Using Lie algebras to parametrize certain types of algebraic varieties II

Willem A. de Graaf, University of Trento, Italy
Janka Pílniková, RICAM Linz / RISC Linz, Austria
Josef Schicho, RICAM Linz, Austria

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Outline

(i) Surfaces in algebraic geometry, parametrizing surfaces

(ii) The Lie algebra of a surface.

(iii) Using the method to parametrize non-trivial cases of Del Pezzo surfaces of degree 8:

   – "non-split" case: the unit sphere as a twist of $\mathbb{P}^1 \times \mathbb{P}^1$

   – "non-semisimple" case: the blowup of $\mathbb{P}^2$

Surfaces in projective space

The n-dimensional projective space:

$$\mathbb{P}^n = \{(t_0 : \cdots : t_n) \mid (t_0 : \cdots : t_n) \neq (0 : \cdots : 0)\}.$$

Let $f_1, \ldots, f_k \in \mathbb{Q}[x_0, \ldots, x_n]$ are forms over $\mathbb{Q}$.
Projective variety S is the set of solutions to $f_1, \ldots, f_k$, i.e. all points in $\mathbb{P}^n$ such that

$$f_1(a_0, \ldots, a_n) = \cdots = f_k(a_0, \ldots, a_n) = 0.$$

Let S be a surface: $\dim(S) = 2$.
Finding a rational parametrization over $\mathbb{Q}$ means finding all rational solutions to the system.

We need a map $\varphi : \mathbb{P}^2$ (or $\mathbb{P}^1 \times \mathbb{P}^1$) $\rightarrow S \subset \mathbb{P}^n$ subject to the following:

Parametrization of a surface

$S \subset \mathbb{P}^n$ a surface.
$\varphi : \mathbb{P}^2$ (or $\mathbb{P}^1 \times \mathbb{P}^1$) $\to \mathbb{P}^n$ is a rational parametrization of S over $\mathbb{Q}$, if

(i) $\varphi = (p_0 : \cdots : p_n)$ and $p_0, \ldots p_n \in \mathbb{Q}[t_0, t_1, t_2]$ are forms of the same degree,

(ii) $\varphi(p) \in S$ for all p from the domain of φ,

(iii) the image of φ is a dense subset of S.

Preprocessing: By embeddings of S we reduce to few basic cases:

- tubular surfaces
- some trivial cases
- Del Pezzo surfaces of degree 5, 6, 7, 8, 9.

The Lie algebra of a surface

For a given surface $S \subset \mathbb{P}^n$,
$G(S, \mathbb{Q}) := \{g \in \mathrm{GL}_{n+1}(\mathbb{Q}) \mid \forall p \in S \; gp \in S\}$.
$L(S, \mathbb{Q}) := \mathrm{Lie}(G(S, \mathbb{Q}))$ – the Lie algebra of S.

How to compute $L(S, \mathbb{Q})$?

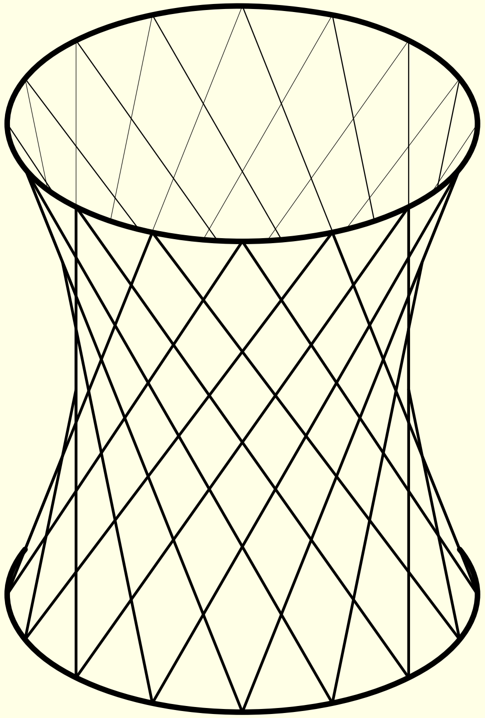
If $S \subset \mathbb{P}^3$ is given by a quadratic form $f(x) = x^T A x$ over $\mathbb{Q}$:
$S = \{p = (x_0 : x_1 : x_2 : x_3)^T \mid p^T A p = 0\}$,
then $G(S, \mathbb{Q}) = \{g \in \mathrm{GL}_4(\mathbb{Q}) \mid g^T A g = \lambda A\}$
and $L(S, \mathbb{Q}) = \{X \in \mathfrak{gl}_4(\mathbb{Q}) \mid X^T A + AX = \lambda A\}$.

If $S \subset \mathbb{P}^n$ is given by a set of quadratic forms $f_1, \ldots, f_k$ over $\mathbb{Q}$, $f_i(x) = x^T A_i x$:
$S = \{p = (x_0 : \cdots : x_n)^T \mid p^T A_i p = 0 \; \forall i\}$,
then $G(S, \mathbb{Q}) = \{g \in \mathrm{GL}_{n+1}(\mathbb{Q}) \mid g^T A_i g \in \langle A_1, \ldots A_k \rangle_{\mathbb{Q}} \; \forall i\}$
and $L(S, \mathbb{Q}) = \{X \in \mathfrak{gl}_{n+1}(\mathbb{Q}) \mid X^T A_i + A_i X \in \langle A_1, \ldots A_k \rangle_{\mathbb{Q}} \; \forall i\}$.

Parametrizing twists of $\mathbb{P}^1 \times \mathbb{P}^1$

The canonical surface $S_0 : x_1 x_2 = x_0 x_3$ is parametrized by $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\varphi_0 : (s_0 : s_1;\ t_0 : t_1) \mapsto (s_0 t_0 : s_0 t_1 : s_1 t_0 : s_1 t_1) = (x_0 : x_1 : x_2 : x_3).$$



$$G(S_0, \mathbb{Q}) = \{g \in \mathrm{GL}_4(\mathbb{Q}) \mid \forall p \in S_0\ gp \in S_0\}.$$

The Lie algebra $L(S_0, \mathbb{Q}) = \mathrm{Lie}(G(S_0, \mathbb{Q}))$ is isomorphic to $\mathfrak{sl}_2(\mathbb{Q}) \oplus \mathfrak{sl}_2(\mathbb{Q}) \oplus \mathbb{Q}$.
$L(S_0, \mathbb{Q}) \subset \mathfrak{gl}_4(\mathbb{Q})$.

The module V_0 of $L(S_0, \mathbb{Q})$ is 4-dimensional irreducible $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$-module with the heighest weight $(1, 1)$.

Short review of the basic method

EXAMPLE: $S: x_0^2 - x_1^2 - x_2^2 + x_3^2$ is projectivelly equivalent to S_0 over $\mathbb{Q}$:

- the Lie algebra $L(S, \mathbb{Q}) \cong \mathfrak{sl}_2(\mathbb{Q}) \oplus \mathfrak{sl}_2(\mathbb{Q}) \oplus \mathbb{Q}$.
- the module W of $L(S, \mathbb{Q})$ is 4-dimensional irredcible $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$-module with the heighest weight $(1, 1)$.

The isomorphism $\psi : V_0 \to W$:
$$\begin{aligned} e_0 &\mapsto v_3 + v_0, \\ e_1 &\mapsto v_3 - v_0, \\ e_2 &\mapsto v_2 + v_1, \\ e_3 &\mapsto v_2 - v_1 \end{aligned}$$
is unique, up to multiplication by scalars.

Therefore ψ is also the projective equivalence of S_0 and S

$\psi : (x_0 : x_1 : x_2 : x_3) \mapsto (x_3 + x_0 : x_3 - x_0 : x_2 + x_1 : x_2 - x_1)$

and gives us a parametrization of S:

$\varphi = \psi \circ \varphi_0 : (x_0 : x_1 : x_2 : x_3) = (s_1 t_1 + s_0 t_0 : s_1 t_1 - s_0 t_0 : s_1 t_0 + s_0 t_1 : s_1 t_0 - s_0 t_1)$.

Sphere as a twist of $\mathbb{P}^1 \times \mathbb{P}^1$

The unit sphere

$$S : x_1^2 + x_2^2 + x_3^2 = x_0^2$$

is not isomorphic to $S_0(x_1 x_2 = x_0 x_3)$ over $\mathbb{Q}$: $L(S, \mathbb{Q}) = L_0(S, \mathbb{Q}) \oplus I_4$, where $L_0(S, \mathbb{Q})$ is 6-dimensional simple Lie algebra which is a twist of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

But still S has a rational parametrization.

We find a splitting field E of $L_0(S, \mathbb{Q})$ as the centroid of the algebra:

Let E be the centralizer of $\mathrm{ad}(L_0(S, \mathbb{Q}))$ in $\mathfrak{gl}(L_0(S, \mathbb{Q}))$.
Then $E = \mathbb{Q}(i)$ and $L_0(S, E) \cong \mathfrak{sl}_2(E) \oplus \mathfrak{sl}_2(E)$. The corresponding module becomes $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$-module over E with maximal weight $(1, 1)$.

We get a parametrization of S over E: $\psi : S_0 \to S$:

$$(x_0 : x_1 : x_2 : x_3) = (-s_0 t_1 + s_1 t_0 : s_0 t_0 - s_1 t_1 : i s_0 t_0 + i s_1 t_1 : s_0 t_1 + s_1 t_0).$$

Sphere as a twist of $\mathbb{P}^1 \times \mathbb{P}^1$ – continued

$\mathcal{F}_1$, $\mathcal{F}_2$ – the two families of lines on the surface.
$\mathcal{F}_1$ is a 1-dimensional family of lines over E: $\forall (s:t) \in \mathbb{P}^1(E) \quad l_{(s:t)} \in \mathcal{F}_1$.

For the centroid E we have $[E:\mathbb{Q}] = 2$.
Let σ be the nontrivial automorphism of E over $\mathbb{Q}$.

If $l \in \mathcal{F}_1$ then $\sigma(l) \in \mathcal{F}_2$. Therefore $l \cap \sigma(l) = \{p\}$.
p is fixed under σ, hence p is a *rational* point
and $(s:t) \mapsto l_{(s:t)} \cap \sigma(l_{(s:t)})$ is a map $\mathbb{P}^1(E) \to S(\mathbb{Q})$.

The projective line $\mathbb{P}^1(E)$ can be parametrized by the projective plane $\mathbb{P}^2(\mathbb{Q})$:
$(a:b:c) \mapsto (a+ib:c)$.

This leads to a rational parametrization of the sphere

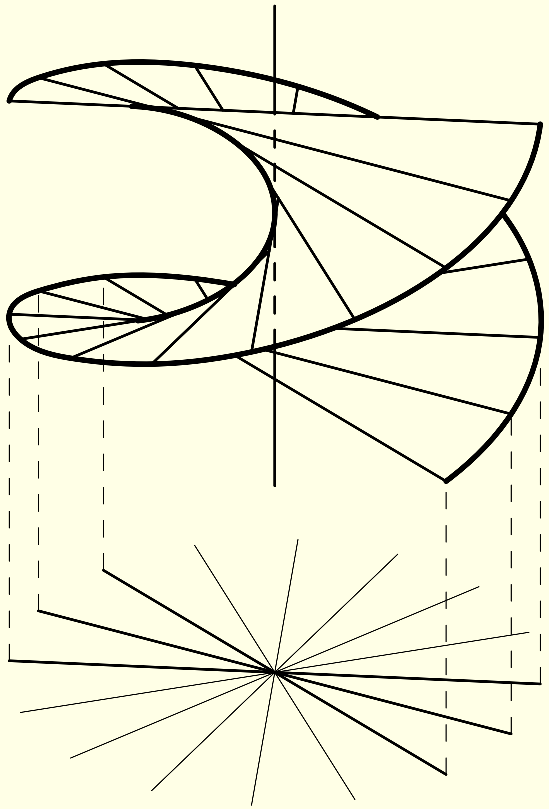
$$(a:b:c) \mapsto (c^2+a^2+b^2 : 2ac : -2bc : c^2-a^2-b^2)$$

with $a, b, c, \in \mathbb{Q}$.

Parametrizing blowups of $\mathbb{P}^2$

The canonical blowup $S_0 \subset \mathbb{P}^8$ is parametrized

$$(s : t : u) \mapsto (s^2t : s^2u : st^2 : stu : su^2 : t^3 : t^2u : tu^2 : u^3).$$



Let $S \subset \mathbb{P}^8$ be projectively equivalent to S_0 over $\mathbb{Q}$.

The Lie algebras of S_0 and S decompose as a sum of $\mathfrak{sl}_2(\mathbb{Q})$ and a 3-dimensional radical R:

$$\varphi_0 : \mathfrak{sl}_2(\mathbb{Q}) + R \to L(S_0, \mathbb{Q}),$$
$$\varphi : \mathfrak{sl}_2(\mathbb{Q}) + R \to L(S, \mathbb{Q}).$$

As $\mathfrak{sl}_2$-modules:

$$V(\varphi_0) = W_2(\varphi_0) \oplus W_3(\varphi_0) \oplus W_4(\varphi_0),$$
$$V(\varphi) = W_2(\varphi) \oplus W_3(\varphi) \oplus W_4(\varphi)$$

with $\dim(W_i(\varphi_0)) = \dim(W_i(\varphi)) = i$.

Parametrizing blowups of $\mathbb{P}^2$ – continued

(1)

Any isomorphism $\psi : V(\varphi_0) \to V(\varphi)$ maps $W_i(\varphi_0)$ to $W_i(\varphi)$, $i = 2, 3, 4$.

$\mathbb{P}(W_2(\varphi_0))$ ($\mathbb{P}(W_2(\varphi))$) is the exceptional line of S_0 (S). One can use geometric methods to parametrize S.

(2)

Consider $V(\varphi_0)$ as an $(\mathfrak{sl}_2 + R)$-module: Elements of the radical carry $W_i(\varphi_0)$ to $W_{i-1}(\varphi_0)$, $i = 3, 4$, so $V(\varphi_0)$ is irreducible. The same with $V(\varphi)$.

The isomorphism $\psi : V(\varphi_0) \to V(\varphi)$ as $(\mathfrak{sl}_2 + R)$-modules is unique up to multplication by scalars.

Therefore it is also an isomorphism of S_0 and S and hence a parametrization of S.