Small maximal spaces of non-invertible matrices

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Special case: $r = n - 1$, *maximal singular space*

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3. $r = n - 1$, $A_1, \ldots, A_n$ generic skew-symmetric matrices and $\mathcal{A} = \{(A_1 x| \ldots |A_n x) \mid x \in \mathbb{C}^n\}$ (Bob Paré)

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4. $\exists$ many sufficient conditions for a matrix space $\mathcal{A}$ to be contained in a compression space: $\dim \mathcal{A}$ large enough, $\mathcal{A}$ spanned by rank one matrices, $r = 1, 2, 3$ while n large, etc. (Fillmore, Laurie, and Radjavi; Dieudonné; Eisenbud-Harris; Lovász)

A sufficient condition for rank-criticality

Setting: $\mathcal{A}$ a matrix space, $\mathrm{rk}(\mathcal{A}) = r$

Goal: decide whether $\mathcal{A}$ is rank-critical

Theoretically: Groebner basis computations. Not feasible!

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Define: $\mathrm{RND}(\mathcal{A}) := \bigcap_{A \in \mathcal{A}} T_A X_r$, *rank-neutral directions* of $\mathcal{A}$

Conclusion: if $\mathrm{RND}(\mathcal{A}) = \mathcal{A}$, then $\mathcal{A}$ is rank-critical

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Setting: G an algebraic group, $\rho : G \to \mathrm{GL}(V)$ a representation, $\mathfrak{g}$ the Lie algebra of G

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Conclusion: decomposition of $\mathrm{End}(V)$ into irreducible G-modules can be used for proving that $\rho(\mathfrak{g})$ is rank-critical

Implementation: in GAP, using Willem de Graaf's Lie algebra algorithms

Two results

Small maximal singular spaces: $G := \mathrm{SL}_m,\ m \geq 3$ acting on $V :=$homogeneous polynomials of degree $me,\ e \geq 1$ yields a $(m^2 - 1)$-dimensional maximal space of singular $n \times n$-matrices, where $n = \dim V$.

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Adjoint representation: for any semisimple $\mathfrak{g}$, $\mathrm{ad}(\mathfrak{g}) \subseteq \mathrm{End}(\mathfrak{g})$ is rank-critical of rank $\dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g}$.

In fact: linear equations 'cutting out' $\mathrm{ad}(\mathfrak{g})$ in $\mathrm{End}(\mathfrak{g})$: $A \in \mathrm{End}(\mathfrak{g})$ lies in $\mathrm{ad}(\mathfrak{g})$ if and only if A maps *every* Cartan $\mathfrak{h}$ into $\bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha$.

See you in Basel!?

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