Lie Algebras, their Classification and Applications

University of Trento

25-27 July 2005

Wreath Lie Algebras

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Lie algebras associated with a pro-p-group

It is well-known that it is possible to attach a Lie ring $L(G)$ to any pro-p-group G, defining a suitable Lie bracket on the sum of the factors of a strongly central series of G.

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If such factors have exponent p, then the Lie ring turns out to be a Lie algebra over the prime field $\mathbb{F}_p$.

Structure of $L(G \wr C_p)$

Let $2 \neq p \in \mathbb{P}$; let C_p be the cyclic group of order p, $F = \mathbb{F}_p$, $F(\epsilon)$ the divided power algebra and δ its canonical derivation.

Let G be a p-group. Then $L(G \wr C_p)$ depends on $L(G)$.

Theorem 1 *Let G be a finitely generated (pro-)p-group with $\gamma_i(G)^p \subseteq \gamma_{i+1}(G)$ for each $i \geq 1$. Then*

$$L(G \wr C_p) \cong (L(G) \otimes F(\epsilon)) \rtimes < d >,$$

where $d = id_{L(G)} \otimes \delta$ is a derivation of order p.

Iterated wreath algebras

Define the wreath operator **w** that associates to any Lie algebra L the *wreath algebra* of L:

$$\mathbf{w}(L) := (L \otimes F(\epsilon)) \rtimes < d >,$$

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where, as above, $d = id_L \otimes \delta$.

Let $W(n) = \underbrace{C_p \wr \cdots \wr C_p}_{n}$; by Theorem 1,

$$L(W(n)) = \mathbf{w}^{n-1}(F) := \omega^n(F)$$

$$\uparrow$$

n-steps wreath algebra

The rôle of $W(n)$

$W(n)$ is the Sylow p-subgroup both of $\mathrm{Sym}(p^n)$ and $SL(p^{n-1}(p-1), \mathbb{Z})$; by Cayley's Theorem, if G is a group of order p^n, then

$$G \hookrightarrow W(n).$$

Moreover, by Vol'vačev's Theorem (1963), if P is a finite p-group with an irreducible representation on the $\mathbb{Q}$-module V, then

$$P \hookrightarrow W(n)$$

and $V = M|_P$, where M is the canonical module of $W(n)$.

The rôle of $\omega^n(F)$

Is there an analogous result about modular Lie algebras and $\omega^n(F)$? Yes, there is; the $\{\omega^n(F)\}_n$ are the "containers" of the finite-dimensional nilpotent "absolutely irreducible" linear Lie algebras.

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Definition $\rho : L \to gl(V)$ is an *absolutely irreducible* representation if it is irreducible over any extension of the base field K, or, equivalently, if it is irreducible and

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Example $\omega^n(F)$ has a faithful absolutely irreducible representation on its canonical module.

Absolutely irreducible case over perfect field

Several results about irreducible representations over algebrically closed fields (["Modular Lie algebras and their representations", H.Strade-R.Farnsteiner]) can be extended to the absolutely irreducible case over perfect fields.

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However, these results can be extended to non-restricted algebras, because each algebra is embedded in a restricted one, preserving finite-dimensionality, nilpotency and the properties of the associated representations.

Embedding for nilpotent algebras

As a consequence, a nilpotent algebra $L = I \oplus Fx$, with a maximal p-ideal I having a faithful absolutely irreducible representation (with character S) on W and such that $x^{[p]} \in I$, can be embedded in $\mathbf{w}(I)$:

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$$\begin{array}{ccc} L & \hookrightarrow & \mathbf{w}(I) = (I \otimes F(\epsilon)) \rtimes < d > \\ x & \mapsto & d + (x^{[p]} + S(x)^p Id_W) \otimes \epsilon^{(p-1)} \\ I \ni y & \mapsto & \sum_{i=0}^{p-1} [y,_i \ x] \otimes \epsilon^{(i)} \end{array}$$

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$$\begin{array}{rcl} W \otimes F(\epsilon) & \cong & Ind_I^L(W,S) = \bigoplus_{i=0}^{p-1} W \otimes x^i \\ w \otimes \epsilon^{(i)} & \mapsto & w \otimes x^{p-1-i} \end{array}$$

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$$L \hookrightarrow \omega^{k+1}(F),$$

$$V = M \otimes F(\epsilon_1) \otimes \cdots \otimes F(\epsilon_k),$$

and

$$\dim V = p^k.$$

A possible application

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- $\exists\,!$ finite-dimensional Lie algebra $\mathcal{L}$ such that $L \hookrightarrow \text{Loop}(\mathcal{L}) = \mathcal{L} \otimes tF[t]$, with $\mathcal{L}$ minimal with respect to this embedding (the MNUA);
- $\exists\,!$ maximal abelian ideal A (the nilradical) of L such that

$$L/A \ltimes A \to \mathcal{L}/\mathcal{I} \ltimes \mathcal{I},$$

  where $\mathcal{I}$ is the unique minimal ideal of the MNUA, $\mathcal{L}/\mathcal{I}$ is nilpotent and $\mathcal{I}$ is an irreducible $\mathcal{L}/\mathcal{I}$-module.

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If $\mathcal{I}$ is an absolutely irreducible module, then

$$\mathcal{L}/\mathcal{I} \hookrightarrow \omega^n(F).$$

A study of the absolutely irreducible subalgebras of $\omega^n(F)$ gives information on the structure of L/A.