“The Topological Weight in Profinite Spaces and Applications to Groups”

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TO DAN SEGAL ON HIS 60th YEAR

Levico Terme
September 18, 2007

1. THE WEIGHT OF A TOPOLOGICAL SPACE

The

$$\textit{weight}\ w(X) \text{ of } X$$

is the smallest cardinal of a base of open sets of X.

If X is an infinite profinite space, then $w(X)$ is the cardinal of any base of clopen sets of X.

2. WEIGHT OF G vs WEIGHT OF X

Proposition Let G be a profinite group and let X be a profinite G-space such that

$$G_{x_1} \cap G_{x_2} = 1 \text{ whenever } x_1, x_2 \in X,\ x_1 \neq x_2.$$

Assume that the subspace

$$X' = \{x \in X \mid G_x \neq 1\}$$

of X is dense in X. Then there exists a profinite G-space $\tilde{X}$ and an epimorphism

$$\mu : X \longrightarrow \tilde{X}$$

of G-spaces such that

(a) $w(\tilde{X}) \leq \max\{w(G), \aleph_0\}$;

(b) $\mu_{|X'} : X' \longrightarrow \tilde{X}$ is $1-1$;

(c) $G_x = G_{\mu(x)}$, for every $x \in X$;

(d) for every $x \in X$, the restriction of μ to the orbit Gx of x in X is a homeomorphism onto the orbit $G\mu(x)$ of $\mu(x)$ in $\tilde{X}$.

3. Pro-p Groups Acting on Profinite Trees

Here $\mathcal{C}$ denotes a variety of finite groups (i.e., a nonempty class of finite groups closed under subgroups, homomorphic images and finite direct products) which is closed under extensions with abelian kernel, i.e., if

$$1 \to K \to G \to H \to 1$$

is an exact sequence of groups with K abelian and $H, K \in \mathcal{C}$, then $G \in \mathcal{C}$.

For example $\mathcal{C}$ is the class of all finite groups.

Theorem Let p be a prime number such that the cyclic group C_p of order p is contained in $\mathcal{C}$, and let G be a second countable pro-p group that acts on a pro-$\mathcal{C}$ tree Υ so that $G_e = 1$, for every edge $e \in E = E(\Upsilon)$. Put $V = V(\Upsilon)$ and $Q_V = G\backslash V$. Then there exists a section (not necessarily continuous)

$$\sigma_V : Q_V = G\backslash V \longrightarrow V$$

of the canonical projection map $\eta_V : V \longrightarrow Q_V$ such that

(a) The collection

$$\{G_{\sigma_V(\underline{v})} \mid \underline{v} \in Q_V\}$$

of closed subgroups of G is continuously indexed by the profinite space Q_V, and

(b) G is a free pro-p product

$$G = \Big(\coprod_{\underline{v} \in Q_V} G_{\sigma_V(\underline{v})} \Big) \amalg F,$$

where F is a free pro-p group.

4. Structure of Subgroups of Free Pro-$\mathcal{C}$ Products

As a consequence of the above theorem we have a somewhat strengthened form of a Kurosh-type theorem due to D. Haran describing the structure of certain closed subgroups of free pro-p products. In a slightly weaker form this was also proved by O. Mel'nikov and in a still weaker form by W. Herfort and L. Ribes.

Again $\mathcal{C}$ is a variety of finite groups which in addition is closed under extensions with abelian kernel.

Theorem Assume that p is a prime number with $C_p \in \mathcal{C}$. Let

$$H = \coprod_{t \in T} H_t$$

be the free pro-$\mathcal{C}$ product of a collection $\{H_t \mid t \in T\}$ of closed subgroups continuously indexed by a profinite space T.

Let G be a second countable pro-p subgroup of H. Then, for each $t \in T$, there is a complete set $\{h_{i,t}\}_{i \in I_t}$ of representatives of the double cosets $G \backslash H / H_t$ such that

$$\{G \cap h_{i,t} H_t h_{i,t}^{-1}\}_{i,t}$$

is a continuously indexed family of subgroups of G, and G is a free pro-p product

$$G = \Big[\coprod_{i,t} (G \cap h_{i,t} H_t h_{i,t}^{-1})\Big] \amalg F,$$

where the free product is taken over the quotient space

$$\bigcup_T G \backslash H / H_t = \{G h_{i,t} H_t \mid t \in T, i \in I_t\}$$

of $H \times T$ consisting of double cosets, and where F is a free pro-p subgroup of G.

For more details see

L. Ribes,
Pro-p Groups that Act on Profinite Trees,
to appear in J. Group Theory.