Discrete images and subgroups of profinite groups

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Question (Serre) Suppose that G is a finitely generated profinite group. Is it true that every finite index subgroup of G is open?

If YES we say that G is *strongly complete*.

Positive answers were obtained by

- Serre 1960s for pro-p groups
- Anderson (1976) for abelian by pronilpotent groups
- B. Hartley (1979) for poly-pronilpotent groups
- Saxl + Wilson and Martinez + Zelmanov for Cartesian products of simple groups.
- Dan Segal (2000) for prosoluble groups.

Our Theorems with Dan:

Theorem A: If G is a finitely generated profinite group then G' and each term of the lower central series of G are closed.

Theorem B: Every finitely generated group G is strongly complete! In fact if w is a locally finite word then the verbal subgroup $w(G)$ is open in G.

The verbal subgroup $w(G)$ is the group generated (abstractly) by all values of w and w^{-1} in G.

The Finite Theorems:

Theorem A': Let d be an integer. There exists an integer $h = h(d)$ such that if Γ is any d generated finite group any element of $\gamma \in \Gamma'$ is a product of h commutators.

Theorem B': Let d be an integer and let w be a d-locally finite word. There exists an integer $f = f(d, w)$ such that if Γ is any d-generated finite group then each $\gamma \in w(\Gamma)$ is a product of f values of w or w^{-1}.

Question For which words w it is true that the verbal subgroup $w(G)$ is closed in every finitely generated profinite group?

We don't know the answer even for the power word $w = x^q$ when $q \geq 5$.

However there is a complete answer for pro-p groups:

Theorem (Andrei Jaikin-Zapirain) Let F be a free group. The words $w \in F$ such that $w(G)$ is closed in every finitely generated pro-p group G are precisely those words which are outside $F''(F')^p$.

Dan Segal Suppose that G is a finitely generated prosoluble group such that

$G = \overline{\langle g_1, \ldots, g_d \rangle} G'$ for some $g_i \in G$.

There is a function $t = t(d)$ depending only on d such that every element of G' is a product of at most h commutators of the form $[x, g_i]$ for various $x \in G$.

Is there a theorem like the above for all finitely generated G? (Subject to $G = \overline{\langle g_1, \ldots, g_d \rangle}$)?

Corollary A finitely generated prosoluble group cannot have any nontrivial perfect images.

Observation Our Theorem B shows that a finitely generated profinite group cannot have an infinite residually finite image.

Vague question (Blaubeuren'07 in discussions with Bader, Caprace, Barnea, Gelander, Breuillard, Wilson, etc.): Can a finitely generated group have any 'strange' images?

More precisely:

Question Can a finitely generated profinite group G have an infinite finitely generated abstract image?

From the observation above it follows that it is enough to answer if G can have an infinite finitely generated simple image.

We know there are no such images if

- G is prosoluble, or if

- G is a Cartesian product of finite simple groups (with Y. Barnea and J. Wilson)

WARNING: G may have infinite countable images! For example C_{p^∞} is an image of the q-adic integers Z_q if $q \neq p$.

Stupid (?) **question**:

Is there some (not finitely generated) pro-p group which has a nontrivial perfect image?

MORE SUBGROUPS

Recall that a limit group is a finitely generated fully residually free group.

Theorem (Breuillard, Gelander, Souto and Storm (or a subset?)). If Γ is a nonabelian limit group then Γ embeds as a dense subgroup in every locally compact group which contains a dense nonabelian free group.

Question: Can we describe the finitely generated subgroups of a free pro-p group?

Not all of them are limit groups.

Here is a natural question motivated by the last theorem:

Question Suppose Γ is a finitely presented group which embeds into every profinite group which contains a nonabelian free group. Must Γ then be a limit group?