Lie nilpotent group algebras and central series

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Lie algebras, their Classification and Applications

Outline

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Associated Lie ring

Let $(R, +, \cdot)$ be an associative ring. We consider the operation $[,]$ defined in R in the following manner:

$$\forall x, y \in R \qquad [x, y] := xy - yx$$

and we call the element $[x, y]$ the *Lie commutator* or the *Lie product* of x and y.
The structure $(R, +, [,])$ is easily verified to be a *Lie ring*

- $\forall a \in R \qquad [a, a] = 0$;
- $\forall a, b, c \in R \qquad [[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$

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Lie nilpotent rings

- The *lower Lie power series* of R is the series

$$R^{[1]} \geq R^{[2]} \geq R^{[3]} \geq \cdots$$

whose n-th term $R^{[n]}$ is the associative ideal generated by all the Lie commutators $[x_1, \ldots, x_n]$, with the assumption that $R^{[1]} := R$.

- The *upper Lie power series* of R is the series

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whose n-therm $R^{(n)}$ is defined by induction as the associative ideal generated by $[R^{(n-1)}, R]$, with the assumption that $R^{(1)} := R$.

- R is called *Lie nilpotent* (*strongly Lie nilpotent*) if there exists m such that $R^{[m]} = 0$ ($R^{(m)} = 0$) .

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If R is Lie nilpotent, (strongly Lie nilpotent) the smallest integer m for which $R^{[m]} = 0$ ($R^{(m)} = 0$) is called the *Lie nilpotency index* (*upper Lie nilpotency index*) of R and it is denoted by $t_L(R)$ ($t^L(R)$) .

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- Clearly, $R^{[n]} \subseteq R^{(n)}$ for all integer n and thus, if R is strongly Lie nilpotent, it is Lie nilpotent and $t_L(R) \leq t^L(R)$.
- A. Giambruno and S.K. Sehgal (1989) proved that the exterior algebra on a countable infinite-dimensional vector space over a field of characteristic not 2 is Lie nilpotent, but not strongly Lie nilpotent.

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Lie nilpotent group algebras

Theorem (Passi-Passman-Sehgal, 1973)

Let KG be a non-commutative group algebra. The following statements are equivalent:

(i) *KG is strongly Lie nilpotent;*

(ii) *KG is Lie nilpotent;*

(iii) *K has positive characteristic p, G is a nilpotent group and its commutator subgroup G' is a finite p-group.*

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Let $U(KG)$ be the unit group of a group algebra KG.

Theorem (Passi-Passman-Sehgal, 1973 + Khripta, 1972)

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According to a result by N.D. Gupta and F. Levin (1983) for arbitrary associative unitary rings, if KG is Lie nilpotent $cl(U(KG)) \leq t_L(KG) - 1$.

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Computation of $\mathrm{cl}(U(KG))$

- A. Shalev (1989) began a systematical study of the nilpotency class of the unit group of a group algebra of a finite p-group over a field with p elements.
  - Using the idea by D.B. Coleman and D.S. Passman (1968), the attempts by Shalev were based on seeing if a wreath product of the type $C_p \wr H$ was involved in $V(KG)$ (in fact, according to an observation by Buckley, in this case $t(H) = \mathrm{cl}(C_p \wr H) \leq \mathrm{cl}(U(KG))$).
  - Shalev conjectured that $V((KG))$ always possesses a section isomorphic to the wreath product $C_p \wr G'$.
  - He proved the result in 1990 when G' is cyclic and the characteristic of the ground field is odd and A.B. Konovalov (2001) confirmed the statement in the case in which G is a 2-group of maximal class.
- Du's Theorem (1992) gave a great contribution since it reduced the computation of the nilpotency class $\mathrm{cl}(U(KG))$ to that of the Lie nilpotency index $t_L(KG)$ of the group algebra.

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Du's Theorem

- S.A. Jennings (1955) proved that if R is a radical ring, the adjoint group R° is nilpotent if, and only if, R is Lie nilpotent.
- Jennings conjectured that if R is radical, $\mathrm{cl}(R^\circ) = t_L(R) - 1$.
- H. Laue (1984) conjectured that if R is radical, for every non-negative integer n, $Z_n(R) = \zeta_n(R^\circ)$.
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Let R be an associative ring. For all $a, b \in R$ we set

$$a \circ b := a + b + ab.$$

It is well known that $(R, \circ)$ is a monoid (with 0 as neutral element). The group R° of all the invertible elements of $(R, \circ)$ is called the adjoint group of R. If $R = R^\circ$, which means that R coincides with its Jacobson radical, then the ring R is called radical.

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- X. Du (1992) proved Laue conjecture's.

Let R be an associative ring. For all $a, b \in R$ we set

$$a \circ b := a + b + ab.$$

It is well known that $(R, \circ)$ is a monoid (with 0 as neutral element). The group R° of all the invertible elements of $(R, \circ)$ is called the adjoint group of R. If $R = R^\circ$, which means that R coincides with its Jacobson radical, then the ring R is called radical.

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Application to the Unit Group

Applying Du's Theorem to group algebras we obtain that if K is a field of positive characteristic p and G is a finite p-group, then

$$\mathrm{cl}(U(KG)) = t_L(KG) - 1.$$

- The computation of $\mathrm{cl}(U(KG))$ is reduced to that of $t_L(KG)$.
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We set for all positive integer n

$$\mathfrak{D}_{(n)}(KG) := G \cap (1 + KG^{(n)}) = G \cap (1 + \omega(G)^{(n)}),$$

the so called *n-th upper Lie dimension subgroup* of G.
Put $p^{d_{(k)}} := |\mathfrak{D}_{(k)}(G) : \mathfrak{D}_{(k+1)}(G)|$, where $k \geq 1$. If KG is Lie nilpotent,

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Outline

The definition

Let KG be the group algebra of a group G over a field K. We consider the upper Lie central series of KG,

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$$\forall n \in \mathbb{N}_0 \qquad \mathfrak{C}_n(G) := G \cap (1 + Z_n(KG)) = G \cap (1 + Z_n(\omega(G))).$$

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Du's Theorem and ULC subgroups

Let K be a field of positive characteristic p and let G be a finite p-group. Then

- $\forall i \in \mathbb{N} \qquad \mathfrak{C}_i(G) = G \cap \zeta_i(V(KG))$;
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Theorem (Catino, S.)

Let K be a field of positive characteristic p. Then $\mathrm{cl}(U(KG)) = t_L(KG) = t^L(KG)$ if

- *G is in $CF(4, n, p)$;*
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According to Blackburn's definition, a finite group G belongs to $CF(m, n, p)$ if $|G| = p^n$, $\mathrm{cl}(G) = m - 1$ and

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According to a result by R.K. Sharma and Vikas Bist (1992), $t_L(KG) \le t^L(KG) \le |G'| + 1$.

- A. Shalev (1993) proved that, if $p \ge 5$, $|G'| = p^n$ for some integer n and $t_L(KG) < |G'| + 1$, then $t_L(KG) \le p^{n-1} + 2p - 1$ and the equality holds if, and only if, G' has a cyclic subgroup of index p and $\gamma_3(G) \not\le G'^p$.
  - Assume that G is a $CF(5, n, 2)$ group and K is a field of even characteristic. Then

$$t_L(KG) = t^L(KG) = 8 > 2^{3-1} + 4 - 1 = 7.$$

    In this sense, Shalev's inequality does not hold in characteristic 2.
  - Let $f(2, n)$ be a function such that $t_L(KG) \le f(2, n)$ when $t_L(KG)$ is not maximal. The upper bound is exact when G is a $CF(5, n, 2)$ group and G' is elementary abelian. In this case G' does not contain any cyclic subgroup of index 2, thus also the group-theoretical condition required by Shalev's result does not hold.

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Outline

Lie nilpotent group algebras and central series
Lie nilpotency index
Computation of $\mathrm{cl}(U(KG))$
Upper Lie codimension subgroups
Open questions

Open questions and research lines

- The conjecture $t_L(KG) = t^L(KG)$
  - We tested for all finite 2-groups of GAP library (that is, for all finite 2-groups of order $\leq 2^9$).
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Open questions and research lines

- The conjecture $t_L(KG) = t^L(KG)$
  - We tested for all finite 2-groups of GAP library (that is, for all finite 2-groups of order $\leq 2^9$).
  - A possibile approach is to describe in terms of the elements of G the upper Lie codimension subgroups, providing us of a means to compute $t_L(KG)$.
- To find the function $f(2, n)$ such that $t_L(KG) \leq f(2, n)$ when $t_L(KG)$ is not maximal and to study when the upper bound is achieved.

- Are the assumptions on the centres of G of the previous theorems essential?
  - Computer investigation by GAP confirms the results without assumptions on the centres of G.
- To go on describing the terms of the series of the upper Lie codimension subgroups of G when G is a $CF(m,n,p)$ group proving if, in this case, the terms of the series coincide with those of the upper central series of G.
  - Computer investigation by GAP confirms the result, which is in general not true.

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Example

Let $G := \langle x, y \,|\, x^4 = y^4 = (x, y, y) = (x, y, x) = 1 \rangle$. G is a group of order 64 with $|\zeta(G)| = |G'| = 4$ and $|\Phi(G)| = 16$. In this case

(1) $\mathfrak{C}_1(G) = \mathfrak{C}_2(G) = \zeta(G)$;

(2) $\mathfrak{C}_3(G) = \Phi(G)$;

(3) $\mathfrak{C}_4(G) = G$.

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