Pro-$p$ Groups with Few Normal Subgroups

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1. On $p$-Groups of Finite Coclass

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A $p$-group $G$ has **coclass** $r$ if $G$ is of class $c$ and $|G| = p^{c+r}$.
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In 1980 Charles Leedham-Green and Mike Newman came with the five coclass conjectures in decreasing order of difficulty:
Conjecture A. For some function $f(p, r)$, every finite $p$-group of coclass $r$ has a normal subgroup $K$ of class at most 2 and index at most $f(p, r)$. 
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The key point (for us) is that pro-$p$ groups of finite coclass are $p$-adic analytic.
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A point worth noticing: finite rank implies PSG is easy. The other direction is harder.
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4. Pro-$p$ groups of finite coclass are not closed under direct sum.
4. Pro-\(p\) Groups of Finite Width

Let \(G\) be a pro-\(p\) group. We say that \(G\) has width \(w\) if for all \(n\)

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Examples:

1. Let $\mathbb{Z}_p$ be the $p$-adic integers.

$$G_n = SL_d^n(\mathbb{Z}_p) = \ker(SL_d(\mathbb{Z}_p) \rightarrow SL_d(\mathbb{Z}_p/(p^n))).$$

$G = G_1$ is a pro-$p$ group, $G_n = \gamma_n(G)$ and

$$|G_n/G_{n+1}| = p^{d^2-1}.$$
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3. The Nottingham group

$$J = \{t + a_1t^2 + a_2t^3 + \cdots | a_i \in \mathbb{F}_p\},$$

where the product is by composition.

$$|\gamma_n(J)/\gamma_{n+1}(J)| = \begin{cases} p & n \not\equiv 1 \mod p - 1 \\ p^2 & n \equiv 1 \mod p - 1. \end{cases}$$
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Goal: Find a good definition to avoid all the more difficult examples.
5. Few Normal Subgroups

A Pro-$p$ group $G$ has **Polynomial Normal Subgroup Growth (PNSG)** if there exists $c$ such that $a_n^<(G) \leq n^c$ for all $n$, where $a_n^<(G)$ is the number of normal subgroups of index $n$. 
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A Pro-$p$ group $G$ has **Polynomial Normal Subgroup Growth (PNSG)** if there exists $c$ such that $a_n^{\triangleleft}(G) \leq n^c$ for all $n$, where $a_n^{\triangleleft}(G')$ is the number of normal subgroups of index $n$.

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**Lemma:** A pro-$p$ group with CNSG has finite normal rank.

**Problem 1:** A pro-$p$ group with finite normal rank has PNSG. What about the other direction? There is a soluble counter example, what about just infinite?
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A group \( G \) is called \textit{r-sandwich} if there is \( r \) such that for all normal subgroup \( N \) of \( G \) there exists \( i \) such that \( \gamma_i(G) \geq N \geq \gamma_{i+r}(G) \).
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A group $G$ is called $r$-sandwich if there is $r$ such that for all normal subgroup $N$ of $G$ there exists $i$ such that $\gamma_i(G) \geq N \geq \gamma_{i+r}(G)$.

**Theorem 1:** Let $G$ be a non-nilpotent pro-$p$ group. Then $G$ has finite obliquity if and only if it is sandwich. Moreover, in such a case, $G$ is just infinite of finite width and has CNSG.
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**Theorem 2:** Let $G$ be a non-nilpotent pro-$p$ group with CNSG. Then $G$ has a maximal finite normal subgroup $K$ and $G/K$ is just infinite. Moreover, $G$ has finite width.
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**Theorem 2:** Let $G$ be a non-nilpotent pro-$p$ group with CNSG. Then $G$ has a maximal finite normal subgroup $K$ and $G/K$ is just infinite. Moreover, $G$ has finite width.

**Problem 2:** Suppose $G$ is hereditarily just infinite pro-$p$ group with CNSG. Is it sandwich?
6. Periodicity

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A period on a pro-$p$ group $G$ is a map $\tau : M \to G$, where $M$ is an open normal subgroup of $G$ such that

1. $\tau(M)$ is an open subgroup of $G$;

2. for every open normal subgroup $H$ of $G$ contained in $\tau(M)$ we have that $\tau^{-1}(H)$ is an open normal subgroup of $G$ and

$$[G : H] > [G : \tau^{-1}(H)].$$
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2. for every open normal subgroup $H$ of $G$ contained in $\tau(M)$ we have that $\tau^{-1}(H)$ is an open normal subgroup of $G$ and

$$[G : H] > [G : \tau^{-1}(H)].$$

We say that a period is uniform if there is a constant $c$ such that for all $H$ as above,

$$[G : H] = p^c[G : \tau^{-1}(H)].$$
**Proposition:** If $G$ admits a period it admits a uniform period.
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Theorem 3: Suppose $G$ is a non-abelian just infinite pro-$p$ group which admits a period. Then $G$ is sandwich, in particular it has CNSG. Moreover, there is $d$ such for all big enough $n$, $a_{p^n}(G) = a_{p^{n+d}}(G)$. 
Theorem 4: Suppose $G$ is one of the known examples of hereditarily just infinite pro-$p$ groups with CNSG. Then $G$ has a period.
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Moreover, every subgroup of finite index of $G$ has all of the above properties too.
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Moreover, every subgroup of finite index of $G$ has all of the above properties too.

In addition, Branch groups and all the other known examples of hereditarily just infinite pro-$p$ groups are all not CNSG.
7. Conjectures?

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Do I dare to conjecture that if $G$ is hereditarily just infinite pro-$p$ group, then:

1. If $G$ has CNSG, then $G$ has a period and, in particular, there exists $d$ such that for all big enough $n$ we have $a_{p^n}(G) = a_{p^{n+d}}(G')$.

2. If $G$ has finite obliquity or CNSG or a period, then every subgroup of finite index of $G$ has finite obliquity or CNSG or a period respectively.
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2. If $G$ has finite obliquity or CNSG or a period, then every subgroup of finite index of $G$ has finite obliquity or CNSG or a period respectively.

3. If $G$ has few normal subgroups, then there exists a constant $c$ such that for all $n$, $a_n(G) \leq n^{c \log n}$. 
8. The Nottingham Group

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Let $H$ be an open subgroup of $J$. Then $H$ contains some

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It is easy to see that there exists $m$ such that for all $N$ normal subgroups of $H$ of big enough index there exists $n$ such that $J_{n+p^m} \leq N \leq J_n$. 
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We define the period on $J_k$ by

$$\tau_m(t(1 + f(t))) = t(1 + t^{p^m} f(t)).$$
Lemma: Let $\phi = a(t) \in J$ and $\psi = t + s(t) \in J_k$. Then

$$\phi \psi \phi^{-1} \equiv t + \frac{s(a(t))}{a'(t)} \mod t^{2k+2}.$$
Lemma: Let $\phi = a(t) \in J$ and $\psi = t + s(t) \in J_k$. Then

$$\phi\psi\phi^{-1} \equiv t + \frac{s(a(t))}{a'(t)} \mod t^{2k+2}.$$ 

Corollary: For $k \geq p^m$, the map $\tau_m$ induces a $J$-isomorphism from $J_k/J_k+p^m$ onto $J_{k+p^m}/J_{k+2p^m}$. 
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Corollary: For \( k \geq p^m \), the map \( \tau_m \) induces a \( J \)-isomorphism from \( J_k/J_k+p^m \) onto \( J_k+p^m/J_k+2p^m \).

The fact that \( \tau_m \) is a period follows from the sandwich property on the previous slide.