Workshop ``Lie algebras, their classification and applications''

An explicit formula for Casimir's ghost

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(part of a joint work with M. Duflo*)*

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$\mathbb{K} \supseteq \mathbb{Q}$ is a field

$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ a Lie $\mathbb{K}$-superalgebra (*i.e a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra*)

$U(\mathfrak{g})$ its enveloping algebra (with the standard graduation)

A Casimir's ghost $d \in U(\mathfrak{g}) \setminus \{0\}$ is A SPECIAL invariant for the action

$$\begin{array}{ccc} \mathfrak{g} \otimes U(\mathfrak{g}) & \xrightarrow{\mathrm{ad}'} & U(\mathfrak{g}) \\ a \otimes w & \to & \begin{cases} a * w - w * a, & a \in \mathfrak{g}_0 \\ a * w + (-1)^{deg(w)} w * a, & a \in \mathfrak{g}_1 \end{cases} \end{array}$$

(i.e $(\mathrm{ad}' a)(d) = 0 \ \forall a \in \mathfrak{g}$*).*

N.B.

- $d^2 \in center(U(\mathfrak{g}))$ (origin of its name).
- It has been introduced by physicists interested in representations of quantum groups and its application to Physics.
- Its existence is important in Representation Theory of Lie superalgebras.

Symmetrization map $\beta : S(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1) \to U(\mathfrak{g})$

$$\beta(X_1 \cdots X_n \otimes X_{n+1} \wedge \cdots \wedge X_m) = \frac{1}{m!} \sum_{s \in \Sigma_m} \pm X_{s(1)} * \cdots * X_{s(m)}$$

Theorem 1. (Gorelik '00)

$$\begin{cases} \mathbb{K} = \mathbb{C}, \quad tr_{\mathfrak{g}_1}(\text{ad}a) = 0 \ \forall a \in \mathfrak{g}_0, \\ \dim \mathfrak{g}_1 < \infty \end{cases}$$

$\Rightarrow \exists d \in \beta(\Lambda(\mathfrak{g}_1))$ such that it is ad'-invariant, its component in $\beta(\Lambda^{max}(\mathfrak{g}_1))$ is not zero.

Gorelik '00: $\begin{array}{l} \{\text{ad}' - invariant\} = \{(\text{ad}'d)(Center(\mathfrak{g}_0))\}; \\ dim\mathfrak{g} = \infty \Rightarrow \{\text{ad}' - invariant\} = \emptyset \end{array}$.

Arnaudon, Bauer, Frappat '97:

- a beautiful formula for d when $\mathfrak{g} = osp(1, 2n) =$

$$= \left\{ \begin{pmatrix} 0 & -v^t & w^t \\ v & A & B \\ w & C & -A^t \end{pmatrix} | B = B^t; \ C = C^t; v, w \in \mathbb{C}^n \right\}.$$

- It uses a special basis of $\mathfrak{g}_1$.

PURPOSE OF THIS COMMUNICATION.

To present a formula for d valid for EVERY

- field $\mathbb{K} \supseteq \mathbb{Q}$,
- Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$,
- basis of $\mathfrak{g}_1$.

EXAMPLES. (Difficult to compute by hand !!!)

Let $v_1, v_2, ..., v_n$ be a basis for $\mathfrak{g}_1$,

$(-1)^{|s|} =$ parity of a permutation s

$\dim \mathfrak{g}_1 = 2$

$$\beta^{-1}(d) = v_1 \wedge v_2 + -\frac{1}{24} \sum_{s\in\Sigma_2} (-1)^{|s|} tr_{\mathfrak{g}_1}(\mathrm{ad} v_{s(1)} \circ \mathrm{ad} v_{s(2)})$$

$\dim \mathfrak{g}_1 = 3$

$$\beta^{-1}(d) = v_1 \wedge v_2 \wedge v_3 + -\frac{1}{24} \sum_{s\in\Sigma_3} (-1)^{|s|} tr_{\mathfrak{g}_1}(\mathrm{ad} v_{s(1)} \circ \mathrm{ad} v_{s(2)}) v_{s(3)}$$

$\dim \mathfrak{g}_1 = 4$

$$\beta^{-1}(d) = v_1 \wedge v_2 \wedge v_3 \wedge v_4 +$$
$$-\frac{1}{24} \sum_{s\in\Sigma_4} (-1)^{|s|} tr_{\mathfrak{g}_1}(\mathrm{ad} v_{s(1)} \circ \mathrm{ad} v_{s(2)}) v_{s(3)} \wedge v_{s(4)} +$$
$$+\frac{1}{1152} \sum_{s\in\Sigma_4} (-1)^{|s|} tr_{\mathfrak{g}_1}(\mathrm{ad} v_{s(1)} \circ \mathrm{ad} v_{s(2)}) tr_{\mathfrak{g}_1}(\mathrm{ad} v_{s(3)} \circ \mathrm{ad} v_{s(4)}) +$$
$$+\frac{1}{2880} \sum_{s\in\Sigma_4} (-1)^{|s|} tr_{\mathfrak{g}_1}(\mathrm{ad} v_{s(1)} \circ \mathrm{ad} v_{s(2)} \circ \mathrm{ad} v_{s(3)} \circ \mathrm{ad} v_{s(4)})$$

HOW TO PROVE THESE FORMULAS.

Let $v_1, v_2, , ..., v_n$ be any basis of $\mathfrak{g}_1$.

We look for $f \in \Lambda(\mathfrak{g}_1)^*$ such that

$$\begin{aligned} \beta^{-1}(d) &= f(1)v_1 \wedge \cdots \wedge v_n + \\ &+ \sum_j \pm f(v_j) v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_n + \\ &+ \sum_{i<j} \pm f(v_i \wedge v_j) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_n + \\ &+ \cdots + f(v_1 \wedge \cdots \wedge v_n) \end{aligned}$$

and $f(1) = 1$.

Theorem 2. (independent subject)
We get an explicit formula for the action $\mathfrak{g} \otimes \Lambda(\mathfrak{g}_1) \xrightarrow{\beta^{-1} \circ \mathrm{ad}' \circ \beta} \Lambda(\mathfrak{g}_1)$ *solving functional equations by formal power series.*

Then
- $\beta^{-1}(d)$ is a $\beta^{-1} \circ \mathrm{ad}' \circ \beta$-invariant
- we work with the dual action to show that

$$g(t) := t \coth\left(\frac{t}{2}\right), \qquad h(t) := -\tanh\left(\frac{t}{4}\right)$$

Theorem 3. $i)$ *A formal power series* $\rho(t) = 1 + \rho_2 t^2 + \cdots \in \mathbb{K}[[t^2]]$ *such that*

$$\frac{\ln(\rho)(t) - \ln(\rho)(u)}{t-u} g(t-u) - \frac{g(t) - g(t-u)}{u} + \\ -h(t-u) = 0 \quad \text{modulo} \quad (t^2 - u^2)$$

provides such an f.

$ii)$ *The only solution is* $\rho(t) = \frac{\sinh(\frac{t}{2})}{\frac{t}{2}} \in \mathbb{Q}[[t]]$.

Examples. Let $X, Y \in \mathfrak{g}_1$.
$f(X) = 0$,
$f(X \wedge Y) = \rho_2 tr_{\mathfrak{g}_1}(\mathrm{ad}X \circ \mathrm{ad}Y - \mathrm{ad}Y \circ \mathrm{ad}X)$.

In general $f = e^{-tr_{\mathfrak{g}_1}\left(\ln(\rho)(M)\right)}$,
$M =$ nilpotent matrix with coefficients in $\Lambda(\mathfrak{g}_1^*)$
$M = \sum_{i=1}^{\dim \mathfrak{g}_1} \mathrm{ad} v_i \otimes v_i^* \in End(\mathfrak{g}) \otimes \mathfrak{g}_1^*$.

GORELIK'S THEOREM APPLIES WHEN

• $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is a supersymmetric space:

(i.e $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is a Lie superalgebra such that $\mathfrak{h}$ and $\mathfrak{q}$ are $\mathbb{Z}/2\mathbb{Z}$-graded, $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{h},\mathfrak{q}] \subseteq \mathfrak{q}$, $[\mathfrak{q},\mathfrak{q}] \subseteq \mathfrak{h}$)

N.B. A Lie superalgebra $\mathfrak{g}$ is a supersymmetric space such that $\mathfrak{h} = \mathfrak{g}_0$ and $\mathfrak{q} = \mathfrak{g}_1$.

Theorem 4. *.*

$$\begin{cases} tr_{\mathfrak{q}_1}(\mathrm{ad} a) = 0 \ \forall a \in \mathfrak{h} \\ \mathfrak{q} = \mathfrak{q}_1 \\ \dim \mathfrak{q}_1 < \infty \end{cases}$$

$\Rightarrow \exists d \in \beta(\Lambda(\mathfrak{q}_1))$ *such that it is* ad*'-invariant, its component in* $\beta(\Lambda^{max}(\mathfrak{q}_1))$ *is not zero.*

• the hypothesis on the trace is removed: the formula of Casimir's ghost provide an invariant for a more complicated action of $\mathfrak{g}$.