algebras are defined as algebras of special derivations of algebras of divided powers in one variable ([StrFar]), and can be regarded abstractly as vector spaces

$$W_n = W(1; n) = \{y_i : -1 \leq i \leq q - 2\},$$

of dimension q over the field $\mathbf{F}_p$ with p elements, where p is an odd prime and $q = p^n$, with the product defined by

$$[y_i y_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i} \right) y_{i+j} \,.$$

These algebras are graded over the integers, grading the y_i by their indices.

There is another grading of W_1 that turns out to be relevant for the graded Lie algebra associated to the Nottingham group. This is a grading over a cyclic group of order $q-1$, and it is derived from the one over the integers by declaring y_{-1} and y_{q-2} to be of weight 1. In other words, we are viewing modulo $q-1$ the opposite grading to the grading over the integers.

We thus easily get that y_{q-i} acquires weight $i-1$ in this grading, for $2 \le i \le q$. As $[y_{-1} y_0] = y_{-1}$, and $[y_{q-2} y_0] = 2 y_{q-2}$, we see that the new grading is indeed over a cyclic group of order $q-1$, and that all homogeneous component have dimension 1, except that the i-th one, for $i \equiv 1 \pmod{q-1}$, has dimension 2.

Now consider the loop algebra of W_n with respect to the latter grading. That is, take first the Lie algebra

$$W_n \otimes \mathbf{F}_p[t],$$

where $\mathbf{F}_p[t]$ is a polynomial algebra, and consider the subalgebra L generated by the two elements

$$x = y_{-1} \otimes t, \qquad y = y_{q-2} \otimes t.$$

One easily sees that the action of the two generators x and y on L is defined as follows. Define formally an element v_0 with the property

$$[v_0 x] = -x, \qquad [v_0 y] = -2y.$$

and elements v_i, for $i > 0$

$$v_i = [v_{i-1} x y \underbrace{x \cdots x}_{q-3}].$$

Then we have, for $i \ge 0$,

$$\begin{aligned}
&[v_i x y y] = [v_i x y x y] = \cdots = [v_i x y \underbrace{x \cdots x}_{q-4} y] = 0,\\
&[v_{i+1} y x] = -2 [v_{i+1} x y],\\
&[v_{i+1} x x] = [v_{i+1} y y] = 0.
\end{aligned}$$

It follows from the results of the previous section that for $n = 1$ the loop algebra L of the Witt algebra W_1 is indeed isomorphic to the graded Lie algebras associated to the Nottingham group $J(p)$.

We will now need a particular 2-cocycle for the Zassenhaus algebra W_n, for $p > 3$. One may see [Gar] for the facts about cohomology, central extensions and universal coverings that we will be using in the following.

We claim that the following defines a non-trivial cocycle

$$\varphi(y_i, y_j) = \delta(i+j, q)(-1)^{i-1} = \delta(i+j, q)(-1)^{j}.$$

This definition is reminiscent of the L^--associative *symmetric* form of [Far1, Lemma 4.1]. This cocycle appears already in [Dzhu, Theorem 2].

To see that φ is indeed a cocycle, we rewrite it in terms of a different basis. It is a standard fact that the elements e_α, for $\alpha \in \mathbf{F}_q$, defined by

$$\begin{cases}
e_0 = y_{-1} + y_{q-2},\\
e_\alpha = y_{q-2} + \displaystyle\sum_{i=-1}^{q-2} \alpha^{i+1} y_i, \quad \text{for } \alpha \in \mathbf{F}_q^\star
\end{cases}$$

also form a basis for W_n over the field $\mathbf{F}_q$, and that they multiply as $[e_\alpha e_\beta] = (\beta - \alpha) \cdot e_{\alpha+\beta}$. In fact, Witt and Zassenhaus originally defined these algebras in terms of these generators.

We rewrite φ in terms of the e_α. We have, for $\alpha, \beta \neq 0$,

$$\begin{aligned}
\varphi(e_\alpha, e_\beta) &= \varphi(y_{q-2}, \beta^3 y_2) + \varphi(\alpha^3 y_2, y_{q-2}) + \sum_{i=2}^{q-2} \alpha^{i+1} \beta^{q-i+1} \varphi(y_i, y_{q-i}) \\
&= \beta^3 - \alpha^3 - \alpha\beta^2 \sum_{i=2}^{q-2} (-\alpha\beta^{-1})^i \\
&= \beta^3 - \alpha^3 + \alpha\beta^2(1 - \alpha\beta^{-1}) - \alpha\beta^2 \sum_{i=0}^{q-2} (-\alpha\beta^{-1})^i \\
&= \beta^3 - \alpha^3 + \alpha\beta^2 - \alpha^2\beta - \alpha\beta^2 \cdot \begin{cases} 1 & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} -\alpha^3 & \text{if } \alpha+\beta = 0 \\ (\beta-\alpha)(\alpha+\beta)^2 & \text{otherwise.} \end{cases}
\end{aligned}$$

and also

$$\varphi(e_\alpha, e_0) = -\alpha^3,$$

which is compatible with the above formula.

Now it is apparent that φ on the e_α is just the nontrivial cocycle

$$f(e_\alpha, e_\beta) = -\delta(\alpha+\beta, 0) \cdot \alpha^3,$$

as in [Blo], modified by the coboundary $h([e_\alpha e_\beta])$, where

$$h(e_\gamma) = \gamma^2.$$

Note that for $p = 3$ the cocycle $f(e_\alpha, e_\beta) = -\delta(\alpha+\beta, 0) \cdot \alpha^3 = -\delta(\alpha+\beta, 0) \cdot \alpha$ is the coboundary $h([e_\alpha e_\beta])$, where $h(e_\gamma) = 1/2 \cdot \delta(\gamma, 0)$.

The loop algebras of the universal covering

The cocycle φ respects the grading of W_n over the integers. In fact, if one uses φ to construct a central extension

$$0 \to \mathbf{F}_p z \to N \to W_n \to 0,$$

one sees immediately that z acquires weight q. More interestingly φ repects also the grading of W_n over a cyclic group of order $q-1$; here z acquires weight $q-2$, that is, same as y_1, as the only Lie products in N involving z are

$$[y_i y_{q-i}] = (-1)^{i-1} z,$$

for $i = 2, \dots, q-2$, and the resulting weight is indeed consistently

$$(q-i-1) + (i-1) = q-2.$$

It follows from [Blo] and [Far2] that N is in fact the universal covering $\widehat{W}_n$ of W_n, but see the comments in the next section.

Let N be the central extension of W_n we have just constructed. We will continue slightly abusing notation, and write $[\cdot,\cdot]$ for the Lie product in N from now on, so that for instance

$$\begin{aligned}
[y_{-1} y_i] &= y_{i-1}, \text{ for } 0 \le i \le q-2, \text{ and}\\
[y_j y_{q-j}] &= (-1)^{j-1} z, \text{ for } 2 \le j \le q-2,
\end{aligned}$$

where $q = p^n$ here. We take $p > 3$ in the rest of the paper.

As before, take the tensor product over $\mathbf{F}_p$ of N by a polynomial algebra $\mathbf{F}_p[t]$

$$N \otimes \mathbf{F}_p[t]$$

and define M as the Lie subalgebra spanned by $x = y_{-1} \otimes t$ and $y = y_{q-2} \otimes t$.

It is immediate that M has an infinite-dimensional centre, as

$$Z(M) = \left\langle z \otimes t^{q-2+j(q-1)} : j \ge 0 \right\rangle.$$

Once more, it is convenient to define formally an element v_0 such that

$$[v_0 x] = -x, \qquad [v_0 y] = -2y,$$

and then, for $i > 1$,

$$v_i = [v_{i-1} x y \underbrace{x \cdots x}_{q-3}].$$

Then the action of the generators x and y on M is completely described by the following set of relations, for $i \ge 0$,

$$\text{(Rels)} \qquad \begin{cases}
[v_i x y \underbrace{x \cdots x}_{j} y] = 0 \qquad \text{for } 0 \le j < q-5\\
[v_i x y \underbrace{x \cdots x}_{q-5} x y] = [v_i x y \underbrace{x \cdots x}_{q-5} y x] = [v_i x y \underbrace{x \cdots x}_{q-5} y y] = 0\\
[v_{i+1} x x] = [v_{i+1} y y] = 0,\\
[v_{i+1} y x] = -2[v_{i+1} x y],
\end{cases}$$

M has a natural grading over the integers, inherited by the one of N over a cyclic group of order $q-1$. This is determined by letting $M_1 = \langle x, y \rangle$. We have in fact, for $i \ge 0$.

$$\begin{aligned}
M_{i(q-1)+1} &= \langle [v_i x], [v_i y] \rangle,\\
M_{i(q-1)+j} &= \left\langle [v_i y \underbrace{x \cdots x}_{j-1}] \right\rangle, \qquad \text{for } 1 < j \le q-2,\ j \ne q-3,\\
M_{i(q-1)+q-3} &= \left\langle [v_i y \underbrace{x \cdots x}_{q-3}], [v_i y \underbrace{x \cdots x}_{q-4} y] \right\rangle,
\end{aligned}$$

where the displayed elements are linearly independent, so that

$$\dim(M_i) = \begin{cases} 2 & \text{if } i \equiv 1, q-3 \pmod{q-1}, \\ 1 & \text{otherwise.} \end{cases}$$

THE GRADED NOTTINGHAM LIE ALGEBRA IS NOT FINITELY PRESENTED

We are now able to prove that the graded Lie algebra associated to the Nottingham group is not finitely presented. We use the following result, which is a direct analogue of a result of B. H. Neumann ([Neu] or [Rob, 2.2.3]):

Theorem. *Let L be a finitely presented Lie algebra, and M be a finitely generated Lie algebra M. Suppose there is an epimorphism*

$$\psi : M \to L.$$

Then $\ker(\psi)$ *is finitely generated as an ideal of M.*

In our situation L is the graded Lie algebra associated to the Nottingham group, so that L is the loop algebra of the Witt algebra W_1 we have discussed previously. We can actually take more generally L to be the loop algebra of the Zassenhaus algebra W_n. We take M to be the corresponding loop algebra of the central extension N of W_n we have just constructed, and ψ to be the map induced by the canonical epimorphism $N \to W_n$. Now $\ker(\psi)$ is the infinite-dimensional centre of M. It follows from the Theorem above that L is not finitely presented.

We will show in the following section that M itself, however, is finitely presented. Note that this provides an independent, although somewhat involved, proof for the well-known fact ([Blo], [Far2]) that the cocycle constructed in the previous section spans the Schur multiplier of W_n. Suppose in fact that the dimension of the Schur multiplier is greater than 1. The algebra M would thus be obtained from the loop algebra of the universal covering $\widehat{W}_n$ by factoring out an infinite-dimensional centre. But then the Theorem would prevent M from being finitely presented.

The above arguments do not work when $p = 3$. In this case we have first of all that $W_1 = sl(2)$ has trivial Schur multiplier. Still, there exists a Lie algebra M with an infinite-dimensional centre $Z(M)$ such that $M/Z(M) \cong L$. This is shown in [Car]. Similarly, the presentation of the next section does not work for $p = 3$. Again in [Car], an argument of [CMNS] is extended to show that the algebra M just mentioned is finitely presented.

In the general case, the algebra W_n in characteristic 3, for $n > 1$, has Schur multiplier of dimension $n - 1$ ([Far2]), and thus the loop algebra L could be seen not be finitely presented in this case too. Constructing a finitely presented algebra M which is a central extension of L as above is more complicated in this case, and dealing with this here would constitute a substantial digression from our present goals.

As mentioned in the introduction, examples of finitely presented groups whose centre is not finitely generated were given by V. N. Remeslennikov ([Rem]), and by H. Abels ([Ab]) in the soluble case.

The presentation

We will prove that the algebra M, obtained as the loop algebra of a central extension of the Zassenhaus algebra W_n, is isomorphic to the algebra Q given by the finite presentation

$$\text{(Pres)}\qquad \begin{aligned} Q = \Big\langle\ x, y :\ & [yxy], [yxxxy], \dots, [y\underbrace{x\cdots x}_{q-6}y],\\ & [y\underbrace{x\cdots x}_{q-4}yx],\\ & [y\underbrace{x\cdots x}_{q}],\\ & [y\underbrace{x\cdots x}_{q-1}y\underbrace{x\cdots x}_{q-5}yx],\\ & [y\underbrace{x\cdots x}_{q-1}y\underbrace{x\cdots x}_{q-3}yx] = -2[y\underbrace{x\cdots x}_{q-1}y\underbrace{x\cdots x}_{q-3}xy]\ \Big\rangle, \end{aligned}$$

by showing that the defining relations of Q imply all of (Rels). Recall that we are taking $p > 3$.

We note some immediate consequences of (Pres). First, for k even, $0 \le k < q-6$, we have by induction, starting with $[yy] = 0$,

$$\begin{aligned} 0 &= [[y\underbrace{x\cdots x}_{k/2}][y\underbrace{x\cdots x}_{k/2}]]\\ &= \pm[y\underbrace{x\cdots x}_{k}y] \end{aligned}$$

Then

$$\begin{aligned} 0 &= -[[y\underbrace{x\cdots x}_{q-5}][yxy]]\\ &= [y\underbrace{x\cdots x}_{q-5}xyy] - 2[y\underbrace{x\cdots x}_{q-5}yxy] + [y\underbrace{x\cdots x}_{q-5}yyx]\\ &= [y\underbrace{x\cdots x}_{q-4}yy] \end{aligned}$$

so that $[y\underbrace{x\cdots x}_{q-4}y]$ is central, and

$$0 = [[y\underbrace{x\cdots x}_{(q-3)/2}][y\underbrace{x\cdots x}_{(q-3)/2}]] = \pm[y\underbrace{x\cdots x}_{q-3}y].$$

Also

$$\begin{aligned} 0 &= \pm[[y\underbrace{x\cdots x}_{(q-1)/2}][y\underbrace{x\cdots x}_{(q-1)/2}]]\\ &= -\binom{q-1}{2}[y\underbrace{x\cdots x}_{q-2}yx] + [y\underbrace{x\cdots x}_{q-1}y], \end{aligned}$$

so that

$$[y\underbrace{x\cdots x}_{q-2}yx] = -2[y\underbrace{x\cdots x}_{q-1}y].$$

So suppose (Rels) are satisfied for values less than some $i > 2$, and let $v = v_i \in Q_{i(q-1)}$. We have in particular

$$[vxx] = [vyy] = 0, \qquad [vyx] = -2[vxy].$$

We want to show, assuming (Pres), that (Rels) hold for i. We begin with showing, by induction on $1 \le j \le q-6$, that

$$[vxy\underbrace{x\cdots x}_{j-1}y] = 0$$

holds.

For $1 \le k \le q-3$, write $v_{-k} = [v_{i-1}xy\underbrace{x\cdots x}_{q-3-k}]$, so that

$$[v_{-k}\underbrace{x\cdots x}_{k}] = v_i = v.$$

Write $j = \beta p^t - 1$, with $\beta \not\equiv 0 \pmod{p}$, and take $k = p^t$. Suppose for the moment that $j + k \le q - 3$, so that $[y\underbrace{x\cdots x}_{j+k}y]$ is trivial or central, and compute

$$\begin{aligned}
0 &= [v_{-k}[y\underbrace{x\cdots x}_{j+k}y]]\\
&= (-1)^k\binom{j+k}{k}[vyx\underbrace{x\cdots x}_{j-1}y] + (-1)^{k+1}\binom{j+k}{k+1}[vxy\underbrace{x\cdots x}_{j-1}y]\\
&= \pm\left(2\binom{j+k}{k} + \binom{j+k}{k+1}\right)[vxy\underbrace{x\cdots x}_{j-1}y]\\
&= \pm\beta[vxy\underbrace{x\cdots x}_{j-1}y]
\end{aligned}$$

as

$$\binom{j+k}{k} = \binom{\beta p^t + p^t - 1}{p^t + 0} = \beta,$$

and

$$\binom{j+k}{k+1} = \binom{\beta p^t + p^t - 1}{p^t + 1} = -\beta,$$

by Lucas' theorem. Now $\beta < p^{n-t}$, where $q = p^n$. Thus $\beta \le p^{n-t} - 1$, and $j + k \le q - 1$, so the above does not work when $\beta = p^{n-t} - 1$, that is, $j = p^n - p^t - 1$.

But in this case we may just work out

$$\begin{aligned}
0 &= [v[y\underbrace{x\cdots x}_{j}y]]\\
&= [v[y\underbrace{x\cdots x}_{j}]y] - [vy[y\underbrace{x\cdots x}_{j}]]\\
&= [vy\underbrace{x\cdots x}_{j}y] - j[vxy\underbrace{x\cdots x}_{j-1}y] - (-1)^{j}[vy\underbrace{x\cdots x}_{j}y]\\
&= (-2(1-(-1)^{j}) - j)[vxy\underbrace{x\cdots x}_{j-1}y]\\
&= (-2\cdot 2 + 1)[vxy\underbrace{x\cdots x}_{j-1}y]\\
&= -3[vxy\underbrace{x\cdots x}_{j-1}y],
\end{aligned}$$

as requested. One might note that these arguments fail without remedy when q is a power of 3 greater than 9. As mentioned in the previous section, this depends on the behaviour of the Schur multiplier of W_n in characteristic 3.

Now we need to prove $[vxy\underbrace{x\cdots x}_{q-5}y]$ central, and $[vxy\underbrace{x\cdots x}_{q-4}y] = 0$. The easy part is

$$0 = [[vxy\underbrace{x\cdots x}_{q-6}][yxy]] = -[vxy\underbrace{x\cdots x}_{q-5}yy].$$

Proving

$$0 = [vxy\underbrace{x\cdots x}_{q-5}xy] = [vxy\underbrace{x\cdots x}_{q-5}yx].$$

requires more effort.

The next central element. We first obtain a relation between the two commutators we want to kill:

$$\begin{aligned}
0 &= [[vx][y\underbrace{x\cdots x}_{q-4}y]]\\
&= [vx[y\underbrace{x\cdots x}_{q-4}]y] - [vxy[y\underbrace{x\cdots x}_{q-4}]]\\
&= [vxy\underbrace{x\cdots x}_{q-4}y] - (-1)^{q-5}(q-4)[vxy\underbrace{x\cdots x}_{q-5}yx] - (-1)^{q-4}[vxy\underbrace{x\cdots x}_{q-4}y]\\
&= 2[vxy\underbrace{x\cdots x}_{q-4}y] + 4[vxy\underbrace{x\cdots x}_{q-5}yx].
\end{aligned}$$

Let now $u = v_{i-1}$, so that

$$v = [uxy\underbrace{x\cdots x}_{q-3}].$$

We will use the formula, which only depends on the Jacobi identity,

$$
\begin{aligned}
&[z[y\underbrace{x\cdots x}_{s}y\underbrace{x\cdots x}_{t}]] = \\
&\qquad = \sum_{i=0}^{t}(-1)^i\binom{t}{i}[z\underbrace{x\cdots x}_{i}[y\underbrace{x\cdots x}_{s}y]\underbrace{x\cdots x}_{t-i}] \\
&\qquad = +\sum_{i=0}^{t}(-1)^i\binom{t}{i}[z\underbrace{x\cdots x}_{i}[y\underbrace{x\cdots x}_{s}]y\underbrace{x\cdots x}_{t-i}] \\
&\qquad\quad -\sum_{i=0}^{t}(-1)^i\binom{t}{i}[z\underbrace{x\cdots x}_{i}y[y\underbrace{x\cdots x}_{s}]\underbrace{x\cdots x}_{t-i}] \\
\text{(F1)}\quad &\qquad = +\sum_{i=0}^{t}\sum_{j=0}^{s}(-1)^i\binom{t}{i}(-1)^j\binom{s}{j}[z\underbrace{x\cdots x}_{i+j}y\underbrace{x\cdots x}_{s-j}y\underbrace{x\cdots x}_{t-i}] \\
\text{(F2)}\quad &\qquad\quad -\sum_{i=0}^{t}\sum_{j=0}^{s}(-1)^i\binom{t}{i}(-1)^j\binom{s}{j}[z\underbrace{x\cdots x}_{i}y\underbrace{x\cdots x}_{j}y\underbrace{x\cdots x}_{s-j+t-i}].
\end{aligned}
$$

We will apply this to the expansion of

$$
\begin{aligned}
0 &= [[ux][y\underbrace{x\cdots x}_{q-1}y\underbrace{x\cdots x}_{q-5}y]] \\
\text{(A)}\quad &= [ux[y\underbrace{x\cdots x}_{q-1}y\underbrace{x\cdots x}_{q-5}]y] \\
\text{(B)}\quad &\quad -[uxy[y\underbrace{x\cdots x}_{q-1}y\underbrace{x\cdots x}_{q-5}]],
\end{aligned}
$$

distinguishing the values of i and j that give non-trivial commutators. One may note the slight simplification coming from the fact that

$$(-1)^j\binom{q-1}{j} = 1 \qquad \text{for all } j.$$

We get the following nontrivial terms. From (A), (F2), for $i = 0$, $j = 1$

$$(-1)[uxy\underbrace{x\cdots x}_{q-3}xy\underbrace{x\cdots x}_{q-5}xy];$$

from (A), (F2), $i = 0$, $j = 2$

$$(-1)[uxy\underbrace{x\cdots x}_{q-3}yx\underbrace{x\cdots x}_{q-5}xy];$$

from (B), (F1), $i = q-5$, $j = 2$

$$(-1)[uxy\underbrace{x\cdots x}_{q-3}yx\underbrace{x\cdots x}_{q-5}xy];$$

from (B), (F1), $i = q - 6$, $j = 3$

$$-(-1)(-5)[uxy\underbrace{x\cdots x}_{q-3}yx\underbrace{x\cdots x}_{q-5}yx];$$

from (B), (F2), $i = q - 5$, $j = 3$

$$(-1)[uxy\underbrace{x\cdots x}_{q-3}xy\underbrace{x\cdots x}_{q-5}xy];$$

from (B), (F2), $i = q - 6$, $j = 4$

$$-(-1)(-5)[uxy\underbrace{x\cdots x}_{q-3}xy\underbrace{x\cdots x}_{q-5}yx].$$

Keeping in mind that

$$[uxy\underbrace{x\cdots x}_{q-3}yx] = -2[uxy\underbrace{x\cdots x}_{q-3}xy],$$

we get

$$0 = 2[uxy\underbrace{x\cdots x}_{q-3}xy\underbrace{x\cdots x}_{q-5}xy] + 5[uxy\underbrace{x\cdots x}_{q-3}xy\underbrace{x\cdots x}_{q-5}yx];$$

as $2 \cdot 5 - 2 \cdot 4 = 2 \neq 0$, we get

$$0 = [uxy\underbrace{x\cdots x}_{q-3}xy\underbrace{x\cdots x}_{q-5}xy] = [uxy\underbrace{x\cdots x}_{q-3}xy\underbrace{x\cdots x}_{q-5}yx].$$

The next diamond. Now let $w = v_{i+1}$. We are left with proving

$$[wxx] = [wyy] = 0, \qquad [wyx] = -2[wxy].$$

We prove easily $[wxx] = 0$, as

$$0 = [v[y\underbrace{x\cdots x}_{q}]] = [vy\underbrace{x\cdots x}_{q}] = -2[wxx].$$

We have also

$$0 = [w_{-1}[yxy]] = -[w_{-1}xyy] + 2[w_{-1}yxy] - [w_{-1}yyx] = [wyy],$$

as $[w_{-1}y] = 0$. As before, w_{-k} denotes an element such that $[w_{-k}\underbrace{x\cdots x}_{k}] = w$.

We use the formula employed in the previous section as applied to the expansion of

$$(\Theta) \qquad 0 = [u\Big([y\underbrace{x\cdots x}_{q-1}y\underbrace{x\cdots x}_{q-3}yx] + 2[y\underbrace{x\cdots x}_{q-1}y\underbrace{x\cdots x}_{q-2}y]\Big)].$$

Let us first consider the term

$$
\begin{aligned}
&[u[y\,\underbrace{x\cdots x}_{q-1}\,y\,\underbrace{x\cdots x}_{q-2}\,y]] = \\
\text{(A)}\qquad &\qquad = [u[y\,\underbrace{x\cdots x}_{q-1}\,y\,\underbrace{x\cdots x}_{q-2}]y] \\
\text{(B)}\qquad &\qquad\quad - [uy[y\,\underbrace{x\cdots x}_{q-1}\,y\,\underbrace{x\cdots x}_{q-2}]]
\end{aligned}
$$

We expand (A), using the formula. The idea is to study in each case the various possibilities available for i and j in the formula.

We first use (F1) to get the following term:

$$\boxed{\begin{array}{cc} i=0, j=0 & 1 \\ \boxed{|yx|\dots|xy|\dots|xy|} & \boxed{1} \end{array}}\ .$$

A word about notation. The line $i = 0$, $j = 0$ identifies the term being used. We then employ the shorthand

$$\boxed{|z_1 z_2|\dots|z_3 z_4|\dots|z_5 z_6|} = [u z_1 z_2\,\underbrace{x\cdots x}_{q-3}\,z_3 z_4\,\underbrace{x\cdots x}_{q-3}\,z_5 z_6],$$

where $\{z_{2i-1}, z_{2i}\} = \{x, y\}$, for $i = 1, 2, 3$. The boxed number in the right-hand lower corner is the final coefficient, and on top of it is the recipe to obtain it. This includes the various leading signs, and the final signed binomial coefficient.

We continue to list the terms we obtain from (F1).

$$\boxed{\begin{array}{cc} i=0, j=1 & 1 \\ \boxed{|xy|\dots|xy|\dots|xy|} & \boxed{1} \end{array}}\ .$$

Now for (F2) we get:

$$\boxed{\begin{array}{cc} i=0, j=q-1 & (-1)1 \\ \boxed{|yx|\dots|xy|\dots|xy|} & \boxed{-1} \end{array}}\ ,$$

$$\boxed{\begin{array}{cc} i=0, j=q-2 & (-1)1 \\ \boxed{|yx|\dots|yx|\dots|xy|} & \boxed{-1} \end{array}}\ ,$$

$$\boxed{\begin{array}{cc} i = 1, j = q-3 & (-1)(-1)1\binom{q-2}{q-3} \\ \boxed{|xy|\dots|yx|\dots|xy|} & \boxed{-2} \end{array}},$$

$$\boxed{\begin{array}{cc} i = 1, j = q-2 & (-1)(-1)1\binom{q-2}{q-3} \\ \boxed{|xy|\dots|xy|\dots|xy|} & \boxed{-2} \end{array}}.$$

We expand (B), using (F1) first.

$$\boxed{\begin{array}{cc} i = q-2, j = 0 & (-1)(-1)^{q-2}\binom{q-2}{q-2} \\ \boxed{|yx|\dots|yx|\dots|xy|} & \boxed{1} \end{array}}.$$

Recall that we have q odd here.

$$\boxed{\begin{array}{cc} i = q-3, j = 1 & (-1)(-1)^{q-3}\binom{q-2}{q-3} \\ \boxed{|yx|\dots|yx|\dots|yx|} & \boxed{2} \end{array}},$$

$$\boxed{\begin{array}{cc} i = q-2, j = 1 & (-1)(-1)^{q-2}\binom{q-2}{q-2} \\ \boxed{|yx|\dots|xy|\dots|xy|} & \boxed{1} \end{array}},$$

$$\boxed{\begin{array}{cc} i = q-3, j = 2 & (-1)(-1)^{q-3}\binom{q-2}{q-3} \\ \boxed{|yx|\dots|xy|\dots|yx|} & \boxed{2} \end{array}}.$$

Now we use (F2):

$$\boxed{\begin{array}{cc} i = q-2, j = q-1 & (-1)(-1)(-1)^{q-2}\binom{q-2}{q-2} \\ \boxed{|yx|\dots|yx|\dots|xy|} & \boxed{-1} \end{array}},$$

$$\boxed{\begin{array}{cc} i = q-2, j = q-2 & (-1)(-1)(-1)^{q-2}\binom{q-2}{q-2} \\ \boxed{|yx|\dots|yx|\dots|yx|} & \boxed{-1} \end{array}}.$$

We now consider the term

$$[u[y \underbrace{x\cdots x}_{q-1} y \underbrace{x\cdots x}_{q-3} yx]] = [u[y \underbrace{x\cdots x}_{q-1} y \underbrace{x\cdots x}_{q-3} y]x] - [ux[y \underbrace{x\cdots x}_{q-1} y \underbrace{x\cdots x}_{q-3} y]]$$

$$= [u[y \underbrace{x\cdots x}_{q-1} y \underbrace{x\cdots x}_{q-3}]yx] \tag{C}$$

$$- [uy[y \underbrace{x\cdots x}_{q-1} y \underbrace{x\cdots x}_{q-3}]x] \tag{D}$$

$$- [ux[y \underbrace{x\cdots x}_{q-1} y \underbrace{x\cdots x}_{q-3}]y] \tag{E}$$

$$+ [uxy[y \underbrace{x\cdots x}_{q-1} y \underbrace{x\cdots x}_{q-3}]]. \tag{F}$$

We start expanding (C), by way of (F1)

$$\boxed{\begin{array}{cc} i=0, j=0 & 1 \\ \boxed{|yx|\dots|xy|\dots|yx|} & \boxed{1} \end{array}},$$

$$\boxed{\begin{array}{cc} i=0, j=1 & 1 \\ \boxed{|xy|\dots|xy|\dots|yx|} & \boxed{1} \end{array}}.$$

Now with (F2):

$$\boxed{\begin{array}{cc} i=0, j=q-1 & 1(-1)1 \\ \boxed{|yx|\dots|xy|\dots|yx|} & \boxed{-1} \end{array}},$$

$$\boxed{\begin{array}{cc} i=0, j=q-2 & 1(-1)1 \\ \boxed{|yx|\dots|yx|\dots|yx|} & \boxed{-1} \end{array}},$$

$$\boxed{\begin{array}{cc} i=1, j=q-3 & 1(-1)(-1)^1\binom{q-3}{1} \\ \boxed{|xy|\dots|yx|\dots|yx|} & \boxed{-3} \end{array}},$$

$$\boxed{\begin{array}{cc} i=1, j=q-2 & 1(-1)(-1)^1\binom{q-3}{1} \\ \boxed{|xy|\dots|xy|\dots|yx|} & \boxed{-3} \end{array}}.$$

We expand (D), by way of (F1):

$$\boxed{\begin{array}{cc} i = q-3, j = 1 & (-1)1(-1)^{q-3}\binom{q-3}{q-3} \\ \boxed{|yx|\dots|yx|\dots|yx|} & \boxed{-1} \end{array}},$$

$$\boxed{\begin{array}{cc} i = q-3, j = 2 & (-1)1(-1)^{q-3}\binom{q-3}{q-3} \\ \boxed{|yx|\dots|xy|\dots|yx|} & \boxed{-1} \end{array}}.$$

There are no (F2)-terms here.

Now we expand (E). There are no terms with (F1). With (F2) we have:

$$\boxed{\begin{array}{cc} i = 0, j = q-2 & (-1)(-1)1 \\ \boxed{|xy|\dots|xy|\dots|xy|} & \boxed{1} \end{array}},$$

$$\boxed{\begin{array}{cc} i = 0, j = q-3 & (-1)(-1)1 \\ \boxed{|xy|\dots|yx|\dots|xy|} & \boxed{1} \end{array}}.$$

Now we expand (F) with (F1):

$$\boxed{\begin{array}{cc} i = q-3, j = 1 & 1 \cdot 1(-1)^{q-3}\binom{q-3}{q-3} \\ \boxed{|xy|\dots|xy|\dots|xy|} & \boxed{1} \end{array}},$$

$$\boxed{\begin{array}{cc} i = q-4, j = 2 & 1 \cdot 1(-1)^{q-4}\binom{q-3}{q-4} \\ \boxed{|xy|\dots|xy|\dots|yx|} & \boxed{3} \end{array}},$$

$$\boxed{\begin{array}{cc} i = q-3, j = 0 & 1 \cdot 1(-1)^{q-3}\binom{q-3}{q-3} \\ \boxed{|xy|\dots|yx|\dots|xy|} & \boxed{1} \end{array}},$$

$$\boxed{\begin{array}{cc} i = q-4, j = 2 & 1 \cdot 1(-1)^{q-4}\binom{q-3}{q-4} \\ \boxed{|xy|\dots|yx|\dots|yx|} & \boxed{3} \end{array}}.$$

Finally (F) with (F2) gives:

$$\boxed{\begin{array}{cc} i = q-3, j = q-1 & 1(-1)(-1)^{q-3}\binom{q-3}{q-3} \\ \boxed{|xy|\dots|yx|\dots|xy|} & \boxed{-1} \end{array}},$$

$$\boxed{\begin{array}{cc} i = q-3, j = q-2 & 1(-1)(-1)^{q-3}\binom{q-3}{q-3} \\ \boxed{|xy|\dots|yx|\dots|yx|} & \boxed{-1} \end{array}}.$$

Collecting all the terms thus obtained, and substituting into (Θ), we get

$$-3 \cdot ([wyx] + 2[wxy]) = 0$$

as requested. This is another instance where the case of characteristic 3 exhibits a different behaviour.

Computations

The results of this paper do not rely on any computer calculations. However, computations have played an important role in understanding the structure of the algebras studied here.

We made use of the ANU p-Quotient Program ([HNO]) to compute with graded Lie algebras over a prime field as explained elsewhere ([CMNS]). We have used a small `C` preprocessor, called `ppi` (for prepare `pQ input`), which produces the input needed by the ANU p-Quotient Program from a compact description of the algebra under study.

We have also used a small GAP library ([S+]) to compute in finite quotients of the Nottingham group. This was developed together with P. P. Pálfy while he was visiting Trento within the framework of the Tempus project "Using Computer Algebra" of the European Union.

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