The LEX Game
and some applications

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Standard monomials and leading monomials

$\mathbb{F}$ field, $\mathbb{F}[x_1, \ldots, x_n] = \mathbb{F}[\mathbf{x}]$ polynomial ring in n variables

monomial: polynomials of the form $\mathbf{x}^{\mathbf{w}} := x_1^{w_1} x_2^{w_2} \ldots x_n^{w_n}$

lexicographic order of monomials: $\mathbf{x}^{\mathbf{w}} \prec \mathbf{x}^{\mathbf{u}} \iff \mathbf{w} <_{\text{lex}} \mathbf{u}$

complete order, refinement of divisibility, compatible with multiplication

leading monomial $\ell\mathrm{m}(f)$ of $f(\mathbf{x})$ is its greatest monomial

Let $I \trianglelefteq \mathbb{F}[\mathbf{x}]$ be an ideal.

leading monomials of I: $\mathrm{Lm}(I) = \{\ell\mathrm{m}(f) : f \in I\}$;

upwards closed (with respect to divisibility)

standard monomials of I: $\mathrm{Sm}(I)$ the others;

Examples

Example $I = \langle x_1 + x_2 \rangle$
leading monomials: monomials divisible by x_1
standard monomials: powers of x_2

Example $I = \langle x_1 x_2 + x_1,\ \ x_1 x_2 + x_2 \rangle$
since $x_1 - x_2 \in I$ and $x_2^2 + x_2 \in I$
standard monomials can only be: 1 and x_2

Vanishing ideal of a finite set of points

Let $V \subseteq \mathbb{F}^n$ be finite.

$I(V) := \{f(\mathbf{x}) \in \mathbb{F}[\mathbf{x}] \ : \ f \text{ vanishes on } V\}$

$\mathbb{F}[\mathbf{x}] / I(V)$ is the vector space of functions $V \to \mathbb{F}$.

Theorem $\mathrm{Sm}(I)$ *is a linear base of the vector space* $\mathbb{F}[\mathbf{x}] / I$*.*

Corollary $|V| = |\mathrm{Sm}(I(V))|$

Example $\langle x_1 x_2 + x_1, \ x_1 x_2 + x_2 \rangle = I(\{(0,0), (-1,-1)\})$.

The LEX Game

$V \subseteq \mathbb{F}^n$ finite, $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$.

Lex(V; $\mathbf{w}$) game:

0 Stan thinks of a $\mathbf{y} = (y_1, \ldots, y_n) \in V$.

1 Lea has w_n guesses for y_n. If she finds out she wins. Otherwise, Stan reveals y_n.

2 Lea has w_{n-1} guesses for y_{n-1}. If she finds out she wins. Otherwise, Stan reveals y_{n-1}.

...

n Lea has w_1 guesses for y_1. If she finds out she wins. Otherwise, Stan reveals y_1 and he wins the game.

Example $n = 5$,

V consists of α-β sequences of length 5 in which there is exactly 1, 2 or 3 α.

With $\mathbf{w} = (11100)$ Lea has winning strategy.

With $\mathbf{w} = (01110)$ she does not have.

The theorem

Theorem *In* $\mathsf{Lex}(V;\mathbf{w})$ *Lea has winning strategy* $\iff \mathbf{x}^{\mathbf{w}} \in \mathsf{Lm}\,(I(V))$*.*

Stan may win $\mathsf{Lex}(V;\mathbf{w}) \iff \mathbf{x}^{\mathbf{w}} \in \mathsf{Sm}\,(I(V))$*.*

Example

With $\mathbf{w} = (11100)$ Lea wins, thus $x_1 x_2 x_3 \in \mathsf{Lm}\,(I(V))$;
but with $\mathbf{w} = (01110)$ Stan can win, so $x_2 x_3 x_4 \in \mathsf{Sm}\,(I(V))$.

Proof of the theorem ($\Rightarrow$)

Theorem *Lea wins* $\mathsf{Lex}(V; \mathbf{w}) \iff \mathbf{x}^{\mathbf{w}} \in \mathsf{Lm}\,(I(V))$.

Proof:

$$l(\mathbf{x}) = \prod_{j=1}^{n} \left(\prod_{i=1}^{w_j} \left(x_j - f_{j,i}(x_{j+1}, \ldots, x_n) \right) \right)$$

where $f_{j,1}$, $f_{j,2} \ldots f_{j,w_j}$ are Lea's guesses for y_j.

$f_{j,i}$ can be considered as a polynomial $\Rightarrow$ $l(\mathbf{x}) \in \mathbb{F}\,[\mathbf{x}]$

$l(\mathbf{x})$ vanishes on V $\Rightarrow$ $l(\mathbf{x}) \in I(V)$

$\ell\mathsf{m}\,(l(\mathbf{x})) = \mathbf{x}^{\mathbf{w}}$ $\Rightarrow$ $\mathbf{x}^{\mathbf{w}} \in \mathsf{Lm}\,(I(V))$

Standard monomials of permutations

Let $n = 4$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{F}$ distinct elements.

$V = \{(\pi(\alpha_1), \pi(\alpha_2), \pi(\alpha_3), \pi(\alpha_4))$: π is a permutation of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\}$.

(Thus $4! = |V| = |\mathrm{Sm}\,(I(V))|$.)

Who wins the game $\mathrm{Lex}(V; \mathbf{w} = (0, 1, 2, 3))$?

And the games
$\mathrm{Lex}(V; (0, 0, 0, 4))$,
$\mathrm{Lex}(V; (0, 0, 3, 0))$,
$\mathrm{Lex}(V; (0, 2, 0, 0))$,
$\mathrm{Lex}(V; (1, 0, 0, 0))$?

Answering these questions we have determined the set $\mathrm{Sm}\,(I(V))$.

'Independence' of $\mathrm{Sm}\,(I(V))$ from V

Corollary

If we apply the injective functions $p_i\colon \mathbb{F} \longrightarrow \mathbb{F}'$ ($i = 1, \ldots, n$) on the elements of $V \subseteq \mathbb{F}^n$ coordinatewise then the image $V' \subseteq \mathbb{F}'^n$ has the same standard monomials as V.

Standard monomials have recursive structure

$V_\alpha = \{(v_1, \ldots, v_{n-1}) \ : \ (v_1, \ldots, v_{n-1}, \alpha) \in V\}$ (prefixes of V)

If Lea doesn't find out $y_n = \alpha$ in Lex(V; $\mathbf{w}$) then comes Lex(V_α; $(w_1, \ldots, w_{n-1})$).

Corollary

Let $n > 1$. Then $\mathbf{x}^\mathbf{w} \in$ Sm $(I(V)) \iff$
there exist at least $w_n + 1$ such α that $x_1^{w_1} \ldots x_{n-1}^{w_{n-1}} \in$ Sm $(I(V_\alpha))$.

Proof:

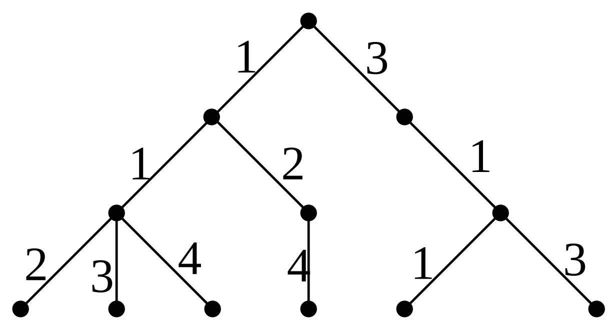
Stan wins Lex(V; $(w_1, \ldots, w_n)$) $\iff$
there exist at least $w_n + 1$ such α that
Stan wins Lex(V_α; $(w_1, \ldots, w_{n-1})$). $\square$

Algorithm for computing standard monomials

word tree: rooted tree of depth n, with elements of $\mathbb{F}$ on the edges. Each leaf corresponds to an element of $\mathbb{F}^n$: consider the way from the leaf to the root.

Example The word tree of

$V = \{(2,1,1),(3,1,1),(4,1,1),(4,2,1),(1,1,3),(3,1,3)\}$ is



Observation:

The subtree belonging to the child α of the root is the word tree of V_α.

Algorithm:

Write on every vertex v the standard monomial set of the subtree of v.

Thus on the $(n-i)^{th}$ level there are monomials of $x_1 \ldots x_i$.

Algorithm for computing standard monomials

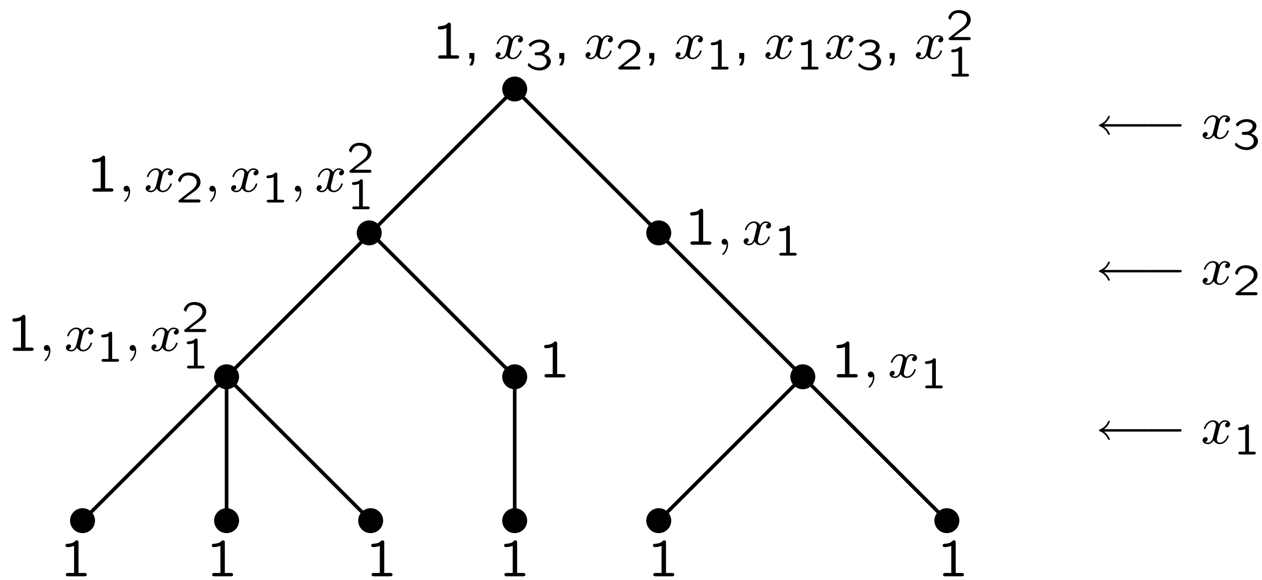
Algorithm

n^{th} level: write 1 on every vertex (leaf)

...

$(n-i)^{th}$ level: if the monomial $m(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_{i-1}]$ can be found in at least $w+1$ sets of children of v then we put $m(\mathbf{x})x_i^w$ in the set of vertex v

Example



Runtime: (implementing the above carefully)

$O(n \cdot |V| \cdot r)$, where r is the maximum degree of the wordtree.