Using Lie algebras to parametrize certain types of algebraic varieties I

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The known curve

Set

$$C_0 = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{Q}) \mid x_0 x_2 - x_1^2 = 0\}$$

then C_0 is isomorphic to $\mathbb{P}^1(\mathbb{Q})$ by

$$(s : t) \longrightarrow (s^2 : st : t^2).$$

Such a map is called a *parametrization* of C_0.

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Question: is C isomorphic to $\mathbb{P}^1(\mathbb{Q})$ as well?

If so can we find a parametrization of C?

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There are algorithms for this by Cremona and Rusin (2003) and Simon (2005). These are based on finding a rational point on C.

In our approach we try to find a 3×3 matrix M with the property that

$$p \in C_0 \text{ iff } Mp \in C.$$

This will then give us a parametrization of C:

$$\mathbb{P}^1(\mathbb{Q}) \longrightarrow C_0 \xrightarrow{M} C.$$

It can be shown that such an M always exists if C is isomorphic to $\mathbb{P}^1(\mathbb{Q})$.

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Set

$$A_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then

$$C_0 = \{p = (x_0 : x_1 : x_2) \mid p^T A_0 p = 0\}.$$

Set

$$G(C_0, \mathbb{Q}) = \{g \in \mathrm{GL}_3(\mathbb{Q}) \mid g^T A_0 g = \lambda A_0\}$$

(i.e., the group consisting of all invertible linear maps that map C_0 into itself).

Then

$$G(C_0, \mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{Q})/\langle \pm I_2 \rangle.$$

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$G(C_0, \mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{Q})/\langle \pm I_2 \rangle$, Why?

Let V be the two-dimensional vector space over $\mathbb{Q}$ with basis v_0, v_1.

Let $\mathrm{Sym}^2(V)$ be the symmetric square of V with basis v_0^2, $2v_0v_1$, v_1^2.

Let $\phi : V \to Sym^2(V)$ be defined by $\phi(v) = v^2$, or in coordinates, $\phi(sv_0 + tv_1) = s^2v_0^2 + st(2v_0v_1) + t^2v_1^2$.

Hence in coordinates the image of ϕ is $\{(s^2, st, t^2)\}$, which is C_0.

The group $G = \mathrm{GL}_2(\mathbb{Q})$ acts on V. Hence also on $\mathrm{Sym}^2(V)$ by $g \cdot vw = (g \cdot v)(g \cdot w)$.

This implies that $\phi(g \cdot v) = g \cdot \phi(v)$. And hence G leaves the image, C_0, invariant.

So we get a surjective group homomorphism $G \to G(C_0, \mathbb{Q})$, with kernel $\pm I_2$.

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The Lie algebra of $G(C_0, \mathbb{Q})$

Set

$$L(C_0, \mathbb{Q}) = \{X \in \mathfrak{gl}_3(\mathbb{Q}) \mid X^T A_0 + A_0 X = \lambda A_0\}.$$

Then $L(C_0, \mathbb{Q}) \cong \mathfrak{gl}_2(\mathbb{Q})$.

Why?

$G(C_0, \mathbb{Q})$ is an algebraic group and

$\mathrm{Lie}(G(C_0, \mathbb{Q})) \cong \mathrm{Lie}(\mathrm{GL}_2(\mathbb{Q})/\langle \pm I_2 \rangle) = \mathfrak{gl}_2(\mathbb{Q})$.

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A basis of $L(C_0, \mathbb{Q})$:

$$I_3, a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, a_2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$a_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Let x, y, h be the standard basis elements of $\mathfrak{sl}_2(\mathbb{Q})$, then we have an injective Lie algebra homomorphism $\varphi_0 : \mathfrak{sl}_2(\mathbb{Q}) \to L(C_0, \mathbb{Q})$, given by

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In this case with $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & -2 \end{pmatrix}$ we have

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Just as for C_0 we define

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Isomorphism

FACT: if $M : C_0 \to C$ exists then $X \mapsto MXM^{-1}$ is an isomorphism $L(C_0, \mathbb{Q}) \to L(C, \mathbb{Q})$.

So we check whether $L(C_0, \mathbb{Q}) \cong L(C, \mathbb{Q})$.
If not, then C cannot be parametrized. We stop (maybe slightly distressed).
If yes, then we proceed.

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Basis of $L(C, \mathbb{Q})$:

$$I_3, b_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, b_2 = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

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The modules

Both $\varphi_0 : \mathfrak{sl}_2(\mathbb{Q}) \to L(C_0, \mathbb{Q})$ and $\varphi : \mathfrak{sl}_2(\mathbb{Q}) \to L(C, \mathbb{Q})$ yield 3-dimensional $\mathfrak{sl}_2(\mathbb{Q})$-modules.

Let V_0 be the $\mathfrak{sl}_2(\mathbb{Q})$-module corresponding to φ_0, with basis e_1, e_2, e_3.
Let V be the $\mathfrak{sl}_2(\mathbb{Q})$-module corresponding to φ_0, with basis v_1, v_2, v_3.
Then there is an isomorphism of $\mathfrak{sl}_2(\mathbb{Q})$-modules, given by

$$e_1 \mapsto v_1$$
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The matrix of this isomorphism is

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Now $N \begin{pmatrix} s^2 \\ st \\ t^2 \end{pmatrix} = \begin{pmatrix} s^2 \\ st + t^2 \\ st \end{pmatrix}$

And if we define $\psi(s : t) = (s^2 : st + t^2 : st)$ then we get a parametrization of C (easy to check directly).

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The matrix of this isomorphism is

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now $N \begin{pmatrix} s^2 \\ st \\ t^2 \end{pmatrix} = \begin{pmatrix} s^2 \\ st + t^2 \\ st \end{pmatrix}$

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What happens if $\psi \neq \varphi$? Then we set $g = \varphi^{-1} \circ \psi$. Then g is an automorphism of $\mathfrak{sl}_2(\mathbb{Q})$.
FACT: then g is a product of $\exp(\mathrm{ad} u)$, where $u \in \mathfrak{sl}_2(\mathbb{Q})$ is such that $\mathrm{ad} u$ is nilpotent. (Well, over $\overline{\mathbb{Q}}$, but that desn't matter here.)

Suppose that $g = \exp(\mathrm{ad} u)$. And set $h = \exp(\psi(u))$.
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But $h \in G(C, \mathbb{Q})$ (also easy to show) so

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This can be generalised to $\mathbb{P}^2(\mathbb{Q})$. There we work with varieties in $\mathbb{P}^9(\mathbb{Q})$, and we ask whether they are isomorphic to $\mathbb{P}^2(\mathbb{Q})$.

There we get a Lie algebra isomorphic to $\mathfrak{gl}_3(\mathbb{Q})$, and we deal with $\mathfrak{sl}_3(\mathbb{Q})$-modules.

Everything is analogous, except that the automorphism of $\mathfrak{sl}_3(\mathbb{Q})$ could be a "diagram automorphism". For such an automorphism we cannot perform the trick with the exponents.

However, the highest weight of the modules is $(3, 0)$. Inserting a diagram automorphism would change that to $(0, 3)$. So, since the modules are isomorphic, this cannot occur.

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Generalisation to $\mathbb{P}^1 \times \mathbb{P}^1$

Same story again. Here we work inside $\mathbb{P}^3(\mathbb{Q})$ or $\mathbb{P}^8(\mathbb{Q})$. The Lie algebras are isomorphic to $\mathfrak{sl}_2(\mathbb{Q}) \oplus \mathfrak{sl}_2(\mathbb{Q})$. And the modules have highest weights $(1,1)$ (in the case of $\mathbb{P}^3(\mathbb{Q})$) or $(2,2)$ (in the case of $\mathbb{P}^8(\mathbb{Q})$).

So in this case the diagram automorphism leaves the highest weights invariant. So it can pay a role.

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