Notes of Algebraic Geometry

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a.y. 2013/2014

Contents

Chapter 1

Varieties, the Picard Group and the Riemann-Roch Theorem

1.1 Varieties

Throughout this course we will deal with complex geometry, hence the field we will use will be $\mathbb{K} = \mathbb{C}$; most of the results we will see hold for a general algebraically closed field.

1.1.1 Affine varieties

By $\mathbb{A}^n = \mathbb{K}^n = \mathbb{C}^n$ we will denote the *affine space*.
A polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ can be easily seen as a function $f : \mathbb{A}^n \to \mathbb{K}$.

Definition 1.1. *For $S \subset \mathbb{K}[x_1, \dots, x_n]$ we define the set*

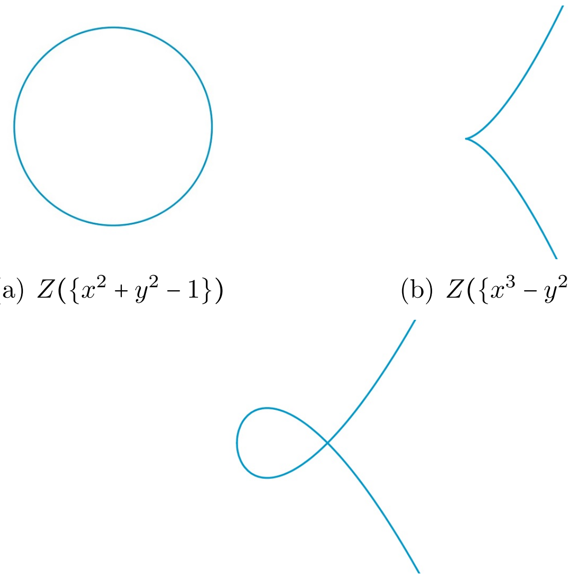
$$Z(S) := \{p \in \mathbb{A}^n : \forall f \in S \ f(p) = 0\},$$

such a set is called affine algebraic set *(a.a.s).*
$Z := Z(S)$ is said to be irreducible *if $\nexists$ Y_1, Y_2 proper affine algebraic subsets of Z such that $Z = Y_1 \cup Y_2$.*

By the *Hilbert basis Theorem* it can be proved that to define an affine algebraic set it suffices a finite number of polynomials, thus we can always suppose that $\#S < \infty$.

Definition 1.2. *An* affine variety *is an irreducible affine algebraic set.*

Remark 1.3. *By this definition, an affine variety can either be smooth (i.e. an embedded complex submanifold of the affine space) or admit some* singular *points, such cusps or nodes; for example all of the three affine algebraic sets represented in Figure 1.1 are affine varieties.*



(a) $Z(\{x^2 + y^2 - 1\})$ (b) $Z(\{x^3 - y^2\})$

(c) $Z(\{x^2 + x^3 - y^2\})$

Figure 1.1: Some examples of affine varieties.

Definition 1.4. *Let $Z \subset \mathbb{A}^n$; we define the set $I(Z)$ to be the set of all polynomials vanishing on Z, that is*

$$I(Z) := \{f \in \mathbb{K}[x_1, \dots, x_n] : \forall p \in Z \ f(p) = 0\}.$$

It is easy to prove that $I(Z)$ is an ideal. The quotient ring

$$\mathbb{K}[x_1, \dots, x_n]/I(Z)$$

is called structure ring *of Z.*

Basically, the structure ring of Z is the set of the restrictions of polynomials to the variety, since two polynomials that differs by an element of $I(Z)$ assume the same values on the variety.

An affine variety $Z \subset \mathbb{A}^n$ can be endowed with two different topologies:

- the *natural* topology induced over Z by $\mathbb{A}^n$;
- the topology whose closed sets are the algebraic subset of Z.

The latter topology - which is the one we will work the most with - is the *Zariski topology* and it is immediate to see that it is coarser than the one induced by $\mathbb{A}^n$.

1.1.2 Projective varieties

In this case the space we are working in is the projective space

$$\mathbb{P}^n_{\mathbb{K}} := \left(\mathbb{K}^{n+1} \smallsetminus \{0\}\right)/\mathbb{K}^*;$$

let $f \in \mathbb{K}[x_0, \dots, x_n]$ be an homogeneous polynomial of degree d, that is

$$f = \sum a_{k_1 \dots k_n} x_0^{d-\Sigma k_i} x_1^{k_1} \dots x_n^{k_n}.$$

Watch out! f is not a function defined over $\mathbb{P}^n$, since

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^n f(x_0, \dots, x_n). \tag{1.1}$$

Asking the value of f in a generic point makes no sense, nevertheless it is possible to see whether a polynomial vanishes in a certain point.

Definition 1.5. *Let $S \subset \mathbb{K}[x_0, \dots, x_n]$ such that $\forall f \in S$ f is homogeneous. By (1.1), the set*

$$Z(S) := \{p \in \mathbb{P}^n : \forall f \in S \ f(p) = 0\}$$

is well defined. The set $Z(S)$ is said to be a projective algebraic set *(p.a.s).*

$Z := Z(S)$ is said to be irreducible *if $\nexists$ Y_1, Y_2 proper projective algebraic subsets of Z such that $Z = Y_1 \cup Y_2$.*

In this case too, by the Hilbert basis Theorem, a finite number of polynomials suffices to define a projective algebraic set.

Definition 1.6. *A* projective variety *is an irreducible projective algebraic set.*

Definition 1.7. *Let $Z \subset \mathbb{P}^n$; we define the set $I(Z)$ to be*

$$I(Z) := \textit{ideal generated by } \{f \in \mathbb{K}(x_0, \dots, x_n) \textit{ homogeneous } : \forall p \in Z \ f(p) = 0\}.$$

The structure ring *of Z is the quotient ring*

$$\mathbb{K}[x_0, \dots, x_n]/I(Z).$$

Analogously to the affine case, a projective variety can be endowed with the standard topology induced by $\mathbb{P}^n$ or with the Zariski topology.

1.1.3 Quasi-projective varieties

Definition 1.8. *Z is said to be a* quasi-projective variety *if Z is a Zariski open subset of a projective variety.*

Example 1.9. *Let us see some example of quasi-projective varieties.*

1. *Every projective variety is trivially a quasi-projective variety.*

2. *$\mathbb{A}^n$ is a quasi-projective variety; indeed*

$$\begin{array}{ccc} \mathbb{P}^n \smallsetminus \{x_0 = 0\} & \cong & \mathbb{A}^n \\ (x_0 : \dots : x_n) & \mapsto & \left(\dfrac{x_1}{x_0}, \dots, \dfrac{x_n}{x_0}\right). \end{array}$$

3. *Every affine variety is quasi-projective. Let us prove this statement.*

   Let $V = Z(\{f_1, \dots, f_l\})$ be an affine variety, where $f_i \in \mathbb{K}[z_1, \dots, z_n]$ with $\deg f_i = d_i$. For every $j = 1, \dots, l$, let us define f_j^h, the homogenization *of f_j, in the following way:*

$$\begin{aligned} \textit{if } f_j &= \sum a_{k_1 \dots k_n} z_1^{k_1} \dots z_n^{k_n} \\ \textit{then } f_j^h &:= \sum a_{k_1 \dots k_n} z_0^{d_j - \Sigma k_i} z_1^{k_1} \dots z_n^{k_n} \end{aligned}$$

   The set $Z := Z(\{f_1^h, \dots, f_l^h\}) \subset \mathbb{P}^n$ is a projective variety and it is called projective closure *of V.*

   Since $Z \subset \mathbb{P}^n$, we can work just like in the second example; we remove from Z the set $\{x_0 = 0\}$ to get a subset of $\mathbb{P}^n \smallsetminus \{x_0 = 0\} \cong \mathbb{A}^n$.

Exercise 1.10. *Prove that $Z \smallsetminus \{x_0 = 0\} \cong V$.*

In these latter examples we have seen how it is possible to move from the affine case to the projective case by homogenization. We can move in the other way too, by *dehomogenization.* Let $f \in \mathbb{K}[x_0, \dots, x_n]$ be a homogeneous polynomial; a dehomogenization of f is

$$f_0 \in \mathbb{K}[z_1, \dots, z_n] \qquad f_0(z_1, \dots, z_n) := f(1, z_1, \dots, z_n).$$

We have $n + 1$ different dehomogenizations of f.

Since $\mathbb{P}^n_{\mathbb{C}}$ is a complex variety of dimension n, a quasi projective variety $Z \subset \mathbb{P}^n_{\mathbb{C}}$ can be a submanifold.

Definition 1.11. *$Z \subset \mathbb{P}^n_{\mathbb{C}}$ quasi projective variety is said to be* smooth *if it is an embedded submanifold.*

From now on by *variety* we will always mean a quasi-projective variety.

1.1.4 Maps

Definition 1.12. *Let X, Y be two varieties; $f : X \to Y$ is said to be* regular *(or a* morphism of varieties*) if*

$$\forall x \in X \quad \exists \quad U(x) \text{ Zariski open, } U \subset \mathbb{A}^n$$
$$V(f(x)) \text{ Zariski open, } V \subset \mathbb{A}^m$$

such that $f(U) \subset V$ and $f|_U = (g_1, \dots, g_m)$ with

$$g_i = \frac{n_i}{d_i}, \qquad n_i, d_i \in \mathbb{K}[x_1, \dots, x_n]$$

and $d_i(p) \neq 0$ $\forall p \in U$ $\forall i = 1, \dots, m$.

Basically this means that locally a morphism of algebraic varieties can be expressed by rational functions.

Definition 1.13. *Let X and Y be two quasi-projective varieties; $f : X \to Y$ is said to be* biregular *if it is regular, invertible and with regular inverse.*

In algebraic geometry biregular mappings play the role of the diffeomorphisms in differential geometry and homeomorphisms in topology; that is, if there exists a biregular map between two algebraic varieties, it means that *they look like the same.*

Definition 1.14. *Let X and Y be two quasi-projective varieties, a* rational map *$f : X \dashrightarrow Y$ is the equivalence class of pair (U, f_U) where U is Zariski open, $f_U : U \to Y$ is regular, modulo the equivalence relation*

$$(U, f_U) \sim (U', f_{U'}) \Leftrightarrow f_U|_{U \cap U'} = f_{U'}|_{U \cap U'}.$$

Example 1.15. *Let us see an example of a biregular mapping.*
Let us define $Q := \{x_0 x_2 = x_1^2\}$[1]$\subset \mathbb{P}^2$ and the map

$$F : \begin{array}{ccc} \mathbb{P}^1 & \to & Q \\ (t_0 : t_1) & \mapsto & (t_0^2 : t_0 t_1 : t_1^2) \end{array}.$$

It is easy to see that this map is well defined (the things to check are that $F(\mathbb{P}^1) \subset Q$, that $F(\lambda t_0, \lambda t_1) = F(t_0, t_1)$ and that no points are mapped into $(0 : 0 : 0)$ that is not a point of $\mathbb{P}^2$).

[1] With a slight abuse of notation, by this we mean that $Q = Z(\{x_0 x_2 - x_1^2\})$.

Let us prove that this is a regular map. Let $x = (t_0 : t_1) \in \mathbb{P}^1$ and suppose that $t_0 \neq 0$, hence $x = (1 : t_1)$; in order to satisfy the conditions of Definition 1.12, we can define

$$\begin{aligned} U(x) = U_0 &:= \{(t_0 : t_1) \in \mathbb{P}^1 : t_0 \neq 0\} \\ V(f(x)) = V_0 &:= \{(z_0 : z_1 : z_2) \in Q : z_0 \neq 0\}. \end{aligned}$$

U and V are Zariski open subsets of the respective projective subspaces. Restricting F to U we get

$$F|_U : (1 : t) \mapsto (1 : t : t^2).$$

Working in an analogous way for point with $t_0 = 0$ and $t_1 \neq 0$, we get that F is locally defined by rational functions, hence it is a regular map.

Is this map invertible? Let us then consider the map

$$\begin{array}{cccc} G : & Q & \to & \mathbb{P}^1 \\ & (x_0 : x_1 : x_2) & \mapsto & \left(\dfrac{x_1}{x_2} : 1\right) \end{array}$$

Apparently, this function is defined only for $x_2 \neq 0$; but

$$\left(\frac{x_1}{x_2} : 1\right) = \left(1 : \frac{x_2}{x_1}\right),$$

thus, G can be equivalently defined for $x_1 \neq 0$ as

$$G(x_0 : x_1 : x_2) = \left(1 : \frac{x_2}{x_1}\right).$$

We are still missing the point $p = (0 : 0 : 1) \in Q$; is it possible to define G in an open neighbourhood of p? Exploiting the fact that in Q $x_0 x_2 = x_1^2$ we get

$$\left(\frac{x_1}{x_2} : 1\right) = \left(\frac{x_1^2}{x_2} : x_1\right) = \left(\frac{x_0 x_2}{x_2} : x_1\right) = (x_0 : x_1) = \left(1 : \frac{x_1}{x_0}\right).$$

We have then defined the function G over the whole Q.

Exercise 1.16. *Prove that G is rational and that $G = F^{-1}$. Thus F is a biregular mapping.*

Remark 1.17. *The projection map*

$$\begin{array}{cccc} \pi : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \\ & (x_0 : x_1 : x_2) & \mapsto & (x_0 : x_1) \end{array}$$

is a rational map, in the sense that its restriction to the open set $\mathbb{P}^2 \setminus \{(0:0:1)\}$ *is a regular map. Moreover it's immediate to see that* $\pi = G$ *in the common domain; but* π *cannot be defined in* $p = (0:0:1)$, *whereas* G *is. This is due to the fact that restricting the function to* Q *the limit*

$$\lim_{\substack{z\to p \\ z\in Q}} \pi(z) = (0:1)$$

exists, while

$$\lim_{z\to p} \pi(z)$$

does not exist.

Exercise 1.18. *Let* $l := \{ax_0 + bx_1 = 0\} \subset \mathbb{P}^2$ *with* $a, b \in \mathbb{C}$; *compute using the standard topology* $\lim_{z\to p} \pi|_l$ *and prove that it depends on* a, b.

Definition 1.19. *A rational map* $F : X \dashrightarrow Y$ *is said to be* birational *if there exists a rational map* $G : Y \dashrightarrow X$ *such that* $F \circ G = \mathrm{Id}_Y$ *and* $G \circ F = \mathrm{Id}_X$.

If X *and* Y *are two varieties such that there exists a birational map* $F : X \dashrightarrow Y$, *then they are said to be* birational.

Definition 1.20. *Let* X *be a variety;* X *is* rational *if it is birational to a projective space.*

Definition 1.21. *Let* X *be a variety;* X *is* unirational *if it is* dominated *by a projective space, i.e. there exists a rational map from a projective space to* X *with dense image.*

1.2 Cartier divisors

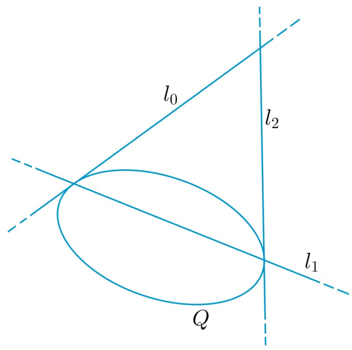
In this section we introduce the Cartier divisors; the formal definition gives rise to objects that seem quite different from the divisors that are concretely used. In order to become familiar with divisors, let us first see a concrete example and then, just after the formal definition, we will see how the first corresponds to the latter.

Example 1.22. *Let us consider the function*

$$\begin{array}{cccc} F: & \mathbb{P}^2 & \dashrightarrow & \mathbb{C} \\ & (x_0 : x_1 : x_2) & \mapsto & \dfrac{x_0 x_1^2}{x_2(x_0x_2 - x_1^2)} \end{array}$$

The function F *is well defined wherever it is defined (since it is the ratio of two homogeneous polynomials of the same degree). This function naturally defines four curves:*

$$l_0 := \{x_0 = 0\}$$
$$l_1 := \{x_1 = 0\}$$
$$l_2 := \{x_2 = 0\}$$
$$Q := \{x_0 x_2 - x_1^2\}.$$



We say that the divisor of F *is*

$$(F) = l_0 + 2l_1 - l_2 - Q.$$

Formally, we are describing the zeroes and the poles of F by considering the zero locus of F and $1/F$ with suitable coefficients:

- *l_0 has coefficient 1, since F vanishes with* multiplicity 1 *along it;*
- *l_1 has coefficient 2, since F vanishes with* multiplicity 2 *along it;*
- *both l_2 and Q have coefficient -1, since F has simple poles along both of them.*

Definition 1.23. *Let X be a quasi-projective variety; a* (Cartier) divisor *over X is a collection $\{(U_i, f_i)\}$ where:*

- *$\{U_i\}$ is an (affine) open cover of X;*
- *$f_i : U_i \dashrightarrow \mathbb{C}$ are such that $\forall i, j$*

$$f_i/f_j : U_i \cap U_j \to \mathbb{C}$$

is regular.

The second condition implies that $f_i/f_j(p) \neq 0 \ \forall p \in U_i \cap U_j$, otherwise f_j/f_i would not be regular.

We say that two divisors $\mathcal{A} = \{(U_i, f_i)\}$ and $\mathcal{B} = \{(V_i, g_i)\}$ are equivalent if and only if $\mathcal{A} \cup \mathcal{B}$ is still a divisor.

Let us see now by an example how these objects can be represented by a formal sum of subvarieties with suitable integer coefficients.

Example 1.24. *Let $U_i := \{x_i \neq 0\} \subset \mathbb{P}^2$ and let us define the following functions:*

$$f_0 : U_0 \to \mathbb{C} \qquad f_0\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) = \frac{x_2}{x_0} - \left(\frac{x_1}{x_0}\right)^2;$$

$$f_1 : U_1 \to \mathbb{C} \qquad f_1\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) = \frac{x_0}{x_1}\frac{x_2}{x_1} - 1;$$

$$f_2 : U_2 \to \mathbb{C} \qquad f_2\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) = \frac{x_0}{x_2} - \left(\frac{x_1}{x_2}\right)^2.$$

It's immediate to see that

$$\frac{f_i}{f_j} = \frac{x_j^2}{x_i^2} : U_i \cap U_j \to \mathbb{C}$$

are regular functions, so $\{(U_i, f_i)\}$ is a divisor. The name we give to this divisor is $Q = \{x_0x_2 - x_1^2 = 0\}$; indeed, it is immediate to see that, if $q = x_0x_2 - x_1^2$, we have

$$f_0 = \frac{q}{x_0^2}, \quad f_1 = \frac{q}{x_1^2}, \quad f_2 = \frac{q}{x_2^2},$$

thus the curve $\{x_0x_2 - x_1^2 = 0\}$ coincides locally with the zero locus of the functions f_i.

We will never give a Cartier divisor via a collection $\{(U_i, f_i)\}$, but always describing its zero locus and the set of its poles with a formal sum of subvarieties with integer coefficients: indeed, we will write

$$D = \sum_{i \in I} a_i D_i,$$

where $a_i \in \mathbb{Z}$, every D_i is an irreducible variety that can locally be expressed as zero locus of a unique regular function[2] and

- D has zeroes of order a_i along D_i if $a_i > 0$;
- D has poles of order $-a_i$ along D_i if $a_i < 0$.

The divisor D is said to be *effective* if $a_i \geq 0 \ \forall i$.

[2]Be aware that not all subvarieties are Cartier divisor, but only the subvarieties which can be locally defined as zero locus of a single rational function.
The simplest example is the origin in $\mathbb{A}^2$: this is not a Cartier divisor, since we cannot describe it using a single function, since its ideal is not principal but needs at least two generators. The following exercise give a different and more interesting example.

Exercise 1.25. *Consider the following affine varieties in $\mathbb{C}^3$:*

$$C := Z(\{y^2 - xz\}), \quad L := Z(\{x, y\});$$

obviously $L \subset C$. Show that L is not a Cartier divisor in C, but $L \smallsetminus \{O\}$ is a Cartier divisor in $C \smallsetminus \{O\}$, where O is the origin of $\mathbb{A}^3$.

The set of Cartier divisors over a quasi-projective variety X is an abelian group and will be denoted by $\mathrm{Div}(X)$.

Example 1.26. *When X is a smooth variety of dimension 1*

$$\mathrm{Div}(X) = \left\{\sum a_i p_i : p_i \in X\right\}.$$

In this particular case if $D = \sum a_i p_i \in \mathrm{Div}(X)$ we define the degree *of D to be the integer*

$$\deg(D) = \sum a_i.$$

Definition 1.27. *If $D \in \mathrm{Div}(X)$ is such that $D = (F)$ for some $F : X \dashrightarrow \mathbb{C}$, we say that D is a* principal divisor.

Remark 1.28. *The divisor defined in Example 1.22 is principal, whereas the divisor Q defined in Example 1.24 is not principal.*

Exercise 1.29. *Show that Q is not principal. [Hint: "restrict Q to a line", and use the well known fact that principal divisors on smooth varieties of dimension 1 have degree 0.]*

Definition 1.30. *Let X be a quasi-projective variety and let $D_1, D_2 \in \mathrm{Div}(X)$. We say that D_1 and D_2 are* equivalent *(denoted by $D_1 \equiv D_2$) if and only if $D_1 - D_2$ is a principal divisor.*

The Picard group *of X is*

$$\mathrm{Pic}(X) := \mathrm{Div}(X)/\equiv.$$

Let us see some example of Picard groups.

Example 1.31. *Let $X = \mathbb{P}^n$. Let $p_0, p_1 \in \mathbb{C}[x_0, \ldots, x_n]$ two homogeneous polynomials of the same degree, then*

$$\frac{p_0}{p_1} : \mathbb{P}^n \to \mathbb{C}$$

is a rational function. Let $D_i = \{p_i = 0\}$, then

$$D_0 - D_1 = \left(\frac{p_0}{p_1}\right).$$

Hence $D_0 \equiv D_1$. It can be proved that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$*, and the isomorphism is given by*

$$\begin{array}{ccc} \mathrm{Pic}(\mathbb{P}^n) & \to & \mathbb{Z} \\ [\sum a_i D_i] & \mapsto & \sum a_i \deg D_i \end{array},$$

where $\deg D_i$ *is the degree of the polynomial that defines the curve D_i.*

Example 1.32. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. First, let us prove that X is a projective variety. Let us define the function*

$$\begin{array}{cccc} \varphi: & \mathbb{P}^1 \times \mathbb{P}^1 & \hookrightarrow & \mathbb{P}^3 \\ & ((t_0 : t_1), (s_0 : s_1)) & \mapsto & (t_0 s_0 : t_0 s_1 : t_1 s_0 : t_1 s_1) \end{array}$$

It's easy to see that this function is well defined; moreover, denoting with $(x_0 : x_1 : x_2 : x_3)$ the homogeneous coordinates of $\mathbb{P}^3$, φ gives an isomorphism between $\mathbb{P}^1 \times \mathbb{P}^1$ and $Q = \{x_0 x_3 - x_1 x_2 = 0\} \subset \mathbb{P}^3$, hence $\mathbb{P}^1 \times \mathbb{P}^1$ is a projective variety.

It can be proved that $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \times \mathbb{Z}$*. If $a, b \geq 0$, the isomorphism maps $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ to the class of effective divisors given by bihomogeneous polynomials with bidegree (a, b) in $\mathbb{C}[t_0, t_1][s_0, s_1]$, that is polynomials of the form*

$$p = \sum \lambda_{\alpha\beta} t_0^{a-\alpha} t_1^{\alpha} s_0^{b-\beta} s_1^{\beta}.$$

Exercise 1.33. *Prove that $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$ is a projective variety.*

Example 1.34. *Let $T = \mathbb{C}^2/\Lambda$ be a complex torus, where Λ is a lattice, that is the subgroup (respect to the sum) generated by two complex numbers which are linearly independent over $\mathbb{R}$. Being the quotient of a group by a subgroup, T is a group; we know that T can be embedded in $\mathbb{P}^2$ as algebraic variety. It is possible to prove that* $\mathrm{Pic}(T) \cong T \times \mathbb{Z}$ *and the isomorphism is given by*

$$\begin{array}{ccc} \mathrm{Pic}(T) & \to & T \times \mathbb{Z} \\ [\sum a_i p_i] & \mapsto & (\sum p_i, \sum a_i) \end{array}.$$

1.3 Sheaves

Definition 1.35. *Let X be a topological space, a* presheaf of rings (groups, abelian groups, modules,...) *$\mathcal{F}$ on X is a collection of rings (respectively, of groups, abelian groups, modules, ...) $\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F})$, one for every open set U, and a collection of rings (respectively groups, abelian groups, modules, ...) morphisms $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ whenever $V \subseteq U$, called* restriction maps*, such that*

- $\mathcal{F}(\varnothing) = \{0\}$;

- $\rho^U_U = \mathrm{Id}_{\mathcal{F}(U)}\ \forall U$;
- $\forall \quad W \subset V \subset U \quad \rho^U_W = \rho^V_W \circ \rho^U_V$.

An element $s \in \mathcal{F}(U)$ is called section *of $\mathcal{F}$ over U. An element of $H^0(\mathcal{F}) := H^0(X, \mathcal{F})$ is called a* global section *of $\mathcal{F}$.*

Example 1.36. *Fix a ring G and define the presheaf G^X as*

$$G^X(U) := \{f : U \to G\} \quad \rho^U_V(f) = f|_V.$$

Example 1.37. *The following are presheaves:*

Continuous $\mathbb{C}$-valued functions on a topological space	$\mathcal{C}^0_X$
Smooth $\mathbb{R}$-valued functions on a real manifold	$\mathcal{C}^\infty_X$
Holomorphic $\mathbb{C}$-valued functions on a complex manifold	${}^h\mathcal{O}_X$
Regular $\mathbb{C}$-valued functions on a variety	$\mathcal{O}_X$
$\mathbb{Z}$-valued constant functions on a topological space	$\mathbb{Z}$
$\mathbb{Q}$-valued constant functions on a topological space	$\mathbb{Q}$
$\mathbb{R}$-valued constant functions on a topological space	$\mathbb{R}$
$\mathbb{C}$-valued constant functions on a topological space	$\mathbb{C}$

in every case, with the only exception of the fourth one, we use the standard topology. For $\mathcal{O}_X$ we use Zariski topology.

Definition 1.38. *A presheaf $\mathcal{F}$ over X is said to be a* sheaf *if it satisfies the so called* sheaf axiom*:*

$\forall U \subset X$ open set, $\forall \{U_i\}$ open covering of U, suppose that are given $\forall i$ $s_i \in \mathcal{F}(U_i)$ such that $\forall i, j$ $\rho^{U_i}_{U_i \cap U_j} s_i = \rho^{U_j}_{U_i \cap U_j} s_j$. Then there exists a unique $s \in \mathcal{F}(U)$ such that $\rho^U_{U_i} s = s_i\ \forall i$.

Basically we are asking that whenever two sections $s_1 \in \mathcal{F}(U)$ and $s_2 \in \mathcal{F}(V)$ agree on the common domain, there exists a unique section $s \in \mathcal{F}(U \cup V)$ whose restrictions to U and V are exactly s_1 and s_2.

Remark 1.39. *All presheaves in 1.37 but $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are sheaves, while constant presheaves are not (take into account - for example - two open disjoint sets). We will consider instead the locally constant functions; in this way we get the locally constant sheaves $\underline{\mathbb{Z}}$, $\underline{\mathbb{Q}}$, $\underline{\mathbb{R}}$, $\underline{\mathbb{C}}$.*

Example 1.40. *The following are sheaves of abelian groups on a smooth variety:*

$$\mathcal{O}^*_X = \{f : X \to \mathbb{C}^* \text{ regular}\}$$
$${}^h\mathcal{O}^*_X = \{f : X \to \mathbb{C}^* \text{ holomorphic}\},$$

where the group action is given by the multiplication on $\mathbb{C}^*$.

Definition 1.41. *Let X be a variety and $D \in \mathrm{Div}(X)$, $D = \sum a_i D_i$. For every Zariski open set U let us define the complex vector space*

$$\Gamma(U, \mathcal{O}_X(D)) := \{f : U \dashrightarrow \mathbb{C} : \ (f) + D|_U \geq 0\} \cup \{0\};$$

we will also denote it by $H^0(U, \mathcal{O}_X(D))$. The set $H^0(X, \mathcal{O}_X(D))$ will be simply denoted by $H^0(\mathcal{O}_X(D))$.

Using the same notation of the previous definition, if $f \in H^0(\mathcal{O}_X(D))$, then

- if $a_i < 0$, f vanishes along D_i with order at least $-a_i$;
- if $a_i > 0$, f admits poles over D_i of order up to a_i.

f is a rational function with a principal divisor (f). Anyway, when considering it as a section of $\mathcal{O}_X(D)$ we will usually denote it by the letter s (to recall that we are not considering it as a function) and attributes to it the effective divisor $(s) := (f) + D$.

Using this notation, it is possible to prove that

$$(s_1) = (s_2) \Leftrightarrow \exists \lambda \in \mathbb{C}^* : s_1 = \lambda s_2. \tag{1.2}$$

The dimension of $H^0(\mathcal{O}_X(D))$ will be denoted by

$$h^0(\mathcal{O}_X(D)) := \dim H^0(\mathcal{O}_X(D)).$$

Proposition 1.42. *If $D_1, D_2 \in \mathrm{Div}(X)$ are linearly equivalent, then $H^0(\mathcal{O}_X(D_1)) \cong H^0(\mathcal{O}_X(D_2))$ via an isomorphism that preserves the divisors. In other words, if $s_1 \in H^0(\mathcal{O}_X(D_1))$ is mapped by this isomorphism to $s_2 \in H^0(\mathcal{O}_X(D_2))$, then $(s_1) = (s_2)$.*

Proof. $D_1 \equiv D_2 \Leftrightarrow D_1 - D_2 = (F)$ with $F : X \dashrightarrow \mathbb{C}$. If $f \in H^0(U, \mathcal{O}_X(D_1))$, then

$$(fF|_U) = (f) + (F) = (f) + D_1 - D_2 \geq -D_2,$$

hence $fF|_U \in H^0(U, \mathcal{O}_X(D_2))$. The sheaves isomorphism is given by

$$\begin{array}{ccc} H^0(U, \mathcal{O}_X(D_1)) & \to & H^0(U, \mathcal{O}_X(D_2)) \\ f & \mapsto & fF|_U \\ g/F|_U & \leftarrow\!\shortmid & g \end{array}$$

Let us prove the second part of the statement: the divisor of f as a section of $H^0(\mathcal{O}_X(D_1))$ is $(f) + D_1$, while the divisor of its image is $(fF) + D_2 = (f) + (F) + D_2 = (f) + D_1 - D_2 + D_2 = (f) + D_1$. □

Example 1.43. *Let* $X = \mathbb{P}^2$*,* $L_0 = \{x_0 = 0\}$*,* $L_1 = \{x_1 = 0\}$*; by Example 1.31,* $L_0 \equiv L_1$*, hence, by Proposition 1.42,* $H^0(\mathcal{O}_X(L_1)) \cong H^0(\mathcal{O}_X(L_2))$*; let* $f \in H^0(\mathcal{O}(L_0))$*, then* $f = L/x_0$*, where* $L \in \mathbb{C}[x_0, x_1, x_2]_1$*.*

It is immediate to see that the isomorphism defined in Proposition 1.42 maps f *into* $g = L/x_1$*; the divisor we get is* $(f) + L_0$ *equal to* $(g) + L_1$ *equal to* $(s) = \{L = 0\}$*.*

By (1.2) the set

$$|D| := \{(s) : s \in H^0(\mathcal{O}_X(D)), s \neq 0\}$$

is a subset of $\mathrm{Div}(X)$ formed by effective divisors having a natural structure of projective space, indeed $|D| \cong \mathbb{P}^{h^0(\mathcal{O}_X(D))-1}$. The set $|D|$ is called *complete linear system* associated with D and gives exactly those effective divisors which are linearly equivalent to D. A *linear system* is a linear subspace of a complete linear system.

Exercise 1.44. *All rational functions on* $\mathbb{P}^n$ *can be obtained by taking the quotient of two homogeneous polynomial of the same degree (we are not asking you to prove it, take it as true).*

Let D *be a (non necessarily effective) divisor on* $\mathbb{P}^n$ *and let* $d \in \mathbb{Z}$ *its degree (i.e. its class in* $\mathrm{Pic}(\mathbb{P}^n)$*). Give an explicit isomorphism among* $H^0(\mathcal{O}(D))$ *and the vector space of the homogeneous polynomials of degree* d*.*

Exercise 1.45. *State and solve an exercise similar to the previous one, but considering* $\mathbb{P}^1 \times \mathbb{P}^1$ *instead of* $\mathbb{P}^n$*. Here you will need that every rational function on* $\mathbb{P}^1 \times \mathbb{P}^1$ *can be obtained by taking the quotient of two bihomogeneous polynomials of the same bidegree.*

1.4 Pull-back and push-forward

Definition 1.46. *Let* X *and* S *be two varieties and* $f : X \to S$ *a morphism; let* $D \in \mathrm{Div}(S)$*,* $D = \sum a_i D_i$*. We define the* pull-back *of* D $f^*D \in \mathrm{Div}(X)$ *as*

$$f^*D := \sum a_i f^* D_i,$$

where, if D_i *is locally defined by* f_i*,* f^*D_i *is locally defined by* $f_i \circ f$*.*

Example 1.47. *Let*

$$\begin{array}{cccc} f: & \mathbb{P}^1 & \to & \mathbb{P}^1 \\ & (x_0 : x_1) & \mapsto & (x_0^2 : x_1^2) \end{array}$$

and let us consider $P_1 := (1:0)$*,* $P_2 := (1:1)$*,* $P_3 := (1:-1)$*. Then*

$$f^*(P_2) = P_2 + P_3, \qquad f^*(P_1) = 2P_1.$$

Remark 1.48. *Using the same notation of previous definition, the following facts hold:*

1. *D effective $\Rightarrow f^*D$ is effective;*
2. *$D_1 \equiv D_2 \Rightarrow D_1 - D_2 = (g)$ for some $g : S \dashrightarrow \mathbb{C} \Rightarrow f^*D_1 - f^*D_2 = (g \circ f) \Rightarrow f^*D_1 \equiv f^*D_2$;*
3. *$f^* : \mathrm{Div}(S) \to \mathrm{Div}(X)$ is a homomorphism of abelian groups; since, by 2., the subgroup of principal divisors of S is mapped into the subgroup of principal divisors of X, f^* induces a homomorphism*
$$\begin{array}{cccc} f^*: & \mathrm{Pic}(S) & \to & \mathrm{Pic}(X) \\ & [D] & \mapsto & [f^*D] \end{array}.$$

The definition of pull-back is natural. The next definition, the *push-forward*, needs some extra assumptions.

Definition 1.49. *Let X be a smooth surface (i.e. a complex variety of dimension 2). Let $f : X \to S$ be a generically finite map of degree d, that is $\exists U \subset S$ open set such that $\forall p \in U$ $\#f^{-1}(p) = d$.*

If $D = \sum a_i D_i \in \mathrm{Div}(X)$, then the push-forward *of D is $f_*D := \sum a_i f_* D_i \in \mathrm{Div}(S)$, where*

- *if $f(D_i)$ is constant, then $f_*D_i := 0 \in \mathrm{Div}(S)$;*
- *otherwise $f(D_i)$ is a Cartier divisor in S and $D_i \to f(D_i)$ is generically finite of degree $r \le d$. In this case we define $f_*D_i := rf(D_i)$.*

Remark 1.50. *Using the notation of previous definition, the following facts hold:*

1. *D effective $\Rightarrow f_*D$ is effective;*
2. *$D_1 \equiv D_2 \Rightarrow f_*D_1 \equiv f_*D_2$;*
3. *$f_* : \mathrm{Div}(X) \to \mathrm{Div}(S)$ is a homomorphism of abelian groups, hence, as in the case of pull-back, f_* induces a homomorphism $f_* : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$;*
4. *$f_* f^* D \equiv dD$.*

Example 1.51. *Let*
$$\begin{array}{cccc} g: & \mathbb{P}^2 & \to & \mathbb{P}^2 \\ & (x_0 : x_1 : x_2) & \mapsto & (x_0^2 : x_1^2 : x_2^2) \end{array}.$$

This map is generally finite of degree 4.

Let us consider $l_0 := \{x_0 = 0\} \in \mathrm{Div}(\mathbb{P}^2)$, *then it is easy to see that* $g^* l_0 = 2l_0$. *Moreover,* $g(l_0) = l_0$ *and* $g|_{l_0} : l_0 \to l_0$ *has degree* 2, *hence* $g_* l_0 = 2l_0$. *Eventually, it is immediate to see that* $g^* g_* l_0 = 4l_0$.

Let us now consider $l = \{x_0 + x_1 + x_2 = 0\} \in \mathrm{Div}(\mathbb{P}^2)$; *first of all, we observe that* $l \equiv l_0$, *hence, by Remark 1.50, we expect* $g_* l \equiv g_* l_0 = 2l_0$. *Indeed* $g(l) = \{x_0^2 + x_1^2 + x_2^2 - 2x_0x_1 - 2x_0x_2 - 2x_1x_2 = 0\}$ *and that* $g|_l : l \to g(l)$ *has degree* 1. *Hence* $g_* l = g(l) \equiv 2l_0$.

1.5 Intersection multiplicity

From now on by *surface* we will always mean a smooth variety of dimension 2 and by *curve in it* an irreducible divisor in it.

Let S be a surface and C, C' curves in it with $C \neq C'$. $\forall x \in S$ we define the set

$$\mathcal{O}_x := \{\text{germ in } x \text{ of regular functions } f : U \to \mathbb{C}, x \in U\};$$

this set is called *local ring of* S at x.

Definition 1.52. *Let* $x \in C \cap C'$ *and* $f, g \in \mathcal{O}_x$ *germs of regular functions at* x *that define* C *and* C' *respectively. We define the* intersection multiplicity *of* C *and* C' *at* x *as*

$$m_x(C, C') := \dim_{\mathbb{C}} \mathcal{O}_x/(f, g)$$

Remark 1.53. *By* Hilbert Nullstellensatz $m_x(C, C') < \infty$.

Example 1.54. *Let* $S = \mathbb{C}^2$, $C = \{x = 0\}$, $C' = \{y = 0\}$, $p = 0$. *Then the map*

$$\begin{array}{ccc} \mathcal{O}_p/(x, y) & \to & \mathbb{C} \\ [f] & \mapsto & f(0) \end{array}$$

is an isomorphism. Therefore $m_p(C, C') = 1$.

Following propositions are given without proof.

Remark 1.55. *Under the hypotheses of Definition 1.52,* $\forall x \in C \cap C'$, $m_x(C, C') \geq 1$.

The following proposition, which we do not have time to prove, gives a clear interpretation of the intersection multiplicity.

Proposition 1.56. *Under the hypotheses of Definition 1.52, $m_x(C, C') = 1 \Leftrightarrow$ both C and C' are smooth in x and intersect transversely.*

Exercise 1.57. *Using the same notation of Example 1.54, prove that*

$$\begin{aligned} m_p(\{x = 0\}, \{x = y^r\}) &= r; \\ m_p(\{x = 0\}, \{x^2 = y^3\}) &= 3; \\ m_p(\{y = 0\}, \{x^2 = y^3\}) &= 2. \end{aligned}$$

Definition 1.58. *Under the hypotheses of Definition 1.52 we define the* intersection number *of C and C' as*

$$C \cdot C' := \sum_{x \in C \cap C'} m_x(C, C'). \tag{1.3}$$

By *Hilbert Nullstellensatz* the sum in (1.3) is finite, hence $C \cdot C' < \infty$.

1.6 Morphisms of sheaves

Definition 1.59. *Let X be a variety and let $\mathcal{F}, \mathcal{G}$ be two sheaves of rings (groups,...). A function*

$$f : \mathcal{F} \to \mathcal{G}$$

is said to be a morphism of sheaves *if $\forall U \subset X$ open set $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is a ring (group,...) homomorphism, and $\forall U, V \subset X$ open sets such that $V \subset U$, the diagram*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ {\scriptstyle \rho^U_V}\downarrow & & \downarrow{\scriptstyle \rho^U_V} \\ \mathcal{F}(V) & \xrightarrow[f_V]{} & \mathcal{G}(V) \end{array}$$

commutes.

Example 1.60. *Let us see some fundamental examples of morphisms of sheaves.*

1. *Let $D_1, D_2 \in \mathrm{Div}(X)$ such that $D_1 \equiv D_2$, then $D_1 - D_2 = (F)$ for some $F : X \dashrightarrow \mathbb{C}$; then $\forall U$ open set we define the morphism*

$$\begin{array}{ccc} \Gamma(U, \mathcal{O}_X(D_1)) & \to & \Gamma(U, \mathcal{O}_X(D_2)) \\ f & \mapsto & fF|_U \end{array};$$

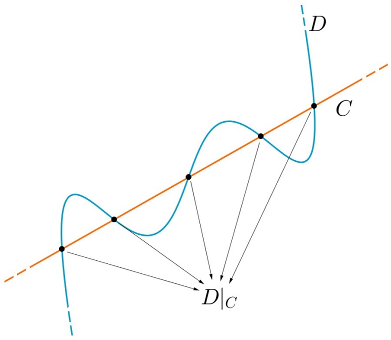
these maps give a morphism of sheaves among $\mathcal{O}_X(D_1)$ and $\mathcal{O}_X(D_2)$.

2. *Let Y be a variety and $X \subset Y$ a subvariety; the* restriction

$$\mathcal{O}_Y \to \mathcal{O}_X$$

is a morphism; actually, we cannot give a morphism among two sheaves defined on different varieties, but we can see $\mathcal{O}_X$ as a sheaf on Y defining $\forall U \subset Y$ Zariski open set $\Gamma(U, \mathcal{O}_X) := \Gamma(U \cap X, \mathcal{O}_X)$.

3. *Let $C, D \in \mathrm{Div}(X)$, with C irreducible, $D = \sum a_i D_i$ with $C \neq D_i$ $\forall i$.*



Under these hypotheses, the divisor $D|_C \in \mathrm{Div}(C)$ is well defined: if locally $D_i = (f_i)$, then $D_i|_C = (f_i|_C)$. The restriction

$$\mathcal{O}_X(D) \to \mathcal{O}_C(D|_C)$$

is a morphism of sheaves if we consider the latter as sheaf on X as in the previous case.

4. *$C, D \in \mathrm{Div}(X)$, $D \geq 0$. Let $U \subset X$ be an open set; if $f \in \Gamma(U, \mathcal{O}_X(C))$, then $(f)+C \geq 0$, thus $(f)+C+D \geq D \geq 0$. This means that $f \in \Gamma(U, \mathcal{O}_X(C+D))$. The function*

$$\begin{array}{ccc} \mathcal{O}_X(C) & \to & \mathcal{O}_X(C+D) \\ f & \mapsto & f \end{array}$$

is well defined and is a morphism.

Definition 1.61. *Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X and let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. $\forall U \subset X$ open set we define*

$$(\ker f)(U) := \{u \in \mathcal{F}(U) : f_U(u) = 0\} \subset \mathcal{F}(U).$$

Exercise 1.62. $\ker f$ *is a sheaf.*

Definition 1.63. *A morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is said to be* injective *if* $\ker f = 0 \Leftrightarrow \forall U \;\ker f_U = 0$.

Remark 1.64. *The morphisms defined in Examples 1.60.1 and 1.60.4 are injective.*

Definition 1.65. *Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X and $f : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves; f is said to be* surjective *if $\forall p \in X$ $\exists U \subset X$ open set $p \in U$ such that $\forall \xi \in \mathcal{G}(U)$ $\exists V \subset U$ open set with $p \in V$ and $\exists \eta \in \mathcal{F}(V)$ such that $f_V(\eta) = \rho^U_V(\xi)$.*

Remark 1.66. *The morphisms defined in Examples 1.60.1, 1.60.2 and 1.60.3 are surjective.*

Exercise 1.67. *The function*

$$\begin{array}{rccc} \exp: & {}^h\mathcal{O}_{\mathbb{C}} & \to & {}^h\mathcal{O}^*_{\mathbb{C}} \\ & f & \mapsto & e^{2\pi i f} \end{array}$$

is a morphism of sheaves of abelian group (where the operation on ${}^h\mathcal{O}_{\mathbb{C}}$ is the sum, while the operation on ${}^h\mathcal{O}^*_{\mathbb{C}}$ is the multiplication). This morphism is surjective, but it is not surjective on $\mathbb{C}^*$ since the function z (the identity) gives a section of $\Gamma(\mathbb{C}^*, {}^h\mathcal{O}_{\mathbb{C}^*})$, which is not of the form $e^{2\pi i f}$: indeed most courses in complex analysis show that $\log z$ cannot be defined on the whole $\mathbb{C}^*$.

Definition 1.68. *An injective and surjective morphism of sheaves is called* isomorphism *of sheaves.*

By the previous remarks, $D_1 \equiv D_2 \Rightarrow \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$.

Definition 1.69. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be three sheaves on X; we say that*

$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0$$

is a short exact sequence of sheaves *if*

- *g is a surjective morphism;*
- *f is an injective morphism;*
- *$f(\mathcal{A}) \subset \ker(g)$ and $f : \mathcal{A} \to \ker(g)$ is surjective.*

Definition 1.70. *A sequence of morphisms of sheaves*

$$\ldots \longrightarrow \mathcal{A}_{i-1} \xrightarrow{f_i} \mathcal{A}_i \xrightarrow{f_{i+1}} \mathcal{A}_{i+1} \longrightarrow \ldots$$

is said to be exact *if f_i surjects to $\ker f_{i+1}$ $\forall i$.*

Exercise 1.71. *Let S be a variety; $\forall C$ irreducible divisor in S and $\forall D \in$* $\mathrm{Div}(S)$ *there are exact sequences.*

$$0 \to \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \to 0$$

and

$$0 \to \mathcal{O}_S(D-C) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_C(D) \to 0$$

1.7 Čech cohomology

Definition 1.72. *Let $\mathcal{F}$ be a sheaf on X; $\forall U \subset X$ open set, $\forall q \in \mathbb{N}$ $\exists$ $\check{H}^q(\mathcal{F})$* q-th Čech cohomology group *such that:*

1. $\check{H}^0(\mathcal{F}) = \Gamma(X, \mathcal{F})$;
2. $\forall 0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ *short exact sequence of sheaves $\exists$ a long exact sequence of cohomology groups*
$$0 \to \check{H}^0(\mathcal{A}) \to \check{H}^0(\mathcal{B}) \to \check{H}^0(\mathcal{C}) \to \check{H}^1(\mathcal{A}) \to \check{H}^1(\mathcal{B}) \to \check{H}^1(\mathcal{C}) \to \dots \tag{1.4}$$
3. *If $\mathcal{F}$ is a* coherent sheaf of $\mathcal{O}_X$−modules *and X is projective, then all $\check{H}^q(\mathcal{F})$ are $\mathbb{C}$-vector spaces of finite dimension and each map in (1.4) is linear;*
4. *Under the hypotheses of previous point, if* $\dim X = n$, *then* $\check{H}^q(\mathcal{F}) = 0$ $\forall q > n$ *(if X is not smooth, then there exists a Zariski open set $U \subset X$ smooth; in this case* $\dim X := \dim U$*).*

Definition 1.73. *Under the hypotheses of the last point of Definition 1.72, the* Euler-Poincaré characteristic of $\mathcal{F}$ *is*

$$\chi(\mathcal{F}) := \sum (-1)^q h^q(\mathcal{F}),$$

where $h^q(\mathcal{F}) := \dim \check{H}^q(\mathcal{F})$.

Exercise 1.74. *If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence of $\mathcal{O}_X$-modules, then $\chi(\mathcal{A}) + \chi(\mathcal{C}) = \chi(\mathcal{B})$.*

1.8 The Canonical sheaf

Definition 1.75. *Let X be a variety of dimension n; by Ω^q_X we denote the sheaf of* regular q-form *on X, that is*

$$\Omega^q_X(U) := \{\omega : \text{ locally } \omega = \sum_{1 \le i_1 \le \dots \le i_q \le n} f_{i_1 \dots i_q} dx_{i_1} \wedge \dots \wedge dx_{i_q},$$
$$\text{where } x_i \text{ are regular local coordinates and } f_{i_1 \dots i_q} \text{ are regular}\}$$

Ω^n_X *is called* canonical sheaf.

Let $\omega \in \Omega^n_X(U)$, then locally $\omega = f dx_1 \wedge \dots \wedge dx_n$ is defined through a single rational function f. Hence the locus of zeroes and poles of ω, locally defined as the locus of zeroes and poles of f, is a Cartier divisor.

Definition 1.76. *Let ω be a rational n–form on X so regular on a Zariski open subset of X, then (ω) is defined as the divisor of its zeroes and poles. A divisor of this kind is said to be* canonical.

Remark 1.77. *Let ω be a rational n–form on X and g a rational function on X, then $g\omega$ is another rational n–form and its Cartier divisor is $(g\omega) = (g) + (\omega) \Rightarrow (\omega) \equiv (g\omega)$. In other words every divisor linearly equivalent to a canonical divisor is still a canonical divisor.*

Vice versa, suppose that ω' is another rational n–form, write locally $\omega = f dx_1 \wedge \cdots \wedge dx_n$ and $\omega' = f' dx_1 \wedge \cdots \wedge dx_n$, then f'/f does not depend on the coordinates and therefore defines a rational function on X. [3] *This implies that $(\omega) \equiv (\omega')$, that is the canonical divisors form an equivalence class for the relation "linear equivalence". This equivalence class is then an element of* $\mathrm{Pic}(X)$*, usually denoted by K (sometimes we will denote - with a slight abuse of notation - by K any canonical divisor in $Div(X)$).*

Example 1.78. *Let $X = \mathbb{P}^1$ and consider the usual open sets*

$$\begin{aligned} U_0 &:= \{x_0 \neq 0\} \qquad \text{with local coordinate } x = x_1/x_0 \\ U_1 &:= \{x_1 \neq 0\} \qquad \text{with local coordinate } y = x_0/x_1. \end{aligned}$$

It is immediate to see that in $U_0 \cap U_1$ $y = 1/x$.

Let us consider $\omega \in \Omega^1_X(U_0)$ defined as $\omega = dx$. As a rational 1–form on X it has no zeroes and a double pole in $(0:1)$ since in the coordinates of U_1:

$$\omega = d\left(\frac{1}{y}\right) = -\frac{dy}{y^2}.$$

Hence $(\omega) = -2P$. As we saw in a previous section, $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$*, hence $K_{\mathbb{P}^1} = -2$.*

Exercise 1.79. *Prove that:*

1. $K_{\mathbb{P}^n} = -(n+1) < 0$;
2. $K_{\mathbb{P}^1 \times \mathbb{P}^1} = (-2, -2)$.

Remark 1.80. *By Exercise 1.79 and Exercise 1.44, it is immediate to see that $H^0(K_{\mathbb{P}^n}) = \{0\}$, since there are no homogeneous polynomials of degree* < 0.

[3] By a slight abuse of notation we could write that $\frac{\omega}{\omega'}$ is a rational function.

Exercise 1.81. *Let $K = (\omega) \in \operatorname{Div}(X)$ with ω rational n–form. Prove that*

$$\begin{array}{ccc} \mathcal{O}(K) & \longrightarrow & \Omega^n_X \\ f & \longmapsto & f\omega \end{array}$$

is an isomorphism of sheaves.

We do not prove the next two statements.

Theorem 1.82. *Let C be a smooth compact curve (that is a Riemann surface) $\Rightarrow h^0(\Omega^1_C) = g$, where g is the genus of C.*

Theorem 1.83 (Serre duality)**.** *Let X be a smooth variety of dimension n, then $\forall q, \forall D \in \operatorname{Div}(X)$*

$$H^q(D) \cong H^{n-q}(K - D)^*.$$

Remark 1.84. *Let C be a smooth compact curve. We have seen that*

$$\begin{aligned} &h^q(\mathcal{O}_C) = 0 \qquad \text{if } q \neq 0, 1; \\ &h^0(\mathcal{O}_C) = 1; \\ &h^1(\mathcal{O}_C) = h^0(K_C) = g; \end{aligned}$$

hence $\chi(\mathcal{O}_C) = 1 - g$.

Definition 1.85. *Let X be a variety, $p_1, \dots, p_r \in X$. A* skyscraper sheaf supported on *$\{p_1, \dots, p_r\}$ is a sheaf $\mathcal{F}$ obtained by associating to each p_i a vector space V_i and then by taking $\forall U \subset X$ open set*

$$\mathcal{F}(U) = \bigoplus_{i: p_i \in U} V_i$$

with the natural restrictions.

By the definition of Čech cohomology, it is easy to prove the following

Proposition 1.86. *Using the same notation of the previous definition, $\check{H}^0(\mathcal{F}) \cong \bigoplus V_i$ and $\forall q \neq 0$ $h^q(\mathcal{F}) = 0$.*

Example 1.87. *Let X be a variety $p \in X$, then $\mathcal{O}_p \cong \underline{\mathbb{C}}$, seen as a sheaf on X (as we did in Example 1.60.2), is a skyscraper sheaf supported on p.*

More generally, $\forall Y \subset X$, if $\mathcal{F}$ is a sheaf on Y, then $H^q(\mathcal{F})$ does not change if $\mathcal{F}$ is considered as a sheaf defined on X.

1.9 Riemann-Roch theorem

Let C be a smooth compact curve; let us define for all points $p \in C$ the *evaluation map at* p to be

$$\begin{array}{rccc} \mathrm{eval}: & \mathcal{O}_C & \longrightarrow & \mathcal{O}_p \\ & f & \longmapsto & f(p) \end{array};$$

it is immediate to see that

$$\begin{aligned} \ker(\mathrm{eval}) &= \{f \text{ regular } : f(p) = 0\} \\ &= \{f \text{ regular } : (f) - p \geq 0\} = \mathcal{O}_C(-p), \end{aligned}$$

hence

$$0 \longrightarrow \mathcal{O}_C(-p) \longrightarrow \mathcal{O}_C \xrightarrow{\mathrm{eval}} \mathcal{O}_p \longrightarrow 0$$

is an exact sequence.

Analogously, $\forall D \in \mathrm{Div}(C)$ $D = \sum a_i p_i$ such that $\forall i$ $p \neq p_i$, the sequence

$$0 \longrightarrow \mathcal{O}_C(D-p) \longrightarrow \mathcal{O}_C(D) \xrightarrow{\mathrm{eval}} \mathcal{O}_p \longrightarrow 0 \tag{1.5}$$

is exact. Since $D \equiv D' \Rightarrow \mathcal{O}_C(D) \cong \mathcal{O}_C(D')$, and since one can always find a divisor D' linearly equivalent to D which does not involve a chosen point p, we can drop the hypothesis for which $p \neq p_i$ $\forall i$ (1.5).

Corollary 1.88. *$\forall p \in C$, $\forall D \in \mathrm{Div}(C)$ $\chi(\mathcal{O}_C(D)) = \chi(\mathcal{O}_C(D-p)) + 1$.*

Proof. The sequence $0 \to \mathcal{O}_C(D-p) \to \mathcal{O}_C(D) \to \mathcal{O}_p \to 0$ is exact. Hence $\chi(\mathcal{O}_C(D)) = \chi(\mathcal{O}_C(D-p)) + \chi(\mathcal{O}_p) = \chi(\mathcal{O}_C(D-p)) + 1$. □

Corollary 1.89. *Let C be a smooth compact curve of genus g, then $\forall D \in$ $\mathrm{Div}(C)$*

$$\chi(\mathcal{O}_C(D)) = \deg(D) - g + 1.$$

Proof. By Corollary 1.88 if the statement is true for a divisor D, then, for every p, the statement is true both for $D-p$ and $D+p$. It is then enough to prove it for a divisor, and indeed it is true for $D = 0$ by Remark 1.84. □

This corollary, known as *Riemann-Roch theorem*, is usually written in a different way; indeed, under our hypotheses

$$\chi(\mathcal{O}_C(D)) = h^0(D) - h^1(D) = h^0(D) - h^0(K-D),$$

where the last equality holds by Serre duality. Hence

$$h^0(D) - h^0(K-D) = \deg(D) - g + 1.$$

However, we prefer the statement of Corollary 1.89 since it does generalize to higher dimensional varieties.

1.10 The intersection form

Let S be a smooth projective surface, $C, C' \in \mathrm{Div}(S)$ irreducible and $C \neq C'$; in a previous section, $\forall x \in C \cap C'$ we have considered the ring

$$\mathcal{O}_{C\cap C',x} := \mathcal{O}_x/(s, s')$$

where s and s' are the local equations of C and C' respectively. More precisely, we defined the intersection multiplicity of C and C' at x as the dimension of it as a complex vector space. We define $\mathcal{O}_{C\cap C'}$ to be the skyscraper sheaf supported on $C \cap C'$ that associates to each $x \in C \cap C'$ the ring $\mathcal{O}_{C\cap C',x}$.

Lemma 1.90. *Let S be a smooth surface, $C, C' \in \mathrm{Div}(S)$ irreducible and $C \neq C'$, then there exists an exact sequence*

$$0 \longrightarrow \mathcal{O}_S(-C-C') \xrightarrow{\alpha} \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C') \xrightarrow{\beta} \mathcal{O}_S \xrightarrow{\gamma} \mathcal{O}_{C\cap C'} \longrightarrow 0. \quad (1.6)$$

Proof. Let us define α, β and γ.

Map $\gamma : \mathcal{O}_S \to \mathcal{O}_{C\cap C'}$ simply maps $f \in \mathcal{O}_S(U)$ in its classes in each $\mathcal{O}_x/(s, s')$. This map is trivially surjective.

By definition $f \in \ker\gamma \Leftrightarrow \forall x \in U \; [f]_{\mathcal{O}_x} \in (s, s')$, that is if and only if locally $f = as + bs'$. Hence, since we want (1.6) to be exact, we define

$$\beta(a, b) = as + bs',$$

so that β surjects on $\ker\gamma$. Finally, we define

$$\alpha(h) = (s'h, -sh).$$

It is immediate to see that α is injective and that $\beta \circ \alpha = 0$. The surjectivity of α onto $\ker\beta$ follows from standard results of commutative algebra using that C and C' are irreducible and distinct (this means that s and s' are relatively prime). $\square$

Now that we have a long exact sequence, we can split it into short exact sequences, and compute χ of the members of such sequences. From (1.6) we get the exact sequences

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_S(-C-C') \longrightarrow \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C') \longrightarrow \ker\gamma \longrightarrow 0 \\ 0 \longrightarrow \ker\gamma \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{C\cap C'} \longrightarrow 0. \end{gathered} \quad (1.7)$$

We recall that

$$\chi(\mathcal{O}_{C\cap C'}) = h^0(\mathcal{O}_{C\cap C'}) = \sum m_x(C, C') = C \cdot C'.$$

From (1.7) and Exercixe 1.74 follows that

$$C \cdot C' = \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(-C - C')) - \chi(\mathcal{O}_S(-C)) - \chi(\mathcal{O}_S(-C')). \tag{1.8}$$

Note that (1.8) gives an expression of $C \cdot C'$ which only depends on the equivalence class of C and C' modulo linear equivalence. In particular $C \cdot C'$ does not change if we substitute C by a different curve linearly equivalent to it, since the right member of (1.8) depends not on $C, C' \in \mathrm{Div}(S)$ but on their classes in $\mathrm{Pic}(S)$. This suggests us to use (1.8) to extend the definition of intersection to a form on $\mathrm{Pic}(S)$.

Definition 1.91. $\forall D_1, D_2 \in \mathrm{Pic}(S)$ *we define*

$$D_1 \cdot D_2 := \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(-D_1 - D_2)) - \chi(\mathcal{O}_S(-D_1)) - \chi(\mathcal{O}_S(-D_2)). \tag{1.9}$$

Theorem 1.92. *(1.9) defines a symmetric bilinear form on the abelian group* $\mathrm{Pic}(S)$ *such that, if* $C_1 \in |D_1|$, $C_2 \in |D_2|$ *are irreducible and* $C_1 \neq C_2$, *then* $D_1 \cdot D_2 = C_1 \cdot C_2 = \sum_{x \in C_1 \cap C_2} m_x(C_1, C_2)$.

We have already remarked that the last part of the theorem holds, and by definition, it is immediate to see that $D_1 \cdot D_2 = D_2 \cdot D_1$. We need only to prove bilinearity. Since the function is defined on an abelian group, by *bilinearity* we mean

$$(D + D') \cdot C = (D \cdot C) + (D' \cdot C) \quad \text{and} \quad D \cdot (C + C') = D \cdot C + D \cdot C'.$$

Lemma 1.93. *Let* C *be a smooth curve on* S, $D \in \mathrm{Div}(S)$, *then*

$$C \cdot D = \deg \mathcal{O}_C(D) = \deg D|_C.$$

Proof. The sequences

$$0 \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_S(-C-D) \longrightarrow \mathcal{O}_S(-D) \longrightarrow \mathcal{O}_C(-D) \longrightarrow 0$$

are exact. Hence using Riemann-Roch on C

$$\begin{aligned} C \cdot D &= \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) - \chi(\mathcal{O}_S(-D)) + \chi(\mathcal{O}_S(-C-D)) \\ &= \chi(\mathcal{O}_C) - \chi(\mathcal{O}_C(-D)) \\ &= 1 - g - (1 - g + \deg(\mathcal{O}_C(-D))) \\ &= \deg(\mathcal{O}_C(D)). \end{aligned}$$

□

$\forall D_1, D_2, D_3 \in \mathrm{Pic}(S)$ let us define the function

$$s(D_1, D_2, D_3) := (D_1 \cdot (D_2 + D_3)) - (D_1 \cdot D_2) - (D_1 \cdot D_3).$$

Exercise 1.94. $s(D_1, D_2, D_3) = s(D_{\sigma(1)}, D_{\sigma(2)}, D_{\sigma(3)})$ $\forall \sigma \in S_3$.

By Lemma 1.93, if $\exists\ C \in |D_1|$ smooth, then

$$s(D_1, D_2, D_3) = \deg \mathcal{O}_C(D_2 + D_3) - \deg \mathcal{O}_C(D_2) - \deg \mathcal{O}_C(D_3) = 0,$$

since, on a curve $\deg(A + B) = \deg A + \deg B$. Hence, by Exercise 1.94, if $\exists\ C \in |D_3|$ smooth, $s(D_1, D_2, D_3) = 0$.

We can now prove Theorem 1.92.

Proof of Theorem 1.92. Let $A, B \in \mathrm{Div}(S)$ be smooth and irreducible and $D \in \mathrm{Div}(S)$, then, since B is smooth,

$$0 = s(D, A - B, B) = (D \cdot A) - (D \cdot (A - B)) - (D \cdot B),$$

thus $D \cdot (A - B) = (D \cdot A) - (D \cdot B) = \deg D|_A - \deg D|_B$, which in turn implies that the map $D \to D \cdot (A - B)$ is linear (and therefore by simmetry also the map $D \to D \cdot (B - A)$).

The theorem follows then since every divisor is linearly equivalent to a difference of smooth effective divisors, by the classical Theorem of Bertini and Serre which we recall immediately after this proof. $\square$

Definition 1.95. *Let X be a variety; $D \in \mathrm{Div}(X)$ is said to be* very ample *if $\exists$ an embedding of X in $\mathbb{P}^n$ such that D is the Cartier divisor locally defined by the restriction to X of (x_0) (equivalently we will say that "D is a hyperplane section" or that $D = X \cap \{x_0 = 0\}$).*

Exercise 1.96. *If D is very ample, then nD is very ample $\forall n \in \mathbb{N}$ with $n \geq 1$.*

The two following fundamental theorems will be stated without proof.

Theorem 1.97 (Theorem of Bertini)**.** *If X is smooth and D is very ample, then $\exists C \in |D|$ smooth and irreducible.*

The geometric idea lying under the theorem of Bertini is that if we cut X with a *generic* hyperplane, we get a smooth subvariety of X.

Theorem 1.98 (Theorem of Serre)**.** *Let X be a variety, $A, D \in \mathrm{Div}(X)$ with A very ample and D effective; then $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ $D + nA$ is very ample.*

Let S be a projective variety, $\forall C \in \mathrm{Div}(S)$ we are interested in computing the value $C^2 = C \cdot C$. Let us see some examples.

Example 1.99 (Fibered surfaces)**.** *Let S be a projective, smooth and irreducible surface and let $\pi : S \to C$ be a surjective morphism, with C smooth irreducible curve. $\forall p \in C$ $p \in \mathrm{Div}(C)$, hence we shall consider the pull-back $\pi^* p = F_p \in \mathrm{Div}(S)$, where by F_p we denote the fiber over p.*

First, let us remark that there exists $\sum a_i p_i \in \mathrm{Div}(C)$ such that $p \equiv \sum a_i p_i$ and $p \neq p_i$ $\forall i$. Since, by Remark 1.48, the pull-back preserve linear equivalence,

$$F_p = \pi^* p \equiv \pi^* \left(\sum a_i p_i\right) = \sum a_i \pi^* p_i = \sum a_i F_{p_i},$$

thus

$$F_p^2 = F_p \cdot \left(\sum a_i F_{p_i}\right) = \sum a_i (F_p \cdot F_{p_i}) \overset{(\#)}{=} \sum a_i 0 = 0,$$

where equality $(\#)$ holds because $F_p \cap F_{p_i} = \varnothing$ $\forall i$.

Example 1.100. *Let $S = \mathbb{P}^2$, $C_1 = \{f_{d_1} = 0\}$, $C_2 = \{f_{d_2} = 0\}$, where $f_{d_i} = \mathbb{C}[x_0, x_1, x_2]_{d_i}$ (possibly equal). Then $C_1 \equiv d_1 l_0$ and $C_2 \equiv d_2 l_1$; hence*

$$C_1 \cdot C_2 = (d_1 l_0) \cdot (d_2 l_1) = d_1 d_2 (l_0 \cdot l_1) = d_1 d_2,$$

where last inequality holds because of Proposition 1.56, since $l_0 \pitchfork l_1 = \{(0:0:1)\}$.

Exercise 1.101. *Let $C_1, C_2 \in \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \times \mathbb{Z}$, with $C_1 = (a_1, b_1)$ and $C_2 = (a_2, b_2)$; compute $C_1 \cdot C_2$.*

(To check if the solution is right, $(2,7)(3,6) = 33$)

Theorem 1.102 (Riemann-Roch theorem for surfaces)**.** *Let S be a projective surface, $D \in \mathrm{Div}(S)$, then*

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - DK). \tag{1.10}$$

Proof. By (1.9)

$$(-D) \cdot (D - K) = \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(K)) - \chi(\mathcal{O}_S(D)) - \chi(\mathcal{O}_S(K - D));$$

by Serre duality $h^q(\mathcal{O}_S(D)) = h^{2-q}(\mathcal{O}_S(K - D))$, hence, since $2 - q$ is even whenever q is, we conclude that $\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S(K - D))$. Therefore

$$(-D) \cdot (D - K) = 2(\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(D)),$$

hence

$$\begin{aligned}\chi(\mathcal{O}_S(D)) &= \chi(\mathcal{O}_S) - \frac{1}{2}(-D)(D - K) \\ &= \chi(\mathcal{O}_S) + \frac{1}{2}D(D - K).\end{aligned}$$

□

Corollary 1.103. *Under the hypotheses of Theorem 1.102, if* $\chi(\mathcal{O}_D) > 0$ *then at least one among* $|D|$ *and* $|K - D|$ *is non empty.*

Proof. $h^0(D) + h^0(K - D) = h^0(D) + h^2(D) \geq \chi(\mathcal{O}_S(D)) > 0$, where the first equality holds by Serre duality. Hence at least one among $h^0(D)$ and $h^0(K - D)$ is strictly greater than zero. □

Corollary 1.104 (Genus formula)**.** *Let* C *be a smooth curve in* S*, then the genus of* C *is*

$$g = 1 + \frac{1}{2}(C^2 + CK).$$

Proof. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Hence

$$\begin{aligned} g &= 1 - \chi(\mathcal{O}_C) && \text{by Corollary 1.89} \\ &= 1 - \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(-C)) \\ &= 1 + 1/2((-C)^2 - (-C)K) && \text{by Theorem 1.102} \\ &= 1 + 1/2(C^2 + CK). \end{aligned}$$

□

Exercise 1.105. *Compute the genus of a smooth plane curve of degree* d*.*

Exercise 1.106. *Compute the genus of a curve in* $\mathbb{P}^1 \times \mathbb{P}^1$ *of bidegree* (a, b)*.*

Definition 1.107. *If* $D \in \mathrm{Div}(S)$ *the* arithmetic genus *of* D *is*

$$p_a(D) := 1 + \frac{1}{2}(D^2 + DK).$$

We state the following result without proof.

Theorem 1.108 (Theorem of Noether)**.**

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K^2 + e), \tag{1.11}$$

where e *is the* Euler topological characteristic *of* S *(in this particular case* $e = \sum_{q=0}^{4}(-1)^q h^q_{DR}(S)$*, where* H^*_{DR} *denotes the De Rham cohomology).*

Chapter 2

Birational maps

Before beginning a classification, we have to decide when we are going to consider two of the objects we are classifying to be equivalent. In algebraic geometry, we classify varieties up to isomorphism or, more coarsely, up to birational equivalence. The problem does not arise for curves, since a rational map from one smooth complete curve to another is in fact a morphism. For surfaces, we shall see that the structure of birational maps is very simple; they are composites of *elementary* birational maps, the blow ups.

2.1 Blow ups

In this section we study a classical example of morphism wich is birational but not biregular, called *blow up*.

Definition 2.1. *Let S be a smooth surface and $p \in S$, then there exists a smooth surface $\hat{S}$*[1]*, called* blow up *of S in p, and a birational morphism $\varepsilon : \hat{S} \to S$ such that*

1. $E := \varepsilon^{-1}(p) \cong \mathbb{P}^1$ *(this set is called* exceptional divisor*);*
2. $\varepsilon|_{\hat{S} \smallsetminus E} : \hat{S} \smallsetminus E \longrightarrow S \smallsetminus \{p\}$ *is biregular.*

Example 2.2. *To describe explicitly the behaviour of the blow up in a neighbourhood of p, it is enough to describe the case of $S = \mathbb{C}^2$ in $p = (0,0)$: indeed all blow ups can be obtained by this example writing everything in suitable*

[1]To be precise $\hat{S}$ exists always as a complex manifold, or as "algebraic variety" (a complex manifold with an atlas whose transition functions are regular maps) but not always as a quasi-projective variety.

local coordinates. Let us denote the coordinate on $\mathbb{C}^2$ *by* (x, y) *and those on* $\mathbb{P}^1$ *by* $(t_0 : t_1)$*. We define*

$$\hat{\mathbb{C}}^2 := \{xt_1 = yt_0\} \subset \mathbb{C}^2 \times \mathbb{P}^1$$

and

$$\begin{array}{rccc} \varepsilon : & \hat{\mathbb{C}}^2 & \longrightarrow & \mathbb{C}^2 \\ & ((x,y),(t_0 : t_1)) & \longmapsto & (x,y) \end{array}$$

First of all, it is immediate to see that

$$\varepsilon^{-1}(0) = \{0\} \times \mathbb{P}^1 \subset \mathbb{C}^2 \times \mathbb{P}^1.$$

Now let us prove that $\hat{\mathbb{C}}^2$ *is smooth.*

Let $p \in \hat{\mathbb{C}}^2$*; if* $t_1(p) \neq 0$*, then* $U_1 := \hat{\mathbb{C}}^2 \cap \{t_1 \neq 0\} \subset \mathbb{C}^3$ *with coordinates* $x, y, t = t_0/t_1$ *and its equation is* $f = x - yt = 0$*. Since* $\partial f/\partial x \neq 0$*, by the local diffeomorphism theorem* y, t *are local coordinates (indeed* $x = yt$*). Hence* $\hat{\mathbb{C}}^2$ *is smooth in* p*. Note that in this open set* E *is the divisor of the function* y*. If* $t_0 \neq 0$*, then in* $U_0 := \hat{\mathbb{C}}^2 \cap \{t_0 \neq 0\}$ *we shall use as local coordinates* $x, y, u = t_1/t_0$*; analogously to the previous case, the local equation of* $\hat{\mathbb{C}}^2$ *is* $g = y - xu$*, thus* x, u *are local coordinates and* $y = xu$*. Moreover, we shall remark that in the first open set* E *is the divisor of the function* x*.*

The last thing we have to prove is the second point of Definition 2.1. Basically, it means that $\varepsilon|_{\hat{\mathbb{C}}^2 \smallsetminus E}$ *is invertible. Indeed*

$$\varepsilon|^{-1}_{\hat{\mathbb{C}}^2 \smallsetminus E} : (x, y) \longmapsto ((x, y), (x : y)).$$

Definition 2.3. *Let* C *be an effective divisor on* S *with* $p \in C$*; locally* $C = (f)$*, with* $f(x, y)$ *regular function, where* x, y *are local coordinates such that* $p = (0, 0)$*. Then we shall consider the Taylor series of* f *in a neighbourhood of* p*;*

$$f = f_m(x, y) + f_{m+1}(x, y) + \dots, \qquad \text{where } f_k \in \mathbb{C}[x, y]_k \forall k \text{ and } f_m \not\equiv 0 :$$

in this case we define the multiplicity of C *in* p *to be equal to* m*.*

Definition 2.4. *If* $C = \sum a_i C_i \in \mathrm{Div}(S)$*, the* strict transform *of* C *is* $\hat{C} = \sum a_i \hat{C}_i$*, where* $\hat{C}_i := \overline{(\varepsilon|^{-1}_{\hat{S} \smallsetminus E})_*(C - p)}^{\hat{S}}$ *is a Cartier divisor in* $\hat{S}$*.*

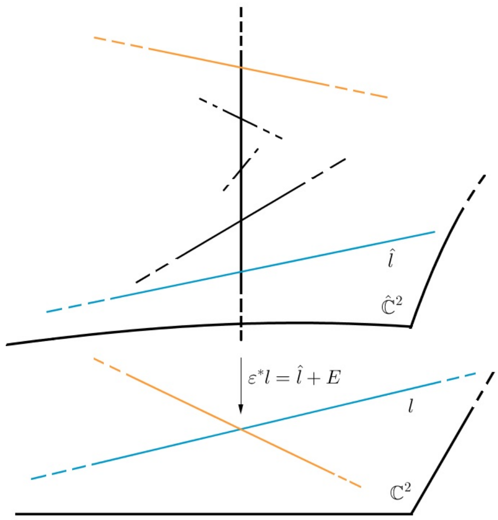
Example 2.5. *In the situation of Example 2.2, let us consider*

$$l = (ax + by = 0) \in \mathrm{Div}(\mathbb{C}^2),$$

then

$$\hat{l} = (ax + by = at_0 + bt_1 = 0),$$

thus $\hat{l} \pitchfork E = ((0,0), (-b : a))$*, thus the intersection point* $\hat{l} \cap E$ *determines a bijection among* E *and the set of the lines through* p*.*
More generally the same computation shows that the strict transform $\hat{C}$ *of every curve smooth at* p *is a curve isomorphic to* C *(through the projection) intersecting* E *transversally at the point given by the tangent direction of* C *at* p*.*



From now on we will assume S and $\hat{S}$ projective, in order to be able to consider their intersection forms.

Lemma 2.6. *Let* $C \in \mathrm{Div}(S)$ *effective and irreducible and suppose that its multiplicity in* p *is* m*, then*

$$\varepsilon^* C = \hat{C} + mE.$$

Proof. Clearly $\varepsilon^* C = \alpha\hat{C} + \beta E$ with $\alpha, \beta \in \mathbb{Z}$ (since if locally $C = (f)$, then $\varepsilon^* C = (\varepsilon^* f) = (f \circ \varepsilon)$ and $\varepsilon^{-1} C = \hat{C} \cup E$). Since ε is biregular out of p, then $\alpha = 1$ (multiplicity cannot change because of biregularity).

Now, let us assume that locally $C = (f)$ for some $f(x,y)$ rational function, where x, y are local coordinates such that $p = (0,0)$. Then

$$f = f_m(x,y) + f_{m+1}(x,y) + \dots, \qquad \text{where } f_k \in \mathbb{C}[x,y]_k \forall k \text{ and } f_m \not\equiv 0.$$

In an open neighbourhood of $q \in E$ with $t_1(q) \neq 0$ we have local coordinates y, t and $x = yt$, so

$$\begin{aligned} \varepsilon^* f &= f_m(yt, y) + f_{m+1}(yt, y) + \dots \\ &= y^m f_m(t,1) + y^{m+1} f_{m+1}(t,1) + \dots \\ &= y^m [f_m(t,1) + y f_{m+1}(t,1) + \dots] \end{aligned}$$

Hence $\varepsilon^* f$ vanishes with multiplicity m in $E = (y)$, thus $\beta = m$. □

Proposition 2.7. *The following properties hold:*

1. *The map*

$$\begin{array}{ccc} \mathrm{Pic}(S) \oplus \mathbb{Z} & \longrightarrow & \mathrm{Pic}(\hat{S}) \\ (C, n) & \longmapsto & \varepsilon^* C + nE \end{array}$$

   is a group isomorphism.

2. $\forall D, D' \in \mathrm{Pic}(S)$

$$\varepsilon^* D \cdot \varepsilon^* D' = D \cdot D';$$
$$E \cdot \varepsilon^* D = 0;$$
$$E^2 = -1.$$

3. $K_{\hat{S}} = \varepsilon^* K_S + E$.

Proof. 2. Since ε is biregular out of p, it is generically finite of degree 1 and therefore $\varepsilon^* D \cdot \varepsilon^* D' = D \cdot D'$. By the Theorem of Serre and Bertini every divisor is linearly equivalent to a divisor involving only curves not containing p, then $\mathrm{Supp}(\varepsilon^* D) \cap E = \varnothing$, hence $E \cdot \varepsilon^* D = 0$.

Let C smooth in p with multiplicity 1 in p. Then $\varepsilon^* C = \hat{C} + E$ by Lemma 2.6; moreover, using local coordinate, we get $\hat{C} \cdot E = 1$. Then

$$0 = E \cdot \varepsilon^* C = E \cdot (\hat{C} + E) = E \cdot \hat{C} + E^2 = 1 + E^2.$$

Hence $E^2 = -1$.

1. Injectivity: Suppose $\varepsilon^* D + nE = 0 \Rightarrow 0 = E(\varepsilon^* D + nE) = 0 - n \Rightarrow n = 0$. Hence $\varepsilon^* D \equiv 0$, thus, $D = \varepsilon_* \varepsilon^* D = \varepsilon_* 0 = 0$, where the first equality holds because ε is generically finite of degree 1.

   Surjectivity: First, we remark that E is the image of $(0, 1)$. Since $\mathrm{Pic}(\hat{S})$ is generated by irreducible curves, it is sufficient to show that the image contains every irreducible curve C, $C \neq E$; if $\varepsilon_* C$ has multiplicity m in p, it is immediate to see that

$$\varepsilon^* (\varepsilon_* C) = (\widehat{\varepsilon_* C}) + mE = C + mE,$$

   thus $(\varepsilon_* C, -m) \longmapsto C + mE - mE = C$.

3. Let ω be a rational form on S. Applying the pull back of forms (watch out: not the pull back of divisors) we get $\varepsilon^* \omega$, that is a 2–form on $\hat{S}$: $K_{\hat{S}} = (\varepsilon^* \omega) = \varepsilon^* (\omega) + \lambda E = \varepsilon^* K_S + \lambda E$. We want to prove that $\lambda = 1$. Since ε is biregular out of E, if a curve C is in the locus of the zeroes

(or of the poles) of ω, its strict transform appears in $(\varepsilon^*\omega)$ with the same multeplicity. Therefore, since $E \cong \mathbb{P}^1$, by Corollary 1.104

$$\begin{aligned} 0 = g(E) &= 1 + \frac{E^2 + KE}{2} \\ &= 1 + \frac{-1 + KE}{2}, \end{aligned}$$

hence $KE = -1$. Thus

$$-1 = E \cdot K_{\hat{S}} = (\varepsilon^* K_S + \lambda E) \cdot E = 0 - \lambda \Rightarrow \lambda = 1.$$

$\square$

Remark 2.8. *An alternative way to prove the last point of Proposition 2.7 is the following: let us consider the 2–form $\omega = dx \wedge dy$, where x, y are local coordinates on S such that $p = (0,0)$, then*

$$\varepsilon^*\omega = d(yt) \wedge dy = tdy \wedge dy + ydt \wedge dy = ydt \wedge dy$$

on the open set of $\hat{S}$ with local coordinates y, t. Hence $\varepsilon^*\omega$ vanishes with multiplicity 1 along $E = (y)$.

Corollary 2.9. *If $D \in Div(S)$ is irreducible with multiplicity m in p, then*

$$0 = E \cdot \varepsilon^* D = E \cdot (\hat{D} + mE) = E \cdot \hat{D} + mE^2 \Rightarrow E\hat{D} = m.$$

Exercise 2.10. *Prove that $\hat{D}^2 = D^2 - m^2$.*

The exceptional divisor E is said to be *rigid*, since there are no other effective divisors linearly equivalent to E. Indeed, let us suppose that $\sum a_i D_i \equiv E$ with $a_i > 0 \forall i$; then

$$0 > -1 = E^2 = E \cdot \left(\sum a_i D_i\right) = \sum a_i E \cdot D_i,$$

thus $\exists i$ such that $D_i = E$ (since if $D_i \neq E$, then $D_i \cdot E \geq 0$). Thus $\sum a_i D_i - E$ is effective and principal, and since we are working on a compact variety $E = \sum a_i D_i$.

Remark 2.11. *It is possible to prove by Meyer-Vietoris Theorem that $e(\hat{S}) = e(S) + 1$ and that $b_1(\hat{S}) = b_1(S)$, $b_3(\hat{S}) = b_3(S)$ [2], while $b_2(\hat{S}) = b_2(S) + 1$. In order to prove this, one exploits the fact that the Poincaré dual of the exceptional curve restricts on the curve itself to a form which is not exact (since its integral is $-1 \neq 0$).*

[2] Recall that the i–th *Betti number* is defined as $b_i(S) := \dim_{\mathbb{R}} H^i_{DR}(S, \mathbb{R})$

2.2 Rational maps

Let S be a smooth projective surface and $\Phi : S \dashrightarrow Y$ rational.
Let $U \subset S$ be an open set such that U is maximal among the open sets on which Φ is defined.

Proposition 2.12. *$F := S \setminus U$ is finite.*

Proof. Since S is compact, we just need to prove that F is discrete. Without loss of generality we shall suppose that S is affine (that is $S \subset \mathbb{C}^k$) and $Y = \mathbb{P}^r$;

$$\begin{aligned} \Phi &= \left(\frac{n_0}{d_0} : \cdots : \frac{n_r}{d_r}\right) \qquad \text{with } n_i, d_i \in \mathbb{C}[x_1, \ldots, x_k] \\ &= (p_0 : \cdots : p_r) \end{aligned}$$

We got the last expression of Φ multiplying every factor by $d_0 \cdots d_r$; the expression we have obtained for Φ is regular on the open set complement of the common zeroes of the polynomial p_i. Note that, if $\exists p \in \mathbb{C}[x_1, \ldots, x_k]$ such that $p | p_i \forall i$

$$\begin{aligned} \Phi &= (p_0 : \cdots : p_r) \\ &= \left(\frac{p_0}{p} : \cdots : \frac{p_r}{p}\right), \end{aligned}$$

gives an expression for Φ which is well defined on a bigger open set, hence we shall suppose that $\gcd(\{p_i\}) = 1$. Standard commutative algebra gives then the result, since a set of regular functions on a surface with infinitely many common zeroes have a common factor. □

This proposition has some important consequences. First of all, $\forall C \subset S$ the set $\Phi(C) = \overline{\Phi(C - F)}$ is well defined. Analogously, we can define $\Phi(S) := \overline{\Phi(S - F)}$.

This allows us to define a pull back map

$$\Phi^* : \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(S) \tag{2.1}$$

even for rational maps. Indeed the map

$$\begin{array}{ccc} \mathrm{Div}(S) & \longrightarrow & \mathrm{Div}(S - F) \\ C & \longmapsto & C - F \end{array}$$

is an isomorphism that preserves linear equivalence, thus we get an isomorphism $\mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(S - F)$; hence, we get

$$\Phi^* : \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(S - F) \xrightarrow{\sim} \mathrm{Div}(S).$$

These maps preserve linear equivalence, thus the map

$$\Phi^* : \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(S)$$

is well defined.

2.3 Linear systems

As we saw in a previous section, if S is a projective variety and $D \in \mathrm{Div}(S)$, the *complete linear system* associated to D is the set

$$\begin{aligned} |D| &:= \{D' \equiv D : D' \geq 0\} \\ &= \{(s) : s \in H^0(\mathcal{O}(D)) \smallsetminus \{0\}\} \cong \mathbb{P}^{h^0(D)-1}, \end{aligned}$$

where the latter isomorphism holds since

$$(s_1) = (s_2) \Longleftrightarrow \exists \lambda \in \mathbb{C}^* : s_1 = \lambda s_2.$$

If P is a projective subspace of $|D|$ it is called *linear system*, thus $P = (V \smallsetminus \{0\})/\mathbb{C}^*$, with $V \subset H^0(\mathcal{O}(D))$ is a vector space. We will sometimes write P^2 for D^2.

Definition 2.13. *Let P be a linear system; if $\exists C \in \mathrm{Div}(S)$ with $C \geq 0$ such that $\forall D \in P$, $C \leq D$, then C is said to be* in the fixed part *of P. If such C does not exist, we say that P* has no fixed part.

Obviously, $\forall P$ linear system $\exists! \Phi \in \mathrm{Div}(S)$ with $\Phi \geq 0$ such that Φ is in the fixed part of P and

$$P(-\Phi) := \{D - \Phi : D \in P\} \subset |D - \Phi|$$

has no fixed part. Such Φ is called *the fixed part of* P.

Though P has no fixed part, it may still have some points which belongs to all of its elements. If $\exists x \in S$ such that $\forall D = \sum a_i D_i \in P$ $\exists i$ such that $x \in D_i$ we say that x is a *base point* for P.

Example 2.14. *The point $(0:0:1) \in \mathbb{P}^2$ is a base point for the linear system $P := \{$lines through $(0:0:1)$ in $\mathbb{P}^2\}$.*

Remark 2.15. *By the Theorem of Bertini, if there exists $D_1, D_2 \in P$ with no common components, then*
$\{$*base points of* $P\} \subset D_1 \cap D_2 \Rightarrow \#\{$*base points of* $P\} \leq \# D_1 \cap D_2 \leq D^2$.

If $P^2 < 0$, P has a fixed part. If P has no fixed part and $P^2 = 0$, then P has no base points.

There exists a 1-to-1 correspondence

$$\left\{\begin{array}{l}\text{Linear system on } S\\ \text{with no fixed parts}\\ \text{and dimension } r\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Rational maps } \Phi : S \dashrightarrow \mathbb{P}^r\\ \text{non degenerate}\\ (\text{that is } \Phi(S) \notin \text{hyperplane})\end{array}\right\} \Big/ \mathrm{Aut}(\mathbb{P}^r),$$

where by *dimension* of P we mean the dimension of P as projective subspace. But how this bijection works?

For a fixed H hyperplane in $\mathbb{P}^r$

$$\Phi^*|H| \longleftarrow\!\shortmid \Phi;$$

vice versa

$$P = \mathbb{P}(\mathrm{span}(s_0, \dots, s_r)) \longmapsto (p \mapsto (s_0(p) : \dots : s_r(p))).$$

a different choice of a basis for P on the left would define the same map on the right modulo isomorphisms of $\mathbb{P}^r$.

Theorem 2.16 (Elimination of indeterminacy locus)**.** *Let $\Phi : S \dashrightarrow Y$ be a rational map from a projective surface to a projective variety. Then there exists a surface S', a morphism $\eta : S' \to S$ which is the composition of a finite number of blow ups such that the composition*

$$\begin{tikzcd} S' \arrow[rr] \arrow[dr, "\eta"'] & & Y \\ & S \arrow[ur, dashed, "\Phi"'] & \end{tikzcd}$$

is a morphism.

Proof. We may assume $Y = \mathbb{P}^n$; moreover we may suppose that $\Phi(S)$ lies in no hyperplane of $\mathbb{P}^n$. By the 1-to-1 correspondence we have seen before, Φ corresponds to a linear system $P \subset |D|$ with no fixed part in S.

Since $P \subset |D|$, $\#\{\text{base points of } P\} \leq D^2$.

If P has no base points, then $s_0(x), \dots, s_n(x)$ do not have common zeroes, hence $(s_0 : \dots : s_n)$ is defined on the whole S, thus Φ is a morphism (and the statement is true since $\eta := \mathrm{Id}_S$ is the composition of zero blow ups).

Otherwise, let $x \in S$ a base point for P and let us consider the blow up S_1 of S in x with the corresponding map $\varepsilon : S_1 \to S$. We get a map

$$\Phi_1 : S_1 \xrightarrow{\varepsilon} S \overset{\Phi}{\dashrightarrow} \mathbb{P}^n$$

$\forall D \in \mathrm{Div}(S)$, since $x \in D$, $\varepsilon^* D \geq E$ and the rational map Φ_1 is given by $(s_0 \circ \varepsilon, \dots, s_n \circ \varepsilon)$; more precisely the fixed part of $\varepsilon^* P := \{\varepsilon^* D : D \in P\}$ is λE for some $\lambda \geq 1$ and the linear system without fixed part $\varepsilon^* P(-\lambda E)$ induces exactly the map $\Phi \circ \varepsilon$.

If $\varepsilon^* P - \lambda E$ has no fixed points, then we are done. Otherwise, we repeat the process.

We need now to prove that this process stops in a finite number of steps.

How many base points does $\varepsilon^* P - \lambda E$ have?

$$\leq (\varepsilon^* D - \lambda E)^2 = D^2 - 2\lambda E \varepsilon^* D + \lambda^2 E^2 = D^2 - \lambda^2 < D^2,$$

hence the maximum number of base points is decreased. This means that the recursive construction of S' stops within P^2 steps. □

Note that this proof gives an explicit construction of S'.

Let us apply this theorem to the following example.

Example 2.17. *Let us consider the rational function*

$$\begin{array}{cccc} \Phi : & \mathbb{P}^1 \times \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^2 \\ & ((x_0 : x_1), (y_0 : y_1)) & \mapsto & (x_0 y_0 : x_0 y_1 : x_1 y_0) \end{array}$$

The linear system associated to Φ is $P \subset |(1,1)|$ (the complete linear system of all bihomogeneous polynomials of bidegree $(1,1)$). By Exercise 1.101, $P^2 = 2$, hence by Theorem 2.16, the resolution of Φ will take up to two blow ups.

First of all, notice that P has a unique base point, that is the only common zero of $x_0 y_0$, $x_0 y_1$ and $x_1 y_0$: $\bar{x} = ((0:1),(0:1))$. Thus, by the Theorem 2.16, we have to blow up $S = \mathbb{P}^1 \times \mathbb{P}^1$ at $\bar{x}$. Let then

$$\varepsilon : S_1 \to S$$

be the blow up of S in $\bar{x}$ and let us study $\Phi \circ \varepsilon$. $\varepsilon^* P$ has fixed part λE, where $\lambda = \min_{D \in P} m_{\bar{x}} D \geq 1$. Since, for example, $x_0 y_1$ vanishes with multiplicity 1 along E, we get that $\lambda = 1$, hence $\varepsilon^* P$ has fixed part E.

Therefore $\Phi \circ \varepsilon$ is induced by $\varepsilon^* P(-E)$. Let us compute it in local coordinates. In a neighbourhood of $\bar{x}$ the local coordinates are $x = x_0/x_1$ and $y = y_0/y_1$ and $\Phi = (xy : x : y)$.

Using the same notation of Example 2.2, we have an open set U_1 on S_1 with local coordinates y, t such that $x = yt$, hence $\varepsilon^* \Phi(y,t) = (y^2 t : yt : y)$. Removing E (that, in terms of the map $\varepsilon^* \Phi$, means that we divide by the equation of $E = (y)$) we get

$$\left(\frac{y^2 t}{y} : \frac{yt}{y} : \frac{y}{y} \right) = (yt : y : 1).$$

It is immediate to see that this map is defined on the whole $E \cap U_1$. The other open set on E we consider is U_0, the one with local coordinates x, u and $y = xu$. A computation similar to the previous one leads to

$$\varepsilon^* \Phi(x,u) = (x^2 u : x : xu)$$
$$\varepsilon^* P(-E) = \left(\frac{x^2 u}{x} : \frac{x}{x} : \frac{xu}{x}\right) = (xu : 1 : u).$$

This implies that this map is well defined on U_0 too. Hence $\varepsilon^* P(-E)$ has no base points on E; moreover, there are no base points out of E too, since the map ε is biregular out of the exceptional divisor.

Finally we observe that we resolved the map Φ with just one blow up, according to Theorem 2.16.

Exercise 2.18. *Resolve the following rational maps.*

1. Φ^{-1}, *where Φ is the map in the previous example.*

2. *The* Cremona transformation

$$\begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ (x_0 : x_1 : x_2) & \mapsto & \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right) \end{array} = (x_1 x_2 ; x_0 x_2 : x_0 x_1)$$

   and its inverse.

3.
$$\begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ (x_0 : x_1 : x_2) & \mapsto & (x_0 x_1 : x_0 x_2 : x_2^2) \end{array}$$

Watch out! In one of these examples some base points will crop up on E (these kind of points are called infinitely near base points*), hence you have to blow up those points too.*

Lemma 2.19. *Let S be a projective irreducible surface (possibly not smooth) and S' a projective smooth surface. Let $f : S \to S'$ be a birational morphism and $p \in S'$ such that f^{-1} is not defined in p. Hence $\exists C \subset S$ curve such that $f(C) = p$.*

Proof. Since S is projective, $f(S) = S'$, therefore $\exists q \in S$ such that $f(q) = p$. Let us choose an affine open set on S containing q (with a slight abuse of notation we will say that $S \subset \mathbb{A}^r$) and an affine open set on S' containing p (let us say that $S' \subset \mathbb{A}^n$). Now that we are working in an affine space, we can write the rational map f as ratio of polynomials:

$$f^{-1}(y_1, \dots, y_n) = (g_1, \dots, g_r), \qquad \text{with } g_i = \frac{u_i}{v_i} \text{ and } u_i, v_i \in \mathbb{K}[y_1, \dots, y_n].$$

We shall suppose that $\forall i$ u_i and v_i have no common factors. Since f^{-1} is not defined in p, $v_i(p) = 0$ for some $i = 1, \dots, n$. Let us suppose, without loss of generality, that $v_1(p) = 0$ and consider

$$D = f^*(v_1) \in \mathrm{Div}(S),$$

which is an effective divisor. Then

$$g_1 = (f^{-1})^* x_1 \Rightarrow x_1 = f^* \left(\frac{u_1}{v_1} \right) \Rightarrow (f^* u_1) \geq (f^* v_1),$$

where the latter implication holds because x_1 is a regular function without poles. Hence, as a set,

$$D = f^{-1}(\{u_1 = v_1 = 0\}) = f^{-1}(Z),$$

where Z is a discrete set (since by hypotheses u_1 and v_1 have no common factors) with $p \in Z$. Up to shrinking S', $Z = \{p\}$, then we can take an irreducible component of D as the curve C of the statement. $\square$

Lemma 2.20. *Let $\Phi : S \dashrightarrow S'$ be a birational map among two smooth projective surfaces. Let $p \in S'$ be such that Φ^{-1} is not defined in p, then $\exists$ a curve $C \subset S$ such that every point of C for which Φ is defined, is mapped into p (by a slight abuse of notation we say that $\Phi(C) = p$).*

Proof. Let us consider the following commutative diagram

$$\begin{tikzcd} & \Gamma(\Phi) \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ S \arrow[rr, dashed, "\Phi"'] & & S' \end{tikzcd}$$

where $\Gamma(\Phi) = \overline{\{(s, \Phi(s)) \in S \times S'\}}^{Zariski}$ is the closure of the *graph* of Φ, which is a projective surface. π is the projection on the first coordinate and it is a birational map; π' is the projection on the second coordinate and, since it is a composition of two birational maps, is birational.

$\Phi^{-1} = \pi \circ (\pi')^{-1}$, therefore $(\pi')^{-1}$ is not defined in p. The map (π') satisfies the hypotheses of Lemma 2.19, hence $\exists$ a curve $\hat{C} \subset \Gamma(\Phi)$ such that $\pi'(\hat{C}) = p$. We define $C := \pi(\hat{C})$. The only thing we need to prove is that C is a curve. Since $\hat{C}$ is a curve, C should be either a point or a curve. If $\pi(\hat{C}) = q \in S$, then $\hat{C} = \{(p, q)\}$, but by Lemma 2.19 $\hat{C}$ is a curve. This is a contradiction, hence C is a curve. $\square$

Theorem 2.21 (Universal property of blow up). *Let $f : X \to S$ be a birational morphism of smooth projective surfaces, and suppose that f^{-1} is not defined in $p \in S$. Let $\varepsilon : \hat{S} \to S$ be the blow up of S in p. Then there exists $g : X \to \hat{S}$ birational morphism such that $f = \varepsilon \circ g$, that is such that the diagram*

$$\begin{tikzcd} & \hat{S} \arrow[d, "\varepsilon"] \\ X \arrow[ur, "g"] \arrow[r, "f"'] & S \end{tikzcd}$$

commutes.

Sketch of the proof. We have to prove that $g := \varepsilon^{-1} \circ f$ is defined on the whole X. Arguing by contradiction, let us suppose that $\exists q \in X$ such that g is not defined; we can apply Lemma 2.20 to $s = g^{-1}$, then $\exists$ a curve $C \subset \hat{S}$ such that $s(C) = q$. Hence $\varepsilon(C) = f(s(C)) = f(q)$. Since E is the only curve of $\hat{S}$ that is contracted to a point by ε, we shall conclude that $C = E$; hence g is not defined in a single point $q = s(E)$. The proof now ends proving that f^{-1} would be defined in p by $f^{-1}(p) = q$, that would lead to a contradiction. □

Theorem 2.22. *Let $f : S \to S_0$ a birational morphism among two smooth surfaces. Then there exists a sequence of blow ups*

$$S_n \xrightarrow{\varepsilon_n} S_{n-1} \xrightarrow{\varepsilon_{n-1}} \cdots \xrightarrow{\varepsilon_2} S_1 \xrightarrow{\varepsilon_1} S_0$$

such that $f = \varepsilon_1 \circ \cdots \circ \varepsilon_n \circ u$ with $u : S \to S_n$ biregular.

Proof. If f is biregular, the theorem is trivially true with $n = 0$. Otherwise, we use Theorem 2.21 to get $f = \varepsilon_1 \circ f_1$; if f_1 is biregular we are done, otherwise we proceed recursively.

We have to prove that this recursive construction stops.

For each point $p \in S_0$ such that f^{-1} is not defined in p, let us consider the curves in the preimage of p: by compactness of S, we get a finite number of curves. Hence, the set of curves in S that are contracted by f to a point is finite. After each blow up, the cardinality of this set decreases (at least by one), hence the recursive procedure stops. □

Corollary 2.23. *If $S \dashrightarrow S'$ is birational among two projective smooth surfaces, then $\exists \hat{S}$ smooth surface, $q : \hat{S} \to S$ and $q' : \hat{S} \to S'$ sequence of blow ups such that the diagram*

$$\begin{tikzcd} & \hat{S} \arrow[dl, "q"'] \arrow[dr, "q'"] & \\ S \arrow[rr, dashed, "\Phi"'] & & S' \end{tikzcd}$$

commutes.

Proof. We construct q with Theorem 2.16, and then we apply Theorem 2.22 to $q' = \Phi \circ q$. □

The thing we are asking now is: when a surface can be seen as blow up of another surface?

Definition 2.24. *A surface S is said to be* minimal *if $\nexists\, S'$ projective smooth surface with $p \in S'$ such that S is the blow up of S' in p. Equivalently S is said to be minimal if every birational morphism $S \to S''$ is biregular.*

Let us prove the equivalence of the two definition.

- $\Leftarrow$ It is immediate using Theorem 2.22.
- $\Rightarrow$ Let us suppose there exists $f : S \to S''$ non biregular. Hence by Theorem 2.22 there exists a sequence $\varepsilon_1, \dots, \varepsilon_n$ with $n \geq 1$ such that $f = \varepsilon_1 \circ \dots \circ \varepsilon_n \circ u$ with $u : S \to S_n$ biregular. Hence S would be isomorphic to the blow up of S_{n-1}. Contradiction.

Definition 2.25. *Let S be a projective smooth surface. A curve $E \subset S$ is said o be* exceptional *if it is the exceptional divisor of a blow up $S \to S'$.*

The following theorem provides us with a powerful tool to understand whether a surface is minimal or not.

Theorem 2.26 (Castelnuovo contractibility criterion)**.** *$E \subset S$ is an exceptional curve$\Leftrightarrow$ $E \cong \mathbb{P}^1$ and $E^2 = -1$.*

First part of the proof. ($\Rightarrow$) is immediate. □

Let us see some consequence of the trivial part of Castelnuovo criterion.

- $\forall C \subset \mathbb{P}^2$ curve, $C^2 = (\deg C)^2 > 0$, hence $\mathbb{P}^2$ is minimal.
- $\forall C \subset \mathbb{P}^1 \times \mathbb{P}^1$ curve, $C^2 \geq 0$ by Exercise 1.101, hence $\mathbb{P}^1 \times \mathbb{P}^1$ is minimal.

Castelnuovo criterion implies the following

Corollary 2.27. *S is minimal$\Leftrightarrow\nexists$ $E \subset S$ smooth such that $E \cong \mathbb{P}^1$ and $E^2 = -1$.*

Lemma 2.28 (Serre vanishing Theorem)**.** *If $H \in \mathrm{Div}(S)$ is very ample, then $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ $h^1(nH) = 0$.*

Second part of the proof of Castelnuovo criterion. Let us suppose that $\exists$ a curve $E \subset S$ such that $E \cong \mathbb{P}^1$ and $E^2 = -1$. By Exercise 1.96, Lemma 2.28 implies that $\exists H$ very ample such that $h^1(\mathcal{O}(H)) = 0$. Let us say that $H \cdot E = k > 0$. We want to prove that the map induced by the linear system $|H + kE|$ is a blow up whose exceptional divisor is E.

$\forall 1 \le i \le k$ let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(H + (i-1)E) \longrightarrow \mathcal{O}_S(H + iE) \longrightarrow \mathcal{O}_E(H + iE) \longrightarrow 0.$$

Since $E \cong \mathbb{P}^1$ and $(H+iE)\cdot E = k-i$, $\mathcal{O}_E(H+iE) \cong \mathcal{O}_{\mathbb{P}^1}(k-i)$. Moreover, since $k - i \ge 0$, by Serre duality $h^1(\mathcal{O}_E(H + iE)) = 0$. Therefore we can consider the long exact sequence

$$\begin{aligned} \ldots \longrightarrow H^0(\mathcal{O}_S(H + iE)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(k - i)) \longrightarrow \\ \longrightarrow H^1(\mathcal{O}_S(H + (i-1)E)) \longrightarrow H^1(\mathcal{O}_S(H + iE)) \longrightarrow 0, \end{aligned}$$

where the last 0 stands for $H^1(\mathcal{O}_E(H + iE))$.

For $i = 1$ the right part of the previous sequence is

$$H^1(\mathcal{O}_S(H)) \longrightarrow H^1(\mathcal{O}_S(H + E)) \longrightarrow 0;$$

since the left term of the exact sequence is 0 by hypothesis, $H^1(\mathcal{O}_S(H+E)) = 0$. By the same argument, we get that $H^1(\mathcal{O}_S(H+iE)) = 0 \ \forall 1 \le i \le k$. Hence we get a short exact sequence

$$0 \longrightarrow H^0(H + (i-1)E) \longrightarrow H^0(H + iE) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(k - i)) \longrightarrow 0. \quad (2.2)$$

Choose a basis $s_0, \ldots, s_n$ of $H^0(H)$, pick a generator s of $H^0(E)$ and for $1 \le i \le k$ elements $a_{i,0}, \ldots, a_{i,k-i} \in H^0(\mathcal{O}_S(H + iE))$ that are mapped onto a basis of $H^0(\mathcal{O}_{\mathbb{P}^1}(k-i))$ (they exist because the map $H^0(H+iE) \to H^0(\mathcal{O}_{\mathbb{P}^1}(k-i))$ is surjective). Then

$$\{s^k s_0, \ldots, s^k s_n, s^{k-1}a_{1,0}, \ldots, s^{k-1}a_{1,k-1}, \ldots, s a_{k-1,1}, a_{k,0}\}$$

is a basis of $H^0(H')$, where $H' = H + kE$. Let $\Phi : S \dashrightarrow \mathbb{P}^N$ be the rational map induced by the linear system $|H'|$. s vanishes on E, while $a_{k,0}$ induces a non zero constant function on E, therefore $\Phi(E)$ is a point, and precisely the point $(0 : 0 : \cdots : 1)$; the map

$$\begin{array}{rccc} \eta : & S - E & \longrightarrow & \mathbb{P}^n \\ & p & \longmapsto & (s^k s_0(p) : \cdots : s^k s_n(p)) \end{array}$$

is the embedding in $\mathbb{P}^n$ associated to $|H|$, since $s \neq 0$ and $\eta = (s_0 : \cdots : s_n)$. This means that the composition

$$\begin{tikzcd} S - E \arrow[r, "\Phi"] \arrow[dr, "\eta"'] & \mathbb{P}^N \arrow[d, dashed] \\ & \mathbb{P}^n \end{tikzcd}$$

is an embedding, hence $\Phi|_{S-E}$ is an embedding too.

The only thing we left to prove is that the image of Φ is smooth. For this part of the proof, that we skip, we need to exploit the hypothesis for which $E^2 = -1$. $\square$

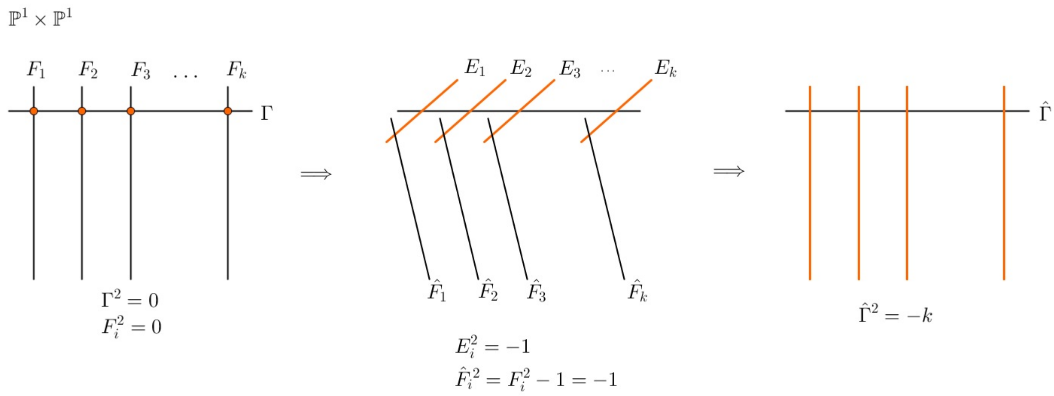


Figure 2.1: In $\mathbb{P}^1 \times \mathbb{P}^1$ let us consider the $k+1$ lines $\Gamma = \mathbb{P}^1 \times (0:1)$, and $F_i = (i:1) \times \mathbb{P}^1$, $i \in \mathbb{N}$, $1 \leq i \leq k$, that intersect as sketched in the figure to the left. Blowing up the surface in the k points of intersection we get the situation outlined in the second figure. By Exercise 2.10 $\hat{F_i}^2 = -1$, hence, by Castelnuovo criterion, we can contract these curves to a point. The surface we get is called *Hirzebruch surface* $\mathbb{F}_k$.

Exercise 2.29. *Let $C \subset S$ a curve in a surface, let $\varepsilon : \hat{S} \to S$ be the blow up of S in $p \in S$ and let $\hat{C}$ be the strict transform of C.*

1. *If $p \notin C$, then $p_a(C) = p_a(\hat{C})$;*
2. *If $p \in C$ is smooth for C, then $p_a(C) = p_a(\hat{C})$;*

3. *Otherwise* $p_a(C) > p_a(\hat{C})$.

   Fact 2.30. $\forall C$ *irreducible* $p_a(C) \geq 0$.

4. $\forall C \subset S$ *irreducible curve in a smooth projective surface* $\exists$ *a sequence of blow ups* $\varepsilon : S' \to S$ *such that the strict transform of* C *is smooth.*

5. $C \subset S$ *as above,* $p_a(C) = 0 \Longrightarrow C \cong \mathbb{P}^1$ *(hence it is smooth).*

Castelnuovo criterion suggests a way to construct new surfaces from old ones: if blowing up some points the strict transform E of a curve has $KE = E^2 = -1$, then by the last exercise is a smooth $\mathbb{P}^1$ and therefore we can contract it to a smooth new surface.

The first application of this idea is the construction of the Hirzebruch surfaces in Figure 2.1 and 2.2 below, which produces the first examples of ruled surfaces, which are the objects of the next chapter.

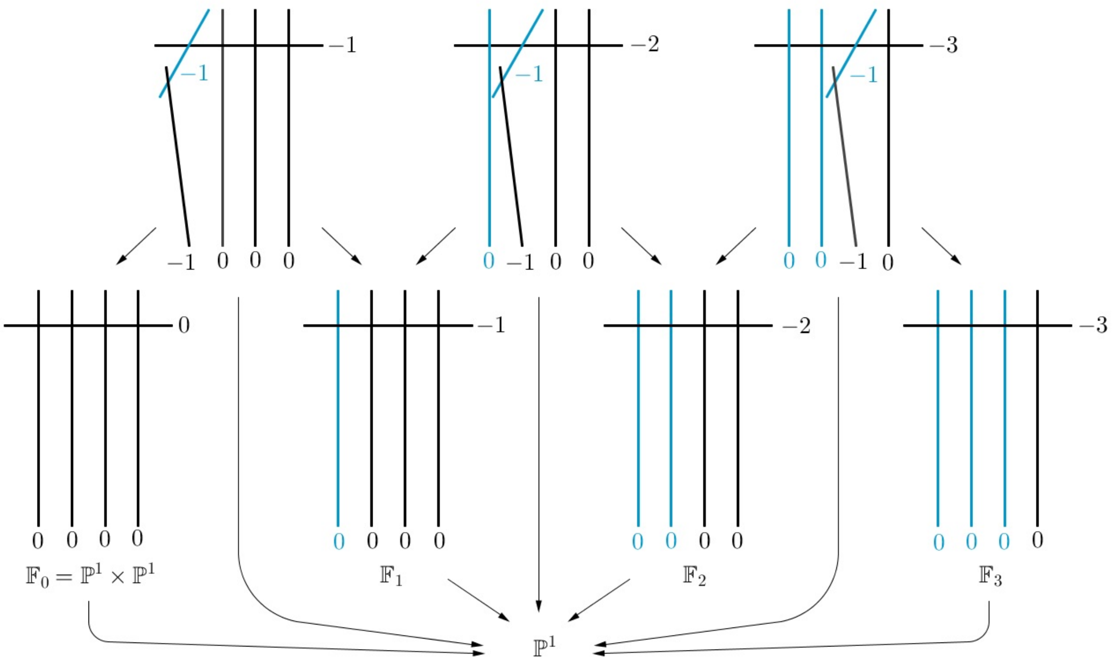


Figure 2.2: Construction of some Hirzebruch surfaces.

Chapter 3

Ruled surfaces

Definition 3.1. *S is said to be* rational *if it is birational to $\mathbb{P}^2$ (or to $\mathbb{P}^1 \times \mathbb{P}^1$).*

Definition 3.2. *S is said to be* ruled *if it is birational to $C \times \mathbb{P}^1$ with C smooth curve.*

Definition 3.3. *S is said to be* geometrically ruled *if $\exists \pi : S \to C$ surjective morphism on a smooth curve C such that every fibre is a smooth curve isomorphic to $\mathbb{P}^1$.*

For example, Hirzebruch surfaces are geometrically ruled.

Remark 3.4. *Let S be a smooth projective surface and C a smooth projective curve. Let $p : S \to C$ be a surjective morphism and $p_1, p_2 \in C$; let us define $F_{p_i} = F_i := p^* p_i$. Then $\forall D \in \mathrm{Div}(S)$ $D \cdot F_1 = D \cdot F_2$. This is obvious if $C \cong \mathbb{P}^1$ since in this case F_1 and F_2 are linearly equivalent; in the general case we can write (as shown in the first chapter by Theorem 1.97 and Theorem 1.98) $D \equiv A - B$ for some smooth $A, B \in \mathrm{Div}(S)$, hence by linearity we may assume D smooth. Then we can conclude by noticing that $p|_D : D \to C$ is a finite morphism and $D \cdot F_i = \deg(F_i|_D) = \deg(p_{|D})$ does not depend on i.*

Theorem 3.5 (Noether-Enriques)**.** *Let S be a smooth projective surface and let $\pi : S \to C$ be a morphism onto a smooth curve with one fiber F_x smooth and isomorphic to $\mathbb{P}^1$. Then $\exists U \subset C$ Zariski open set and a biregular map $\psi : \pi^{-1}(U) \to U \times \mathbb{P}^1$ such that the diagram*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times \mathbb{P}^1 \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

commutes. Moreover $h^2(\mathcal{O}_S) = 0$.

In order to prove Theorem 3.5 we need the following lemma that we state without proof.

Lemma 3.6. *Under the assumptions of Theorem 3.5* $\exists H \in \operatorname{Div}(S)$ *such that* $HF = 1$*, where* F *is a fiber of* π*.*

Proof of Theorem 3.5. First of all $F^2 = 0$ and $F \cong \mathbb{P}^1$, hence, by the genus formula,

$$0 = 1 + \frac{KF + F^2}{2} \Longrightarrow KF = -2.$$

Arguing by contradiction, let us suppose that $h^2(\mathcal{O}_S) > 0$; by Serre duality, this means that $\exists K$ effective canonical divisor. Writing $K = \sum a_i D_i$ with D_i irreducible curve and $a_i > 0$ $\forall i$; therefore

$$KF = \left(\sum a_i D_i\right) \cdot F = \sum a_i (D_i \cdot F).$$

If $F \neq D_i$ then $F \cdot D_i \geq 0$, if $F = D_i$ then $F \cdot D_i = F^2 = 0$; hence $KF \geq 0$. Contradiction! Thus $h^2(\mathcal{O}_S) = 0$.

Let $H \in \operatorname{Div}(S)$ be the divisor whose existence is guaranteed by Lemma 3.6, fix a fiber F_x, and consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(H + (r-1)F_x) \longrightarrow \mathcal{O}_S(H + rF_x) \longrightarrow \mathcal{O}_F(H + rF_x) \longrightarrow 0$$

Since $\mathcal{O}_{F_x}(H+rF_x) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, by Serre duality we get that $h^1(\mathcal{O}_{F_x}(H+rF_x)) = 0$. Hence we get the long exact sequence

$$H^0(H + rF_x) \xrightarrow{\alpha_r} H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \longrightarrow H^1(H + (r-1)F_x) \xrightarrow{\beta_r} H^1(H + rF_x) \longrightarrow 0.$$

The maps β_r are all surjective, hence $\{h^1(H + rF_x)\}_r$ is a non increasing sequence of natural numbers. Thus $\exists r_0$ such that $\forall r \geq r_0$ β_r is an isomorphism. Hence β_r is injective and then α_r is surjective for r sufficiently large. Therefore for a fixed r sufficiently large we can consider the exact sequence

$$H^0(H + rF_x) \xrightarrow{\alpha_r} H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \longrightarrow 0.$$

Let $V \subset H^0(H + rF_x)$ be a vector space of dimension 2 such that $\alpha_r(V) = H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathbb{C}^2$; let $P \subset |H'|$ be the corresponding linear system of dimension 1, where $H' := H + rF_x$. $|\mathcal{O}_{\mathbb{P}^1}(1)|$ is base point free, and therefore P has neither base points on F_x nor F_x is in the fixed part of P.

Let D be a curve, then (as before, since F_x is irreducible with $F_x^2 \geq 0$), for every fiber F (by Remark 3.4) $D \cdot F \geq 0$. If D is in the fixed part of P, then $DF_x = 0$; indeed, if $DF > 0$, then $D \cap F \neq \varnothing$ and a point of this intersection would be a base point for P restricted to F_x, a contradiction. Let $p \in D$, then

$p \in F_y$ for some $y \in \mathbb{P}^1$; if $D \notin F_y$, then $D \cdot F_y > 0$. Contradiction. Hence each fixed component of P is contained in a fiber.

Let $\Phi : S \dashrightarrow \mathbb{P}^1$ be the rational map induced by P; we want to construct an open set $U \subset C$ such that $\Phi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to \mathbb{P}^1$ is well defined. To get U we remove from C the images of the base points, those of the fixed part of P, and the images of the reducible fibers (that is those fibers that can be written in the form $\sum_{i=1}^{k} a_i D_i$ with $\sum a_i \geq 2$). Let us remark that the remaining are irreducible, $F^2 = 0$ and $FK = -2$, hence by the genus formula they all are smooth and isomorphic to $\mathbb{P}^1$.

Finally we get the maps

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi|_{\pi^{-1}(U)}"] \arrow[d, "\pi"'] & \mathbb{P}^1 \\ U & \end{tikzcd}$$

let then $\psi := (\pi, \Phi) : \pi^{-1}(U) \longrightarrow U \times \mathbb{P}^1$. By definition ψ makes the diagram in the statement commute. In order to prove that ψ is an isomorphism, let us construct its inverse.

Let $(x', t) \in U \times \mathbb{P}^1$; for a point $s \in \pi^{-1}(U)$, then $\Psi(s) = (x', t) \Leftrightarrow s \in F_{x'} \cap \Phi^* t$; $\varphi^* t \in P \subset |H'|$. Since $F_{x'}$ is smooth by the choice of U, and $H' \cdot F = 1$, then either $F_{x'} \subset \Phi^* t$, or $\Phi^* t$ and $F_{x'}$ intersect transversely in a single point. Therefore we need to prove that $\forall D \in P$, $\forall x' \in U$ $D \ngeq F_{x'}$. Arguing by contradiction, if $\exists D \in P$ such that $D \geq F_{x'}$, then the map

$$V \longrightarrow H^0(\mathcal{O}_{F'_x}(H')) \cong \mathbb{C}^2$$

would have a non trivial kernel. Hence its image would be of dimension ≤ 1 which implies that P would have a base point on $F_{x'}$. We get a contradiction. □

Theorem 3.5 implies that a geometrically ruled surface is ruled. The inverse is not true.

Lemma 3.7 (Zariski). *Let $p : S \to C$ be a surjective morphism among a smooth projective surface S and a smooth projective curve C with connected fibers (i.e. $\forall x \in C$, F_x is connected). Assume $F = \sum_{i=1}^{k} n_i C_i$ with $k \geq 2$ and $n_i \geq 0 \forall i$. Then $C_i^2 < 0$ $\forall i$.*

Proof.

$$0 = C_i \cdot F = \sum_{j=1}^{k} n_j C_i \cdot C_j = n_i C_i^2 + \sum_{j \neq i} n_j C_j \cdot C_i.$$

Since by hypotheses the fibers are connected, $\forall i \exists j \neq i$ such that $C_i \cdot C_j > 0$. Thus
$$\sum_{j \neq i} n_j C_i \cdot C_j > 0 \Rightarrow n_i C_i^2 + \sum_{j \neq i} n_j C_i \cdot C_j > n_i C_i^2 \Rightarrow 0 > n_i C_i^2.$$
□

Remark 3.8. *Assuming the hypotheses of Lemma 3.7, if $k = 1$, then $F = nC$ for $n \geq 2$, hence*
$$0 = F^2 = n^2 C^2 \Rightarrow C^2 = 0.$$

Proposition 3.9. *Let $p : S \to C$ as in Lemma 3.7, with a fiber $F_x \cong \mathbb{P}^1$. S minimal $\Rightarrow$ S is geometrically ruled, that is $\forall x \in C$ F_x is a smooth irreducible divisor and $F_x \cong \mathbb{P}^1$.*

Proof. $F_x^2 = 0$ and $F_x \cong \mathbb{P}^1$ hence, by the genus formula, $KF = -2$. By Exercise 2.29, every irreducible fiber is smooth and $\cong \mathbb{P}^1$. Let us suppose that there exists a fiber $F = \sum_{i=1}^k n_i C_i$ with $\sum_{i=1}^k n_i \geq 2$.

$k \geq 2$. By Lemma 3.7 $C_i^2 < 0$, thus, by Exercise 2.29, $KC_i \geq -1$. If $KC_i = -1$, then $C_i^2 = -1 \Rightarrow C_i \cong \mathbb{P}^1$. This contradicts the minimality of S. Hence $KC_i \geq 0$ $\forall i$; hence
$$-2 = KF = K \sum n_i C_i = \sum n_i K C_i \geq 0.$$
Contradiction.

$k = 1$ $\Leftrightarrow$ $F = nC$: we want to prove that $n = 1$. Let us suppose that $n \geq 2$, then
$$-2 = KF = nKC \Rightarrow n = 2, KC = -1,$$
but $C^2 = 0$, therefore $p_a(S) \notin \mathbb{Z}$. Contradiction.

□

Theorem 3.10. *Let C be an irrational (that is $g(C) > 0$) smooth curve. Then S is a minimal surface ruled on C (that is birational to $\mathbb{P}^1 \times C$) $\Longleftrightarrow$ S is geometrically ruled on C.*

Lemma 3.11. *Let C_1, C_2 be two smooth curves, $p : C_1 \to C_2$ non constant. Then $g(C_1) \geq g(C_2)$.*

Proof. Let us consider the pull back of regular 1–form as a linear map
$$\begin{array}{rccc} p^* & : H^0(\Omega_{C_2}) & \longrightarrow & H^0(\Omega_{C_1}) \\ & \omega & \longmapsto & p^*\omega \end{array};$$
this map is injective, since it is easy to see that if $\omega \not\equiv 0$ then $p^*\omega \not\equiv 0$. Therefore, since $H^0(\Omega_{C_2}) \cong \mathbb{C}^{g(C_2)}$ and $H^0(\Omega_{C_1}) \cong \mathbb{C}^{g(C_1)}$, we get the thesis.
□

Proof of Theorem 3.10. ⇒ Let us consider the following commutative diagram

$$\begin{tikzcd} S \arrow[r, dashed, "\text{bir.}"] \arrow[dr, dashed] & C \times \mathbb{P}^1 \arrow[d] \\ S' \arrow[u] \arrow[r] & C \end{tikzcd}$$

where the map $S' \to S$ is a sequence $\varepsilon_n \circ \cdots \circ \varepsilon_1$ of n blow ups that resolve the map $S \dashrightarrow C$. Let us suppose n minimal. If $n > 0$, then $\exists E \in \mathrm{Div}(S')$ irreducible such that $E^2 = KE = -1$ ($\Rightarrow E \cong \mathbb{P}^1$) the exceptional divisor of the ε_n. By Lemma 3.11, E is mapped onto a point of C, this means that $\varepsilon_{n-1} \circ \cdots \circ \varepsilon_1$ also resolves $S \dashrightarrow C$, contradicting the minimality of n. Hence $n = 0$ and $S \to C$ is a morphism and therefore by Proposition 3.9, we get the thesis.

⇐ Arguing by contradiction, let us suppose S is not minimal. Then there exists $E \in \mathrm{Div}(S)$ with $E^2 = KE = -1$, $E \cong \mathbb{P}^1$. Let $p : S \to C$ be the ruling morphism. By Lemma 3.11 $p(E) = x \in C$, thus $E \subset F_x \Rightarrow E = F_x$, but $E^2 = -1$ while $F^2 = 0$.

□

By Noether-Enriques Theorem the geometrically ruled surfaces on C are bundles on C with fiber $\mathbb{P}^1 = (\mathbb{C}^2 \smallsetminus \{0\})/\mathbb{C}^*$. If $\mathcal{E} \to C$ is a complex vector bundle of rank 2 and s_0 is he zero section of the bundle, then

$$\mathbb{P}(\mathcal{E}) := (\mathcal{E} \smallsetminus \{s_0(C)\})/\mathbb{C}^* \longrightarrow C$$

is indeed a geometrically ruled surface and one can prove that they all can be constructed in this way.

Hence the classification of complex vector bundle of rank 2 on a Riemann surface leads to a classification of geometrically ruled surfaces.

Example 3.12. *Theorem 3.10 does not hold for a rational curve (that is $g(C) = 0$). Let us see some counterexamples.*

- $\mathbb{P}^2$ *is minimal, it is not geometrically ruled, but it is ruled on* $\mathbb{P}^1$.
- *The Hirzebruch surface* $\mathbb{F}_1$ *is geometrically ruled, but it is not minimal.*

The following formula, that we give without proof, will be useful for the next results.

Proposition 3.13 (Adjunction formula)**.** *Let X be smooth and $Y \subset X$ a (smooth) divisor, then*

$$K_Y = (K_X + Y)|_Y \in \mathrm{Pic}(Y) \tag{3.1}$$

What we are really doing in this operation is taking a divisor *linearly equivalent* to Y, adding it to the canonical divisor of X and restricting the result to Y.

Theorem 3.14. *Let $p : S \to C$ be geometrically ruled, let $H \in \mathrm{Div}(S)$ be such that $H \cdot F = 1$ (it exists by Lemma 3.6). Moreover, we can pick H such that $H > 0$ and $H^0(H - F_x) = 0$ $\forall x$).*

1. *The map*
$$\begin{array}{ccc} \mathrm{Pic}(C) \times \mathbb{Z} & \longrightarrow & \mathrm{Pic}(S) \\ (\delta, n) & \longmapsto & p^*\delta + nH \end{array}$$
is an isomorphism.

2. $b_2(S) := h^2_{DR}(S) = 2$.

3. $K_S = -2H + p^*\delta$ *with* $\deg \delta = H^2 + 2g(C) - 2$

Proof. 1. The non trivial part is surjectivity of the map. Let $D \in \mathrm{Pic}(S)$; then D is in the image of the map $\Leftrightarrow D - (D \cdot F)H$ is in the image of the map. Since $(D - (D \cdot F)H) \cdot F = 0$, we can suppose $D \cdot F = 0$. Let us fix a fiber F_x and define $D_n := D + nF_x = D + p^*(nx)$ $\forall n$. We want to prove that D_n is effective for some n. By Riemann Roch Theorem for surfaces 1.102

$$\chi(\mathcal{O}_S(C)) = \chi(\mathcal{O}_S) + \frac{C^2 - KC}{2}.$$

By the definition of D_n, $D_n^2 - KD_n = D^2 - KD - nKF_x = D^2 - KD + 2n$, hence $\chi(D_n) = \chi(D) + n$. Thus $h^0(D_n) + h^2(D_n) \geq \chi(D_n) = \chi(D) + n$; but $h^0(D_n) + h^2(D_n) = h^0(D_n) + h^0(K - D_n)$, therefore, since $h^0(K - D_n)$ decreases or remains constant as n increases, for n sufficiently big $h^0(D_n) > 0$. Thus $\exists n_0$ such that $\forall n \geq n_0$ $h^0(D_n) > 0$.

Replacing D with D_n, it is enough to prove that $\forall D > 0$ such that $D \cdot F = 0$, then D is in the image of the map. Let then $D = \sum a_i C_i$ with $a_i > 0$ $\forall i$. Then

$$0 = D \cdot F = \sum a_i C_i \cdot F,$$

and, since $C_i \cdot F \geq 0$ $\forall i$, we conclude that $C_i \cdot F = 0$ $\forall i$. Hence $\forall i$ $\exists x \in C$ such that $C_i \subset F_x$. But $\forall x$ $F_x \cong \mathbb{P}^1$ is irreducible, hence $C_i = F_x = p^*x$, thus $D = p^*(\sum a_i x_i)$.

2. To prove this statement, we use Leray-Hirsch Theorem, that states that if S is ruled over C with fiber $\mathbb{P}^1$ (which is topologically homeomorphic to the sphere S^2) then

$$b_2(S) = b_2(C)b_0(\mathbb{P}^1) + b_1(C)b_1(\mathbb{P}^1) + b_0(C)b_2(\mathbb{P}^1). \tag{3.2}$$

Recalling that each compact connected orientable manifold of real dimension 2 has $b_0 = b_2 = 1$ and that $b_1(S^2) = 0$, equation (3.2) implies that $b_2(S) = 2$.

3. By the first point of the Theorem, $K_S = aH + p^*\delta$ for some $a \in \mathbb{Z}$ and $\delta \in \mathrm{Pic}(C)$. By the genus formula $K_S \cdot F = -2$; thus

$$-2 = F \cdot (aH + p^*\delta) = a + 0,$$

that is $a = -2$. Now let us compute $\deg \delta$. H intersects each fiber in a single point, hence the map

$$p|_H : H \to C$$

is an isomorphism. Therefore

$$(p|_H)^* K_C = K_H = (K + H)|_H = (-H + p^*\delta)|_H,$$

where the second equality holds for (3.1). Hence

$$2g(C)-2 = \deg K_C = \deg K_H = H(-H+p^*\delta) = -H^2+(p^*\delta)\cdot H = -H^2+\deg \delta.$$

□

3.1 Numerical invariants

To every projective surface S we can associate several integers.

- The *Betti numbers* $b_i(S) := h^i_{DR}(S)$. $b_i = 0\ \forall i \geq 0, i > 4$, moreover, by Poincaré duality, $b_1 = b_3$ and $b_0 = b_4$. Hence Euler-Poincaré characteristic of S is

$$\begin{aligned} e &= b_0 - b_1 + b_2 - b_3 + b_4 \\ &= 2b_0 - 2b_1 + b_2. \end{aligned}$$

- $h^0(\mathcal{O}_S) = 1$, since S is projective and therefore compact. Being equal for all surfaces, this number is not very interesting.

- The *Irregularity* $q(S) := h^1(\mathcal{O}_S)$
- The *geometric genus* $p_g(S) := h^2(\mathcal{O}_S) = h^0(K_S)$
- The *n−th plurigenus* $P_n := h^0(\mathcal{O}_S(nK))$

It follows

$$\chi(\mathcal{O}_S) = 1 - q + p_g.$$

The following equation comes from Hodge theory and we state it without proof.

Fact 3.15. $q(S) = h^1(\mathcal{O}_S) = h^0(\Omega^1_S) = b_1/2.$

Recall also the Noether's formula (1.11):

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + e(S)).$$

Proposition 3.16. *The integers p_g, q, P_n are birational invariants.*

Sketch of the proof. The proof follows the same idea of the proof of Lemma 3.11. We prove Proposition 3.16 only for p_g (the birational invariance of q and P_n is proved in the same way). Let $\Phi : S \dashrightarrow S'$ be a birational map. If $\omega \in H^0(\Omega^2_{S'})$ and $\omega \not\equiv 0$, then $\Phi^*\omega$ (the pull back of ω as a form) is a non zero regular 2−form on $U \subset S$, where U is the open set where Φ is well defined. By Proposition 2.12 $S \setminus U$ is finite. By results of complex analysis in several variables, a holomorphic form in two variables cannot have isolated poles, hence, by Hartogs' theorem, $\Phi^*\omega$ can be extended to a form in $H^0(\Omega^2_S)$, therefore we have a linear injective map

$$\Phi^* : H^0(\Omega^2_{S'}) \hookrightarrow H^0(\Omega^2_S).$$

Hence $p_g(S) \leq p_g(S')$. Arguing in the same way, using Φ^{-1} instead of Φ we prove the opposite inequality. Thus $p_g(S) = p_g(S')$. □

Proposition 3.17. *Let S be a ruled surface over C; then*

$$q(S) = g(C); \quad p_g(S) = 0; \quad P_n(S) = 0 \quad \forall n \geq 2; \quad b_1 = b_3 = 2q.$$

If S is geometrically ruled, then

$$K_S^2 = 8(1 - g(C)) = 8(1 - q); \quad b_2 = 2.$$

Proof. First of all, we may suppose S geometrically ruled, since, by Proposition 3.16, irregularity, geometrical genus and plurigenera are birational invariants, and by Remark 2.11 the first and the third Betti numbers are birational invariants. Hence by Noether-Enriques theorem 3.5 $p_g = 0$ and $b_2 = 2$ by Theorem 3.14. Now suppose $P_n > 0$; then $\exists nK > 0$, but $nK \cdot F = -2n < 0$. This is a contradiction since F is irreducible and $F^2 \geq 0$.

By Theorem 3.14

$$\begin{aligned} K^2 &= (-2H + p^*\delta)^2 \qquad\qquad \text{with } \deg\delta = H^2 + 2g - 2 \\ &= 4H^2 - 4Hp^*\delta + (p^*\delta)^2 \\ &= 4(H^2 - \deg\delta) = 4(2-2g) = 8(1-g). \end{aligned}$$

Remark 3.18. *Let us consider* $\omega \in H^0(\Omega^1_C)$*, then* $p^*\omega \in H^0(\Omega^1_S)$ *(that is,* $p^*\omega$ *has no poles on* S*). Hence we have an injective map* $p^* : H^0(\Omega^1_C) \hookrightarrow H^0(\Omega^1_S)$*, therefore* $g(C) \leq q(S)$*.*

By Fact 3.15 we get

$$\begin{aligned} \frac{b_1}{2} = q(S) = p_g + 1 - \chi(\mathcal{O}_S) &= 1 - \frac{1}{12}(K^2 + e) \\ &= 1 - \frac{1}{12}(8(1-g) + e) \\ &= 1 - \frac{1}{12}(8(1-g) + 2b_0 - 2b_1 + b_2) \\ &= 1 - \frac{1}{12}(12 - 8g - 2b_1), \end{aligned}$$

hence $6b_1 = 8g + 2b_1$, that is $g(C) = b_1/2 = q(S)$. □

Proposition 3.17 has many consequences; for example, a surface cannot be ruled over two curves with different genera.

Chapter 4

Rational surfaces

If S is a ruled surface, then there exists a birational map $S \dashrightarrow C \times \mathbb{P}^1$ for some curve C. If S is *irregular* (that is $q(S) > 0$), then S is minimal$\Leftrightarrow$ it is geometrically ruled. If S is regular, then S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ (or equivalently to $\mathbb{P}^2$), that is S is *rational*.

Let us then focus our attention on rational surfaces. We have already seen in Figure 2.1 how to construct the Hirzebruch surfaces $\mathbb{F}_n$. The following proposition underlines the importance of such surfaces.

Proposition 4.1. *Let S a rational surface geometrically ruled. Then $\exists n$ such that $S \xrightarrow{\sim} \mathbb{F}_n$.*

Proof. Since S is rational and geometrically ruled, there exists $p : S \to \mathbb{P}^1$, and, by Lemma 3.6, there exists $H \in Div(S)$ with $H > 0$ and $H \cdot F = 1$. Then

$$h^0(H) = h^0(H) + h^2(H) \geq \chi(H),$$

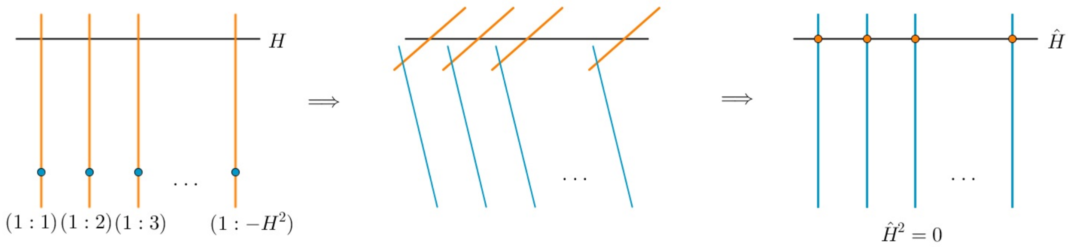
where the first equality holds because $h^2(H) = h^0(K - H) \leq h^0(K) = 0$. Hence, by Theorem 3.14

$$\begin{aligned} h^0(H) \geq \chi(H) &= \chi(\mathcal{O}) + \frac{1}{2}(H^2 - HK) \\ &= \chi(\mathcal{O}) + \frac{1}{2}(h^2 + 2h^2 - \deg \delta) \\ &= \chi(\mathcal{O}) + H^2 + 1 \\ &= H^2 + 2. \end{aligned}$$

Let then consider the natural map

$$\varphi : H^0(H) \longrightarrow H^0(H|_F) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathbb{C}^2$$

If $H^2 > 0$, then $h^0(H) > \dim \mathbb{C}^2 = 2$, then $\ker \varphi \neq \{0\}$. Hence $F_x < H$, that is $H - F_x$ is effective. Replacing H by $H - F$ and iterating the procedure if necessary we may assume $h^0(H - F_x) = 0 \; \forall x$; therefore $H^2 \leq 0$. On the surface S we choose $-H^2$ fibers, the fibers over the points $(1 : i)$, $i \in \{1, \dots, -H^2\}$, we pick a point on each of these fibers, we blow it up, and then contract the strict transforms of the fibers (see Figure 4). We denote by $\hat{H}$ the strict transform of H.



$h^0(\hat{H}) = 2$. Therefore $|\hat{H}|$ defines a map on $\mathbb{P}^1$.

$$\begin{array}{ccc} S' & \xrightarrow{\varphi_{\hat{H}}} & \mathbb{P}^1 \\ \hat{p}\big\downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

The map $\varphi_{\hat{H}}$ is defined on the whole S' because $\hat{H}$ is irreducible (hence $|\hat{H}|$ has no fixed part) and $\hat{H}^2 = 0$ (hence $|\hat{H}|$ has no base points). Then $\varphi_{\hat{H}}$ is a morphism $S' \to \mathbb{P}^1 \times \mathbb{P}^1$.

So S' is biregular to $\mathbb{P}^1 \times \mathbb{P}^1$ and therefore, S is obtained by $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up n points on $-H^2$ different fibers, and then contracting the strict transforms of these fibers: this shows that S is the Hirzebruch surface $\mathbb{F}_{-H^2}$. □

Theorem 4.2. *1.* $\mathrm{Pic}(\mathbb{F}_n) \cong \mathbb{Z}h \oplus \mathbb{Z}f$ *with* $h^2 = n$, $hf = 1$, $f^2 = 0$, *where* f *is the equivalence class of a fiber;*

2. If $n > 0$ $\exists!$ *curve* B *in* $\mathbb{F}_n$ *with* $B^2 < 0$ *and its class* b *in* $\mathrm{Pic}(\mathbb{F}_n)$ *is such that* $b^2 = -n$, $bf = 1$, $b = h - nf$*;*

3. $\mathbb{F}_n \cong \mathbb{F}_m \Leftrightarrow n = m$*;*

4. $\mathbb{F}_n$ *is minimal* $\Leftrightarrow n \neq 1$*;*

5. $\mathbb{F}_1$ *is the blow up of* $\mathbb{P}^2$ *in a point.*

Proof. 1. By Theorem 3.14 $\mathrm{Pic}(\mathbb{F}_n) \cong \mathbb{Z}f \oplus \mathbb{Z}b$, since $\mathbb{Z}f = p^* \mathrm{Pic}(\mathbb{P}_1)$, with $b^2 = -n$, $bf = 1$, $f^2 = 0$. If we take $h = b + nf$ we get the thesis.

2. Note that if $n = 0$ there are no curves with negative autointersection. The only non trivial part of the statement is the uniqueness of B. Let us suppose there exists a curve C such that $C \neq B$ and $C^2 < 0$. In $\mathrm{Pic}(\mathbb{F}_n)$ $C = \alpha h + \beta f$ for some $\alpha, \beta \in \mathbb{Z}$. Then

$$0 \le CF = (\alpha h + \beta f)f = \alpha$$
$$0 \le CB = (\alpha h + \beta f)(h - nf) = \alpha n + (\beta - n\alpha) = \beta.$$

Hence $C^2 = (\alpha h + \beta f)^2 = \alpha n + 2\alpha\beta \ge 0$. Contradiction.

3. $\Leftarrow$ Immediate.

   $\Rightarrow$ If $n \neq m$ the only curves with negative autointersection have different autointersection values. Hence $\mathbb{F}_n \ncong \mathbb{F}_m$.

4. $\mathbb{F}_n$ is not minimal$\Leftrightarrow \exists E \cong \mathbb{P}^1$ with $E^2 = -1$. Hence, by the second point of Theorem 4.2 if $n \neq 1$ $\mathbb{F}_n$ is minimal. Of course, if $n = 1$, B is an exceptional curve and therefore $\mathbb{F}_1$ is not minimal.

5. We have to prove that contracting the curve E in $\mathbb{F}_1$ with $E^2 = -1$ we get $\mathbb{P}^2$. Let us then consider the map

$$\begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \\ (x_0 : x_1 : x_2) & \mapsto & (x_0; x_1) \end{array};$$

this map is the projection with center $(0 : 0 : 1)$. The linear system induced by the map is $P = \{\text{lines through } (0 : 0 : 1)\}$. Let us resolve the map.

$$\begin{tikzcd} & S \arrow[dl, "\varepsilon"'] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^1 \end{tikzcd}$$

The result of the resolution is outlined in Figure 4.1. Since the surface is geometrically ruled and there exists a curve E with $E^2 = -1$, by the second point of the proof, what we get is $\mathbb{F}_1$.

□

4.1 Examples of rational surfaces

A rational map $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^N$ is given by a linear system $P = \Phi^*|H|$ on $\mathbb{P}^2$ without fixed part and (possibly) with some base points. Taken a resolution

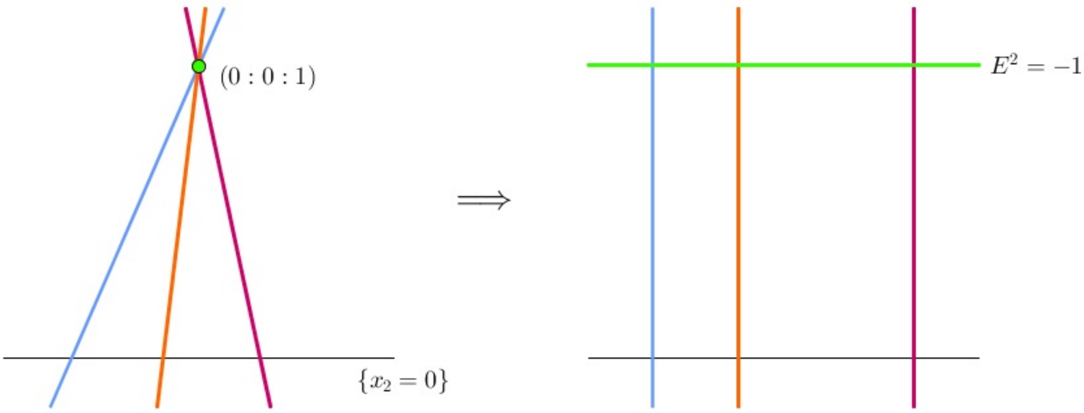


Figure 4.1: Resolution of the projection map

of Φ

$$\begin{tikzcd} & S \arrow[dl, "\varepsilon"'] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\Phi"'] & & \mathbb{P}^N \end{tikzcd}$$

we want to study when $\Phi \circ \varepsilon$ is an embedding. We shall restrict ourselves to simple blow ups and birational Φ. In particular, we will study example where

$$\varepsilon^* \Phi^* |H| \subseteq |dl - \sum m_i E_i|, \tag{4.1}$$

where $d, m_i \in \mathbb{N}$ and $l \in \mathrm{Pic}(\mathbb{P}^2)$ is the class of a line (actually in (4.1) we used a little abuse of notation, in fact by l we mean $\varepsilon^* l$). The cases we are interested in are $d = 2$ and $d = 3$.

Fact 4.3. $h^0(\mathcal{O}_{\mathbb{P}^r}(k)) = \binom{r+k}{r}$ *(it is a simple combinatorial fact; we only have to count the number of monomials in $r+1$ variables of degree k).*

When is the map induced by $|dl - \sum m_i E_i|$ an embedding?

- Injective$\Leftrightarrow$ separates points. That is $\forall x, y \in S$ $\exists C \in |dl - \sum m_i E_i|$ such that $x \in C, y \notin C$ (if $x \in E_1$, this determines a tangent direction on $T_{\varepsilon(x)}\mathbb{P}^2$, hence we have to check that there exists e curve in $\mathbb{P}^2$ passing through $\varepsilon(x)$ with that particular tangent direction).
- Injective differential$\Leftrightarrow$ separates tangents. That is $\forall x \in S$ the curves of P passing through x have not all the same tangent direction.

Remark 4.4. *Let $S \subset \mathbb{P}^N$ smooth surface and $N > 5$. Then $\exists$ a projection $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ such that $\pi|_S : S \to \pi(S)$ is biregular.*

Proof. When the projection map with center $p \in \mathbb{P}^N$ does not work? If either $p \in S$ (in this case the restriction $\pi|_S$ would not be a morphism), or a line through p contains two points of S (in this case $\pi|_S$ would not be injective), or a line through p is tangent to S (in this case $\pi|_S$ would not separates tangents). Hence we can prove that the set of points that do not work is the image of a variety X of dimension 5. In order to prove this, let us consider the Zariski closure of the set $\{(x,y,t) \in S \times S \times \mathbb{P}^N : x \neq y, t \in \overline{xy}\}$; basically, this subvariety of $S \times S \times \mathbb{P}^N$ is the locus of the triples (x,y,t), where either t lies on the line passing through x and y when these are two distinct points, or t belongs the tangent plane at x if $x = y$. This variety surjects on $S \times S$ with a general fiber isomorphic to $\mathbb{P}^1$, hence its dimension is $(\dim S \times S) + \dim \mathbb{P}^1 = 4 + 1 = 5$. □

Remark 4.4 gives the following result.

Proposition 4.5. *Every projective surface is isomorphic, via generic projection, to a smooth surface in $\mathbb{P}^5$.*

4.1.1 Linear systems of conics

Since the number of conics in 2 variables is 6, the linear system $P = |2l|$ induces a map $\Phi : \mathbb{P}^2 \to \mathbb{P}^5$ which is an embedding. The image of Φ is the *Veronese surface V*.

Remark 4.6. *V contains no lines of $\mathbb{P}^5$.*

Proof. Let $C \subset V$, then

$$\begin{aligned}\deg \mathcal{O}_C(H) &= \Phi^{-1}(C) \cdot \Phi^* H \\ &= \Phi^{-1}(C) \cdot 2l = 2 \deg \Phi^{-1}(C),\end{aligned}$$

that is an even number. For a line this number is 1, so C cannot be a line. □

On the other hand, let us consider $l \subset \mathbb{P}^2$ a line. $2l \cdot l = 2$, hence $\Phi(l) \cdot H = 2$. This means that the lines of $\mathbb{P}^2$ are mapped onto conics in V.

Proposition 4.7. *Let $p \in \mathbb{P}^5$ be a generic point. Then projecting away for p $\pi_p : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ induces an isomorphism of V onto its image $V' \subset \mathbb{P}^4$.*

We mention the following important characterization of the Veronese surface, which unfortunately we have no time to prove.

Theorem 4.8. *$\forall S \subset \mathbb{P}^5$ non degenerate*[1]*, if $\exists p \in \mathbb{P}^5$ such that $\pi_p|_S$ is an embedding, then, up to a coordinate change, S is the Veronese surface.*

Proof of Proposition 4.7. The lines of $\mathbb{P}^2$ are parametrized by $(\mathbb{P}^2)^\vee$ (the dual space of $\mathbb{P}^2$). Each line l in $\mathbb{P}^2$ determines a plane containing the conic $\Phi(l)$. This defines a $\mathbb{P}^2$-bundle over $(\mathbb{P}^2)^\vee$ $Y^4 \subset (\mathbb{P}^2)^\vee \times \mathbb{P}^5$. The *secant variety* is

$$X^5 \subset S \times S \times \mathbb{P}^N := \overline{\{(p,q,t) : t \in \overline{pq} \text{ and } p \neq q\}};$$

in our case $N = 5$, $S = V \subset \mathbb{P}^5$. Let $p, q \in S$, then $\overline{\Phi^{-1}(p)\Phi^{-1}(q)}$ is a line in $\mathbb{P}^2$; this line is mapped into a conic $C \subset \mathbb{P}^5$, and $\overline{pq}$ is contained into the plane containing C. Hence the image of $X \to S \times S$ is contained in the image of $Y \to \mathbb{P}^5$. Any six point p out of the image of Y will then do the job. $\square$

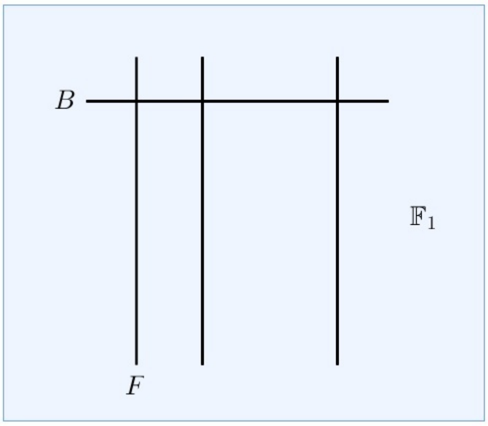
What if we project once more?

$$\begin{tikzcd}
 & V \subset \mathbb{P}^5 \arrow[d, "\text{general projection}"'] \\
\text{smooth} & V' \subset \mathbb{P}^4 \arrow[d, "\text{general projection}"'] \\
\text{singular} & \text{Steiner} \subset \mathbb{P}^3
\end{tikzcd}$$

If we project $\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ away from $p \in V$ the restriction $\pi|_V$ is no longer defined on the whole V. Resolving the map we get

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^4 \\
 & \mathbb{F}_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

and the composition, induced by the linear system $|2l - E|$, embeds $\mathbb{F}_1$ in $\mathbb{P}^4$.



$\subset \mathbb{P}^4$

$f = l - E$
$B = E$

[1] $X \subset \mathbb{P}^N$ is said to be *non degenerate* if $\nexists\ \mathbb{P}^{N-1} \subset \mathbb{P}^N$ a linear subspace $X \subset \mathbb{P}^{N-1}$.

In the description of $\mathrm{Pic}(\mathbb{F}_1)$, the system $|2l - E|$ is given by $|2f + b| = |h + f|$. Where is a fiber mapped?

$$f \cdot H = f \cdot (2f + b) = 1,$$

hence each fiber is mapped into a line of $\mathbb{P}^4$. And where is B mapped?

$$b \cdot H = b \cdot (2f + b) = 2 \cdot 1 - 1 = 1,$$

hence B too is mapped into a line. Moreover, we can compute the degree of these surfaces.

$$\begin{aligned}
&\deg V = (2l)^2 = 4;\\
&\deg V' = (2l)^2 = 4;\\
&\deg \mathbb{F}_1 = (2f + b)^2 = 4bf + b^2 = 4 - 1 = 3.
\end{aligned}$$

With a little abuse of notation we wrote $\deg \mathbb{F}_1$; in fact, this computation would be meaningless for the abstract Hirzebruch surface $\mathbb{F}_1$, it is possible only because in this case $\mathbb{F}_1$ is embedded into the projective space $\mathbb{P}^4$.

Exercise 4.9. *Projecting a smooth surface $S \subset \mathbb{P}^N$ away form a general point, if the map we get is birational, then the degree of the image is the same of the degree of the original surface. If the center of the projection lies on the surface, still assuming the map to be birational, the degree of the image decreases by one.*

Projecting twice from points internal to the Veronese surface we get the situation represented in the following:

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[r, hook, "|2l|"] & V_4 \subset \mathbb{P}^5 \arrow[d, dashed, "\text{projection}"] \\
\mathbb{F}_1 \arrow[u] \arrow[r, "|2l - E|"] & S_3 \subset \mathbb{P}^4 \arrow[d, dashed, "\text{projection}"] \\
\text{further blow up} \arrow[u] \arrow[r, two heads, "|2l - E_1 - E_2|"] & Q_2 \subset \mathbb{P}^3
\end{tikzcd}$$

where, for a fixed $p \in \mathbb{P}^2$ (and, with a slight abuse of notation, $p \in V_4$ is the image of p), E is the exceptional divisor of the blow up $\mathbb{F}_1 \to \mathbb{P}^2$ in p; the map $V_4 \dashrightarrow S_3$ is the projection with center p; $p_2 \in \mathbb{F}_1$ (and, as before, $p_2 \in S_3$

will denote its image as well); E_2 is the exceptional divisor of the blow up of $\mathbb{F}_1$ in p_2 and E_1 the pull back of E; finally, $S_3 \dashrightarrow Q_2$ is the projection with center p_2.

Let us begin from the top of the diagram and let us study the map $\mathbb{P}^2 \hookrightarrow V_4 \subset \mathbb{P}^5$. We denote the coordinates on $\mathbb{P}^2$ by $(x_0 : x_1 : x_2)$ and those on $\mathbb{P}^5$ by $(y_0 : \cdots : y_5)$. Let $q_0, \ldots, q_5$ be a base for $\mathbb{C}[x_0, x_1, x_2]_2$ (its dimension is 6 by Fact 4.3).

The linear system $|2l|$ induces a ring morphism

$$\begin{array}{rccc} \varphi & : \mathbb{C}[y_i] & \longrightarrow & \mathbb{C}[x_j] \\ & y_i & \longmapsto & q_i \end{array}$$

more precisely

$$\varphi_d : \mathbb{C}[y_i]_d \longrightarrow \mathbb{C}[x_j]_{2d}.$$

What is the kernel of this map?

$$\begin{aligned} \ker\varphi &= \{\text{polynomials in } y_i \text{ variables whose pull back vanishes in } \mathbb{P}^2\} \\ &= \{\text{polymomials in } y_i \text{ variables that vanishes on } V_4\} \\ &= \text{ideal that defines } V_4 \text{ in } \mathbb{P}^5 \end{aligned}$$

It is immediate to see that $\ker\varphi_0 = \{0\}$ because $\varphi_0 = \mathrm{Id}_{\mathbb{C}}$ is an isomorphism by definition (regardless the linear system we are considering). Moreover, since $\{q_i\}$ is a basis and V_4 is non degenerate, $\ker\varphi_1 = \{0\}$.

Let us study $\ker\varphi_2$. By 4.3, $\mathbb{C}[y_i]_2 \cong \mathbb{C}^{21}$ and $\mathbb{C}[x_j]_4 \cong \mathbb{C}^{15}$, hence

$$\varphi_2 : \mathbb{C}^{21} \to \mathbb{C}^{15}$$

and $\dim\ker\varphi_2 \geq 6$. Let us choose the the q_i in the simplest possible way.

$$\begin{array}{lll} q_0 = x_0^2 & q_1 = x_0x_1 & q_2 = x_0x_2 \\ q_3 = x_1^2 & q_4 = x_1x_2 & q_5 = x_2^2 \end{array}$$

$\forall (x_0, x_1, x_2) \in \mathbb{C}^3 \smallsetminus \{0\}$ the matrix

$$\begin{pmatrix} x_0^2 & x_0x_1 & x_0x_2 \\ x_0x_1 & x_1^2 & x_1x_2 \\ x_0x_2 & x_1x_2 & x_2^2 \end{pmatrix}$$

has rank 1, hence $\forall (y_0 : \cdots : y_5) \in V_4$,

$$\mathrm{rk}\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_3 & y_4 \\ y_2 & y_4 & y_5 \end{pmatrix} = 1;$$

this means that each 2×2-minor of that matrix is equal to zero on V_4. Hence we get six quadrics

$$y_0y_3 - y_1^2,\ y_0y_4 - y_1y_2,\ y_1y_4 - y_2y_3,\ y_0y_5 - y_2^2,\ y_1y_5 - y_2y_4,\ y_3y_5 - y_4^2 \tag{4.2}$$

that vanishes in V_4.

Exercise 4.10. *What we have seen so far proves that*

$$V_4 \subset V\left(\mathrm{rk}\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_3 & y_4 \\ y_2 & y_4 & y_5 \end{pmatrix} = 1\right),$$

Show that V_4 is exactly the zero locus of the quadrics in (4.2).

Now let us study S_3, the surface we obtain projecting V_4 out of a point $p \in V_4$. By Exercise 4.9, S_3 will have degree 3. The linear system $|2l - E|$ is the system of the conics passing trough the fixed point p. Let us denote the coordinates in $\mathbb{P}^4$ by $(z_0 : \cdots : z_4)$, and fix $Q_0, \ldots, Q_4$ a basis for the set $\{f \in \mathbb{C}[x_0, x_1, x_2]_2 : f(p) = 0\}$. We have a ring morphism

$$\begin{array}{cccc} \psi & : \mathbb{C}[z_i] & \longrightarrow & \mathbb{C}[x_j] \\ & z_i & \longmapsto & Q_i \end{array},$$

more precisely

$$\psi_d : \mathbb{C}[z_i]_d \longrightarrow \mathbb{C}[x_j]_{2d}.$$

Let us study the kernel of this map. As before, $\ker\psi_0 = \ker\psi_1 = \{0\}$.

$$\psi_2 : \mathbb{C}[z_i]_2 \longrightarrow \mathbb{C}[x_j]_4,$$

and $\mathbb{C}[z_i]_2 \cong \mathbb{C}[x_j]_4 \cong \mathbb{C}^{15}$. Anyway ψ_2 is not surjective, since all Q_i vanish in p. Hence every quadric in z_i will be mapped in a linear combination of Q_iQ_j, that vanish in p twice. Therefore

$$\mathrm{Im}\,\psi_2 \subset \{f \in \mathbb{C}[x_j]_4 : f \text{ vanishes at least twice in } p\}.$$

How many conditions these assumptions determine?

Example 4.11. *Let us suppose $p = (1:0:0)$ and let*

$$f_4 = ax_0^4 + bx_0^3x_1 + cx_0^3x_2 + x_0^2A_2(x_1,x_2) + x_0B_3(x_1,x_2) + C_4(x_1,x_2)$$

be a generic quartic. Then

- $f_4(p) = 0 \Leftrightarrow a = 0$;

- *f_4 singular in $p \Leftrightarrow b = c = 0$.*
- f vanishes in $p \Longrightarrow$ one linear condition on the coefficients of f;
- f singular in $p \Longrightarrow$ two linear condition on the coefficients of f.

Hence $\dim \operatorname{Im} \psi_2 \leq 12 \Rightarrow \dim \ker \psi_2 \geq 3$. Therefore $\exists Q_0, Q_1, Q_2 \in \mathbb{C}[z_i]_2$ independent such that

$$S_3 \subset V(Q_i).$$

Remark 4.12. *The quadrics Q_i are irreducible; indeed, suppose that $Q_i = \Pi_1 \cup \Pi_2$, then, since S is irreducible, either $S = S \cap \Pi_1$ or $S = S \cap \Pi_2$. Hence $S \subset \Pi_i$ is degenerate.*

$Q_0 \cap Q_1 =: \Sigma$ is a surface of degree

$$\#\Sigma \cap \underbrace{H_1 \cap H_2}_{\mathbb{P}^2} = 2 \cdot 2 = 4.$$

S is a surface of degree $\#S \cap H_1 \cap H_2 = 3$. Hence, defining $\Pi := \overline{\Sigma \setminus S}$,

$$\#\Pi \cap H_1 \cap H_2 = 4 - 3 = 1,$$

thus Π is a plane. This means that the two quadrics Q_0 and Q_1 cut the surface S and a plane in $\mathbb{P}^4$, the third quartic will get rid of this extra plane.

$\Pi = \{l_0 = l_1 = 0\}$ with $l_i \in \mathbb{C}[z_j]_1$. Q_0 and Q_1 vanishes on $\Pi \Leftrightarrow Q_i \in (l_0, l_1)$ (the ideal generated by l_0 and l_1). Hence

$$Q_0 = l_0 A_1 - l_1 A_0, \qquad Q_1 = l_0 B_1 - l_1 B_0 \qquad \text{with } A_i, B_j \in \mathbb{C}[z_j]_1.$$

We shall rewrite the last equations as

$$Q_0 = \det \begin{pmatrix} l_0 & A_0 \\ l_1 & A_1 \end{pmatrix} \qquad Q_1 = \det \begin{pmatrix} l_0 & B_0 \\ l_1 & B_1 \end{pmatrix}.$$

Let us define

$$M = \begin{pmatrix} l_0 & A_0 & B_0 \\ l_1 & A_1 & B_1 \end{pmatrix}$$

and $Z = \{\operatorname{rk} M = 1\}$. Let $P \in \Sigma$. Let us suppose that $l_0(P) \neq 0$, then in P

$$A_1 = \frac{A_0 l_1}{l_0}, \qquad B_1 = \frac{B_0 l_1}{l_0};$$

$$A_0 B_1 - A_1 B_0 = \frac{A_0 B_0 l_1}{l_0} - \frac{A_0 B_0 l_1}{l_0} = 0,$$

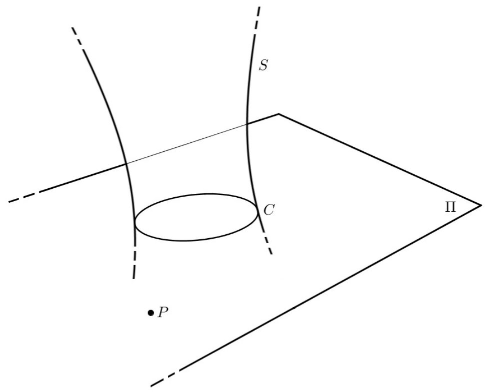
hence $P \in Z$. Analogously we prove that $l_1(P) \neq 0 \Rightarrow P \in Z$. Hence $P \in \Sigma \setminus \Pi \Rightarrow P \in Z$. This means that $S \subset Z$ and therefore $A_0B_1 - A_1B_0$ is in $\ker \psi_2$. One could indeed prove that the 2×2 minors of M generate $\ker \psi$ by proving $S = Z$.

Let $C := S \cap \Pi$; it is defined by $l_0 = l_1 = A_0B_1 - A_1B_0$, hence it is a conic in $\Pi \cong \mathbb{P}^2$. Suppose to have a plane conic in S that is $C_2 \subset \Pi \cong \mathbb{P}^2$ such that $C_2 \subset S$. Let $\Sigma := S \cup \Pi$; we want to compute the dimension of the space W of the quadrics in $\mathbb{C}[z_j]_2$ vanishing on Σ. Clearly $W \subset \mathrm{span}(Q_0, Q_1, Q_2) \subset \mathbb{C}[z_j]_2$.

Choose a point $P \in \Pi \setminus C$, hence $P \notin S$, let

$$\overline{W} := \{f \in \mathbb{C}[z_j]_2 : f \text{ vanishes on} S \cup \{P\}\}$$

$W \subset \overline{W} \subset \mathrm{span}(Q_0, Q_1, Q_2)$ and since $P \notin S$ $\dim \overline{W} = 3 - 1 = 2$.
If $\overline{Q} \in \overline{W}$, then $\overline{Q} \cap \Pi$ is a conic that contains C and P. Contradiction!
Hence $\overline{Q} \cap \Pi = \Pi$, thus $\overline{W} = W$, so $\dim W = 2$.



So we have proved the following proposition.

Proposition 4.13. *The cubic ruled surface $S \subset \mathbb{P}^4$ is contained in a $2-$dimensional linear system of quadrics in $\mathbb{P}^4$, of which it is the intersection. For every pencil of quadrics $\{\lambda Q_1 + \mu Q_2\}$ containing S, $Q_1 \cap Q_2 = S \cup P$, where P is a plane and $S \cap P$ a conic. Conversely, for every conic on S lying in a plane P, $S \cup P$ is the intersection of two quadrics.*

If we project once more, we get the bottom part of the diagram seen above:

$$\text{blow up} \twoheadrightarrow Q_2 \subset \mathbb{P}^3.$$

Fact 4.14. *If $p_2 \notin B$ Q_2 is smooth and $\cong \mathbb{P}^1 \times \mathbb{P}^1$.*
If $p_2 \in B$, then B is contracted into a singular point of Q_2.

4.1.2 Linear system of cubics

Let $p_1, \dots, p_r \in \mathbb{P}^2$ in general position (that is, that no triple of p_is lies on a line, and none 6–tuples of p_is lies on a conic). Let $d = 9 - r$ and S_d the blow up of $\mathbb{P}^2$ in $p_1, \dots, p_r$ with exceptional curves $E_1, \dots, E_r$.

Theorem 4.15. *If $r \leq 6$ (that is $d \geq 3$)$|-K_{S_d}| = |3l - \sum E_i|$ is very ample; this means that such linear system embeds S_d in $\mathbb{P}^{10-r-1} = \mathbb{P}^d$.*

Proof. Exercise. Hint: first study $r = 6$. □

Fact 4.16. *A surface S with $|-K_S|$ very ample is said to be a* Del Pezzo surface*. It is possible to prove that the only del Pezzo surfaces are $S_d \subset \mathbb{P}^d$ with $3 \leq d \leq 9$ and $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^8$.*

Let then study $S_d \hookrightarrow \mathbb{P}^d$; first, let us compute the degree of such surface

$$\begin{aligned}\deg S_d = \# S_d \cap H_1 \cap H_2 &= (3l - \sum E_i)^2 \\ &= 9l^2 + \sum_{i=1}^{r} E_i^2 - 6\sum lE_i - 2\sum_{i\neq j} E_iE_j \\ &= 9 - r = d.\end{aligned}$$

Hence del Pezzo surface of degree d is embedded in $\mathbb{P}^d$.

Let us study $S_3 \subset \mathbb{P}^3$. As before, we have ring morphisms

$$\psi_d : \mathbb{C}[y_i]_d \longrightarrow \mathbb{C}[x_j]_{3d}.$$

$\ker\psi_0 = \ker\psi_1 = \{0\}$. Let us study the map

$$\psi_2 : \mathbb{C}[y_0,\dots,y_3]_2 \longrightarrow \mathbb{C}[x_0,x_1,x_2]_6$$

$\mathbb{C}[y_i]_2 \cong \mathbb{C}^{10}$ and $\mathbb{C}[x_j]_6 \cong \mathbb{C}^{28}$, but we are interested only in those sextics passing trough $p_1,\dots,p_6$ and singular in such points, hence we have $6\times 3 = 18$ linear condition. Thus the map

$$\psi_2 : \mathbb{C}^{10} \longrightarrow \mathbb{C}^{10} \tag{4.3}$$

might be injective. The map ψ_3 is

$$\psi_3 : \mathbb{C}^{20} \to \mathbb{C}^{55-36} = \mathbb{C}^{19}, \tag{4.4}$$

since the triple nonic passing trough $p_1,\dots,p_6$ determines $6\times 6 = 36$ conditions. Hence $\dim\ker\psi_3 \geq 1$.

It is possible to prove that the ideal of S_3 is principal and generated by a cubic; in particular $\dim\ker\psi_2 = 0$ and $\dim\ker\psi_3 = 1$ as the above computation suggested.

Let us study now $S_4 \subset \mathbb{P}^4$; for this surface we need at least two equations. Indeed

$$\psi_2 : \mathbb{C}^{15} \longrightarrow \mathbb{C}^{28-15} \cong \mathbb{C}^{13}, \tag{4.5}$$

since we have 3 linear conditions for each of the 5 points p_i. Actually S_4 is a complete intersection surface of two quadrics in $\mathbb{P}^4$.

In fact, the computations in (4.3), (4.4) and (4.5) are a bit sloppy because we never checked that the conditions imposed by the points are independent. In particular, the target spaces could have dimension bigger than the one we wrote. This is not the case. Let us prove, for example, that in (4.5) the target space actually is $\mathbb{C}^{13}$, that is we want to prove that $h^0(\mathcal{O}_{S_4}(6l - 2\sum_i^5 E_i)) = 13$ ($|6l - 2\sum_i^5 E_i|$ represents the linear system of sextics passing with multiplicity two trough our five points). What we have proved so far is that $h^0(\mathcal{O}_{S_4}(6l - 2\sum_i^5 E_i)) \geq 13$.

Let $H \subset \mathbb{P}^4$ be an hyperplane. If H is generic, $H \cap S_4 \cong C \subset \mathbb{P}^2$ is a smooth cubic passing trough $p_1, \ldots, p_5$, hence an elliptic (hence of genus 1) curve. $\mathcal{O}_{S_4}(6l - 2\sum_i^5 E_i) = \mathcal{O}_{S_4}(2H)$, hence

$$0 \longrightarrow \mathcal{O}_{S_4}(2H - H) \longrightarrow \mathcal{O}_{S_4}(2H) \longrightarrow \mathcal{O}_C(2H) \longrightarrow 0,$$

thus

$$\begin{aligned} h^0(\mathcal{O}_{S_4}(2H)) &\leq h^0(\mathcal{O}_{S_4}(H)) + h^0(\mathcal{O}_C(2H)) \\ &= h^0(\mathcal{O}_{S_4}(3l - \textstyle\sum E_i)) + h^0(\mathcal{O}_C(2H)) \\ &\leq 5 + \chi(\mathcal{O}_C(2H)) \\ &= 5 + \deg(\mathcal{O}_C(2H)) = 5 + 2H \cdot C = 5 + 2\deg S_4 = 13. \end{aligned}$$

Fact 4.17. *Del Pezzo surfaces of degree* 3 *and* 4 *are complete intersections;*

- $S_3 \longrightarrow$ *one cubic equation in* $\mathbb{P}^3$*;*
- $S_4 \longrightarrow$ *two quadric equations in* $\mathbb{P}^4$*.*

$S_5, \ldots, S_9$ *are not complete intersection; actually they need more than* $d - 2$ *equations.*

Let us study S_3, in particular we want to see whether S_3 contains some lines. As we have seen above, we have

$$\begin{tikzcd} E_i \in S_3 \arrow[r, hook, "|3l - \sum E_i| = |-K_S|"] \arrow[d] & \mathbb{P}^3 \\ p_i \in \mathbb{P}^2 & \end{tikzcd}$$

with $p_1, \ldots, p_6$ points in general position. Let $C \subset S_3$ be a line in $\mathbb{P}^3$. Then $C \cong \mathbb{P}^1$, and $C \cdot (3l - \sum E_i) = 1 \Leftrightarrow K_S \cdot C = -1$, hence, by the genus formula, $C^2 = -1$. Thus a line in S_3 is an exceptional curve.

We know six of them, $E_1, \dots, E_6$. Let us define $\forall i < j$

$$l_{ij} \in |l - E_i - E_j|;$$

these 15 curves are the strict transforms of the lines in $\mathbb{P}^2$ passing through p_i and p_j. Moreover

$$(3l - \sum_{k=1}^{6} E_k)(l - E_i - E_j) = 3l^2 - 1 - 1 = 1,$$

hence each l_{ij} is a line. Let us define

$$q_i \in |2l - \sum_{k=1}^{6} E_k + E_i|;$$

this is the strict transform of the conic in $\mathbb{P}^2$ passing through every p_j except for p_i. Then each one of the six q_i is such that

$$q_i^2 = (2l - \sum E_k + E_i)^2 = 4l^2 - 6 - 1 + 2 = -1.$$

Therefore what we have proved so far is that S_3 contains at least $6+15+6 = 27$ lines of $\mathbb{P}^3$. We want to prove that no other line is contained in S_3.

Firs of all, we shall remark that if we take two lines, they either have intersection 0 (if they are disjoint) or 1 (if they intersect). One can easily check that the lines, among these 27, intersecting E_1, are

$$\begin{matrix} l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ q_2 & q_3 & q_4 & q_5 & q_6 \end{matrix}$$

Exercise 4.18. *Verify that each of the above* 27 *lines intersects exactly* 10 *of the other* 26*.*

By 2.7, $\mathrm{Pic}(S) = \mathbb{Z}l \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_6 \cong \mathbb{Z}^7$, hence each $C \in \mathrm{Pic}(S)$ shall be written in the following way:

$$C \equiv al - \sum_{i=1}^{6} m_i E_i.$$

Let us suppose that C is a line and $C \neq E_i$, hence either $C \cdot E_i = 0$ or $C \cdot E_i = 1$. But $C \cdot E_i = m_i$, hence $m_i \in \{0, 1\}$. Furthermore, we know that $C^2 = -1$, then

$$-1 = C^2 = a^2 - \sum m_i^2 \geq a^2 - 6,$$

hence $a^2 \leq 5 \Rightarrow a \in \{0, 1, 2\}$ ($a \geq 0$ otherwise C would not be effective). Hence

$$\begin{array}{lll} a = 0 \Rightarrow & \sum m_i^2 = 1 \Rightarrow & C = E_i \\ a = 1 \Rightarrow & \sum m_i^2 = 2 \Rightarrow & C = l_{ij} \\ a = 2 \Rightarrow & \sum m_i^2 = 5 \Rightarrow & C = q_i \end{array}$$

Theorem 4.19. *Let S be a smooth cubic in $\mathbb{P}^3$, then S is the blow up of $\mathbb{P}^2$ in 6 points in general position embedded by $|3l - \sum E_i|$.*

Proof. First of all, we study the Klein quadric; this is a set that parametrizes the lines in $\mathbb{P}^3$ and has a natural structure of variety.

Let $L \subset \mathbb{P}^3$ be a line, and $a, b \in L$; then

$$a = (a_0 : a_1 : a_2 : a_3)$$
$$b = (b_0 : b_1 : b_2 : b_3).$$

Let us define the map

$$\begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \longmapsto (a_0b_1 - b_0a_1 : a_0b_2 - a_2b_0 : a_0b_3 - a_3b_0 : a_1b_2 - a_2b_1 : a_1b_3 - a_3b_1 : a_2b_3 - a_3b_2)$$

This function, seen as a map $\{\text{lines in } \mathbb{P}^3\} \to \mathbb{P}^5$ is well defined, since if we change the two points and/or their homogeneous coordinates without changing the line, it changes the matrix on the left, but the values at the right change only by a multiplicative constant. This map is not surjective: indeed

$$\{\text{lines in } \mathbb{P}^3\} \xrightarrow{\sim} G := \{y_0y_5 - y_1y_4 + y_2y_3 = 0\} \subset \mathbb{P}^5.$$

Fact 4.20. *The map $\{$lines in $\mathbb{P}^3\} \to G$ is bijective.*

One can easily prove that two different lines are mapped to two different points of G, which is called the Klein quadric and therefore gives a good parameter space for the set of lines in $\mathbb{P}^3$. Computing the derivative, one can easily check that G is a smooth variety of dimension 4. Let us consider $P = |\mathcal{O}_{\mathbb{P}^3}(3H)|$ (that is the linear system of the cubics in $\mathbb{P}^3$) and the set $\mathcal{I} := \{(L, S) : L \subset S\} \subset G \times P$.

$$\begin{tikzcd}[column sep=small] & \mathcal{I} := \{(L,S) : L \subset S\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & P \end{tikzcd}$$

Let us denote by $(x : y : z : t)$ the coordinates in $\mathbb{P}^3$. Let $L \in G$; up to a change of coordinates, we shall suppose that $L = \{z = t = 0\}$, thus $(L, S) \in \pi_1^{-1}(L) \Leftrightarrow S = \{f_3 = 0, \text{where } f_3 \text{ does not contain the monomials } x^3, x^2y, xy^2, y^3\} \cong \mathbb{P}^{\dim P - 4}$. Therefore, $\mathcal{I}$ is a smooth variety of (complex) dimension

$$\dim \mathcal{I} = \dim G + \dim P - 4 = 4 + \dim P - 4,$$

that is $\dim \mathcal{I} = \dim P$. Now let us consider π_2: this is a map among two complex varieties of the same dimension. Then:

- either π_2 has degree strictly positive and π_2 is onto;

- or π_2 has degree zero: in this case $\forall p \in \pi_2(\mathcal{I})$, p is not a regular value and $\pi_2^{-1}(p)$ is not a finite set.

We know that S_3, the del Pezzo surface of degree 3, is a cubic and contains exactly 27 lines. This means that $S_3 \in \pi_2(\mathcal{I})$ and $\pi_2^{-1}(S_3)$ is finite. Hence π_2 cannot have degree 0 and it is onto. This means that each cubic contains a line of $\mathbb{P}^3$: $\forall S \in |\mathcal{O}_{\mathbb{P}^3}(3)|\ \exists L \subset S$ with L line of $\mathbb{P}^3$.

Let S be a smooth cubic in $\mathbb{P}^3$, L a line contained in S. We want to find all the lines in S intersecting L. Since two lines intersecting in a point are contained in the same plane, let us study the pencil of planes $L \subset H$. Up to a change of coordinates, we may suppose that $L = \{z = t = 0\}$, hence we get a natural explicit parametrization of the pencil of planes by associating to each point $(a : b) \in \mathbb{P}^1$ the plane $\{az + bt = 0\}$. It is easy to see that $H \cap S = L \cup C$, where C is a conic; if H is a general plane, C will be smooth, otherwise, C could be the union of two lines (possibly a double line). We want to study in which case the conic we get is non smooth. Let us write the equation of S in the following way:

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F, \tag{4.6}$$

where $A, B, C, D, E, F \in \mathbb{C}[z,t]$ with $\deg A = \deg B = \deg C = 1$, $\deg D = \deg E = 2$ and $\deg F = 3$. Written as (4.6), the equation looks like the equation of a conic, whose rank depends on the matrix

$$M := \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix};$$

$\det M \in \mathbb{C}[z,t]_5$, hence its zero locus is the union of $H_1, \dots, H_5$, 5 planes through L. Since S is smooth, it is possible to prove that

- $H_1, \dots, H_5$ are distinct (that is $\det M$ is square free);

- $\forall H_i$, $H_i \cap S = L \cup L_i \cup L_i'$ are three distinct lines. Moreover, for each plane H in the pencil different from these 5, since the rank of M becomes 3, the intersection $H \cap S$ is the union of a line and a smooth conic.

Hence we have found 11 lines contained in S, and we know that every other line in S would be disjoint form L. If 3 distinct lines in S meet at a point $p \in S$, then, since S is smooth, they lie in the tangent plane to S at p, and so they are coplanar. Now consider L_1 and L_2, they are not coplanar (because

$H_1 \neq H_2$), so $L \cap L_1$ and $L \cap L_2$ are distinct. It easily follows that (again since they are not coplanar) $L_1 \cap L_2 = \varnothing$. Let us define then two birational maps

$$\Phi : L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow S$$
$$\Psi : S \dashrightarrow L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$$

- For p, p' general point in L_1 and L_2 respectively, then $\Phi(p, p') = p''$, where p'' is the only intersection different from p and p' of $\overline{pp'}$ and S. This function is not defined if $\overline{pp'} \subset S$;

- $\forall s \in S \smallsetminus (L_1 \cup L_2)$ let Π_1 be the plane passing through s and L_1, let Π_2 be the plane passing through s and L_2 and $L' := \Pi_1 \cap \Pi_2$, then we define $\Psi(s) = (L' \cap L_1, L' \cap L_2)$.

It is easy to see that Φ is the inverse of Ψ and vice versa. Hence S is rational. Moreover, Ψ can be extended to a morphism: if $s \in L_1$ we take $\Pi_1 := T_s S$, if $s \in L_2$ we take $\Pi_2 := T_s S$. Therefore we have a map

$$\Psi : S \to \mathbb{P}^1 \times \mathbb{P}^1,$$

that is a birational morphism, and so a sequence of blow ups, as many as are the curves contracted by Ψ. Let us count these curves.

Let $s_1, s_2 \in S \smallsetminus (L_1 \cup L_2)$ and suppose $\Psi(s_1) = \Psi(s_2) = (p, p')$, then $s_1, s_2 \in \overline{pp'} \Rightarrow \#\{\overline{pp'} \cap S\} \geq 4 > 3 \Rightarrow \overline{pp'} \subset S$. This means that the curves contracted by Ψ are exactly the lines that intersect both L_1 and L_2. We proceed to calculate the number of these lines. We know that the lines meeting L fall into 5 pair (L_i, L_i') with $1 \leq i \leq 5$. such that L_i, L_i' and L lie in a plane H_i. H_i meets L_2 in one point, which lies on L_i or L_i' (but not both, for else, L_i, L_i' and L_2 would be coplanar). Thus one line in each pair meets L_2 as well, and so Ψ contracts 5 lines.

$$\begin{tikzcd}
 & & & & S \arrow[dl] \\
 & & & S^{IV} \arrow[dl] & \\
 & & S^{III} \arrow[dl] & & \\
 & S^{II} \arrow[dl] & & & \\
S^{I} \arrow[d] \arrow[dr] & & & & \\
\mathbb{P}^1 \times \mathbb{P}^1 & \mathbb{F}_1 \arrow[dr] & & & \\
 & & \mathbb{P}^2 & &
\end{tikzcd}$$

Hence (see the diagram on the left) S is a blow up of $\mathbb{P}^2$ in six points. Moreover, it is not difficult to prove that $H = 3l \smallsetminus (E_1 \cup \cdots \cup E_6)$, that is that S is embedded by the system of cubics through the six points; in other words, S is a del Pezzo surface S_3.

$\square$

Chapter 5

Castelnuovo's Theorem and its applications

5.1 Castelnuovo's Theorem

At the end of 19$^{\text{th}}$ century, it was a well known fact that if C is a projective smooth curve such that $h^0(\Omega^1_C) = 0$, then $C \cong \mathbb{P}^1$. What happens in dimension 2? In the previous chapter we have seen that for a surface S $q(S) := h^0(\Omega^1_S)$ and $p_g(S) := h^0(\Omega^2_S)$ are birational invariants, moreover $q(\mathbb{P}^2) = p_g(\mathbb{P}^2) = 0$, hence it is immediate to conclude that if S is a smooth projective rational surface, then $q(S) = p_g(S) = 0$. Does the opposite implication hold, as in the case of the curves?

Conjecture 5.1 (Max Noether's conjecture)**.** *If S is a smooth projective surface, with $q = p_g = 0$, then S is rational.*

Enriques built a counterexample to Noether's conjecture; in particular, he defined a non rational surface S with $q = p_g = 0$ and $P_2 = 1$.

The construction of Enriques gave the motivation to Castelnuovo to prove its important criterion.

Theorem 5.2 (Castelnuovo's rationality Criterion)**.** *Let S be a projective surface with $q = P_2 = 0$, then S is rational.*

Let us recall that

$$p_g := h^0(\mathcal{O}_S(K_S)) \qquad P_2 := h^0(\mathcal{O}_S(2K_S)).$$

Remark 5.3. *If $p_g \neq 0$, then $P_2 \neq 0$; indeed, if $\sigma \in H^0(\mathcal{O}_S(K_S))$, then $\sigma^2 \in H^0(\mathcal{O}_S(2K_S))$. The opposite implication does not hold by, e.g., the above mentioned Enriques' construction.*

In order to prove Theorem 5.2, we need the following proposition.

Proposition 5.4. *Let S be a minimal projective surface with $q = P_2 = 0$, then $\exists C \subset S$ smooth curve such that $C \cong \mathbb{P}^1$ and $C^2 \geq 0$.*

We do not give the full proof of Proposition 5.4, but only a short sketch, to give an idea on how the hypotheses are used.

Proof of Theorem 5.2. We shall assume, without loss of generality that S is minimal, then, by Proposition 5.4, there exists $C \subset S$ smooth curve such that $C \cong \mathbb{P}^1$ and $C^2 \geq 0$. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0.$$

$\mathcal{O}_C(C) \cong \mathcal{O}_{\mathbb{P}^1}(C^2)$, hence we get the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_S) \longrightarrow H^0(\mathcal{O}_S(C)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(C^2)) \longrightarrow H^1(\mathcal{O}_S) \longrightarrow \dots,$$

but $H^1(\mathcal{O}_S) = H^0(\Omega^1_S) = 0$ by $q = 0$, hence the sequence becomes

$$0 \longrightarrow H^0(\mathcal{O}_S) \longrightarrow H^0(\mathcal{O}_S(C)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(C^2)) \longrightarrow 0;$$

moreover $H^0(\mathcal{O}_S) \cong \mathbb{C}$ (because S is compact) and $H^0(\mathcal{O}_{\mathbb{P}^1}(C^2)) \cong \mathbb{C}^{C^2+1}$. Hence $h^0(\mathcal{O}_S(C)) = C^2 + 2 \geq 2$ (that is the curve C is not rigid) and $\dim |C| = h^0(\mathcal{O}_S(C)) - 1 \geq 1$. This implies that there exists an effective divisor different from C and linearly equivalent to it. Moreover, since C is irreducible, $|C|$ has no fixed part.

Taking a pencil D contained in $|C|$ without fixed part, it defines a map on $\mathbb{P}^1$

$$\begin{tikzcd} & \hat{S} \arrow[dl] \arrow[dr, "\varphi"] & \\ S \arrow[rr, dashed] & & \mathbb{P}^1 \end{tikzcd}$$

Resolving the map, we get a surjective morphism φ with basis $\mathbb{P}^1$ and a fiber isomorphic to $C \cong \mathbb{P}^1$ (the strict transform of C). Hence by Theorem 3.5 $\hat{S}$ is ruled on $\mathbb{P}^1$ and we get the thesis. $\square$

Sketch of the proof of Proposition 5.4. By Serre duality

$$0 = h^0(2K) = h^2(-K)$$

hence $1 + K^2 = \chi(-K) = h^0(-K) - h^1(-K) + h^2(-K) \leq h^0(-K)$. In the proof one distinguishes three cases: $K^2 < 0$, $K^2 = 0$, $K^2 > 0$. We consider only the case $K^2 = 0$. For $K^2 = 0$, we have $h^0(-K) \geq 1$, hence $-K \geq 0$. Let us fix $H \in \mathrm{Pic}(S)$ very ample, and let us study $|H + nK|$: since $-K \geq 0$, $\exists n$ such that $|H + nK| \neq \varnothing$ and $|H + (n+1)K| = \varnothing$: fix $D \in |H + nK|$. D is effective and the following properties hold:

- $|K + D| = \varnothing$;
- $K \cdot D < 0$ indeed $K \cdot D = K \cdot H + nK^2 = -H \cdot (-K) < 0$, where last inequality holds because H is ample and $-K$ is effective.

$D = \sum n_i C_i$ with C_i is curve and $n_i > 0$ $\forall i$. Since $\forall i$ $D - C_i \geq 0$, $h^0(K + C_i) \leq h^0(K + D) = 0 \Rightarrow \forall i$ $|K + C_i| = \varnothing$. Moreover, $\exists i$ such that $K \cdot C_i < 0$, otherwise the second property seen above would not hold. Hence there exists $C < D$ irreducible such that $|K + C| = \varnothing$ and $K \cdot C < 0$. Since C is irreducible, we can exploit 1.104 to compute the arithmetical genus

$$0 \leq p_a(C) = 1 + \frac{K \cdot C + C^2}{2} \Rightarrow K \cdot C + C^2 \geq -2$$

Since $K \cdot C < 0$, $C^2 \geq -1$.

If $C^2 = -1$, then

$$0 \leq 1 + \frac{KC - 1}{2} < 1 \Rightarrow KC = -1, p_a(C) = 0 \Rightarrow C \cong \mathbb{P}^1$$

by Exercise 2.29. But this gives a contradiction, since S by hypothesis is minimal. Thus $C^2 \geq 0$.

Now we prove that $C \cong \mathbb{P}^1$. Since $|K + C| = \varnothing$

$$\begin{aligned} 0 = h^0(K + C) \geq \chi(K + C) &\overset{(1.10)}{=} \chi(\mathcal{O}) + \frac{(K + C)(K + C - K)}{2} \\ &= (1 - q + p_g) + \frac{KC + C^2}{2} \end{aligned}$$

by assumption $q = 0$ and by Remark 5.3 $P_2 = 0 \Rightarrow p_g = 0$, hence $1 - q + p_g = 1$, thus

$$0 \geq 1 + \frac{KC + C^2}{2} = p_a(C) \Rightarrow 0 \geq p_a(C),$$

thus we conclude that $p_a(C) = 0$ and $C \cong \mathbb{P}^1$. □

Remark 5.5. *Let $f : S' \twoheadrightarrow S$ a surjective morphism among two surfaces; it induces an injective pull-back map*

$$f^* : H^0(\Omega_S^2) \hookrightarrow H^0(\Omega_{S'}^2),$$

and more generally injective maps

$$f^* : H^0(\Omega_S^i) \hookrightarrow H^0(\Omega_{S'}^i).$$

Thus

$$p_g(S') \geq p_g(S), \quad q(S') \geq q(S),$$

and analogously one can prove

$$P_2(S') \geq P_2(S), \quad P_n(S') \geq P_n(S).$$

Definition 5.6. *A variety X of dimension n is said to be* unirational *if there exists a map from $\mathbb{P}^n$ to X that is dominant, that means that its image is dense.*

Corollary 5.7. *If S is an unirational surface, than it is rational.*

Proof. Since $p_g(\mathbb{P}^2) = P_2(\mathbb{P}^2) = 0$, by Remark 5.5, $p_g(S) = P_2(S)$ and by Theorem 5.2 S is rational. □

The Corollary holds even for unirational varieties with $n = 1$, but does not hold for $n \geq 3$.

Theorem 5.8. *Let S be a minimal rational surface, then either $S \cong \mathbb{F}_n$ with $n \neq 1$ or $S \cong \mathbb{P}^2$.*

Proof. We fix a very ample divisor H. Let us define

$$\mathcal{A} := \{C \text{ smooth }, C \cong \mathbb{P}^1, C^2 \geq 0\} \neq \varnothing.$$

Let $m := \min_{C\in\mathcal{A}} C^2 \geq 0$ and let us choose $C \in \mathcal{A}$ with $C^2 = m$ and $H \cdot C$ minimal. In other words, for every smooth rational curve C_1 with $C_1^2 = m$, $H \cdot C_1 \geq H \cdot C > 0$ (the second inequality follows as usual since C is effective and H is ample).

1. We first prove that $\forall D \in |C|$, $D \cong \mathbb{P}^1$ smooth. Let $D = \sum n_i C_i \in |C|$.

$$(K + D) \cdot C = (K + C) \cdot C = -2 < 0$$

by genus formula. Since $C^2 \geq 0$, it follows $|K+D| = \varnothing$ and then $|K+C_i| = \varnothing$ $\forall i$. Thus $0 = h^0(K + C_i) \geq \chi(K + C_i) = p_a(C_i)$ $\forall i$, where the last equality holds by the Riemann-Roch Theorem. Hence by Exercise 2.29 $\forall i$ $C_i \cong \mathbb{P}^1$ smooth. $K \cdot C \leq -2 < 0 \Rightarrow \exists i$ such that $K \cdot C_i < 0$ and, by minimality of S (arguing as in the proof of Proposition 5.4), $C_i^2 \geq 0$. Thus $C_i \in \mathcal{A}$.

$$C^2 = C{\cdot}D = C{\cdot}C_i + C{\cdot}(D - C_i) = C_i^2 + (D - C_i){\cdot}C_i + C(D - C_i) \geq C_i^2, \quad (5.1)$$

where last inequality holds since $D - C_i$, C, C_i are effective, and both C and C_i are irreducible with non negative selfintersection. By the minimality of m, (5.1) implies that $C_i^2 = m$. Moreover, by the minimality of $H \cdot C$,

$$C_i \cdot H \geq C \cdot H = H \cdot D = H \cdot C_i + H(D - C_i);$$

since $H \cdot$ (effective divisor) > 0, $D - C_i = 0$, thus $D = C_i$ and we are done.

2. We deduce that $h^0(\mathcal{O}_S(C)) \leq 3$. Let us fix $p \in S$ general enough ($p \notin C$) and, for each element of $H^0(\mathcal{O}_S(C))$ let us consider the first term of its Taylor expansion in a neighbourhood of p; we get a map

$$\mathrm{eval} : H^0(\mathcal{O}_S(C)) \to \mathcal{O}_p/m^2 \cong \mathbb{C}[x,y]/(x,y)^2 \cong \mathbb{C}^3,$$

where x, y are coordinates defined in a neighbourhood of p; basically if $f \in H^0(\mathcal{O}_S(C))$

$$\mathrm{eval}(f) = f(0) + \frac{\partial f}{\partial x}(0)x + \frac{\partial f}{\partial y}(0)y.$$

If $h^0(\mathcal{O}_S(C)) \geq 4$, then $\dim \ker \mathrm{eval} > 0$, thus $\exists f \in H^0(\mathcal{O}_S(C))$ singular in $p \Rightarrow (f) \in |C|$ is singular in p. But this contradicts the first step of the proof.

3. Finally we prove is that $m \in \{0,1\}$. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0;$$

since $q = 0 = h^1(\mathcal{O}_S)$, then $3 \geq h^0(\mathcal{O}_S(C)) = h^0(\mathcal{O}_S) + h^0(\mathcal{O}_C(C)) = 1 + m + 1 = m + 2$, thus $0 \leq m \leq 1$.

   - $m = 0 \Rightarrow h^0(\mathcal{O}_S(C)) = 2$; we have a morphism $S \to \mathbb{P}^1$ (the number of base points is bounded from above by $C^2 = 0$, hence there are no base points) whose fibers are $\mathbb{P}^1$ (since $\forall D \in |C|$, $D \cong \mathbb{P}^1$); therefore S is geometrically ruled so by Proposition 4.1 and Theorem 4.2 $S \cong \mathbb{F}_n$ with $n \neq 1$.
   - $m = 1 \Rightarrow h^0(\mathcal{O}_S(C))) = 3$; we have a morphism $S \to \mathbb{P}^2$ and it is easy to prove that is a biregular morphism (one first prove that: the linear system has no fixed part, then that it has also no base points and finally construct an inverse of the map).

□

5.2 Complex tori

Let V be a complex vector space of finite dimension m; let $\Gamma \subset V$ be a lattice, that is $\Gamma \subset B$ is a subgroup respect to the sum, $\Gamma \cong \mathbb{Z}^{2m}$ and generates V as a real vector space. $T = V/\Gamma$ is a compact variety (with a natural structure of group) said *complex torus*. If T is projective, T is said to be an *abelian variety*.

The projection on the quotient $\pi : V \to T$ is a covering map and

$$d\pi_p : T_pV \cong V \to T_{\pi(p)}T$$

is an isomorphism $\forall p \in V$ and gives a canonical identification between V and T_qT $\forall q \in T$. Let u_i be a coordinate of V; then $du_i \in H^0(\Omega^1_V)$ and $du_i \in V^* \cong \mathbb{C}^m$, and it defines a section $du_i \in H^0(\Omega^1_T)$ (by a slight abuse of notation we will also denote by du_i the pull back $\pi^* du_i$). This implies that $h^0(\Omega^1_T) \geq m$.

Moreover, if $\omega \in H^0(\Omega^1_T)$ then $\pi^*\omega \in H^0(\Omega^1_V)$; $\pi^*\omega = \sum f_i du_i$ where f_i are holomorphic and Γ−periodic functions, thus f_i is constant $\forall i$.

Corollary 5.9. $\pi^* : H^0(\Omega^1_T) \xrightarrow{\sim} H^0(\Omega^1_V) = V^*$.

This implies that if $\dim T = 2$, then $q = 2$.

Theorem 5.10. *Let $T_1 := V_1/\Gamma_1, T_2 := V_2/\Gamma_2$ be two complex tori and $u : T_1 \to T_2$ holomorphic. Then $u = t \circ a$, where t is a translation and a a group homomorphism.*

Proof. We exploit the properties of the universal covering: $\exists!\ \overline{u} : V_1 \to V_2$ such that the diagram

$$\begin{tikzcd} V_1 \arrow[r, "\overline{u}"] \arrow[d, "\pi_1"'] & V_2 \arrow[d, "\pi_2"] \\ T_1 \arrow[r, "u"] & T_2 \end{tikzcd}$$

commutes. It is immediate to prove that $\overline{u}$ is holomorphic because π_1 and π_2 are local diffeomorphisms. $\forall x \in T_1$, $\forall \gamma \in \Gamma_1$ $\overline{u}(x+\gamma) - \overline{u}(x) \in \Gamma_2$, that is a discrete set. Hence for every fixed γ, $\overline{u}(x+\gamma) - \overline{u}(x)$ is constant as function of $x \in V_1$. Hence $\partial\overline{u}/\partial z_j$ are Γ−periodic and therefore (by the maximum principle) constant functions. This means that $\overline{u}$ is affine, that is $\overline{u}(x) = Ax+b$ and clearly $A\Gamma_1 \subset \Gamma_2$; the map $x \mapsto Ax$ goes down to a group homomorphism, and $Ax \mapsto Ax + b$ descends to a translation. □

5.3 Albanese variety

We give the following Theorem, called *universal property of Albanese variety*, without proof.

Theorem 5.11. *Let X be a smooth projective variety. Then $\exists!\ A := \mathrm{Alb}(X)$ abelian variety and a morphism*

$$\alpha : X \to \mathrm{Alb}(X)$$

with connected fibers such that $\forall f : X \to T$ *with* T *complex torus* $\exists!\ \widetilde{f} : A \to T$ *such that the diagram*

$$\begin{tikzcd} X \arrow[d, "\alpha"'] \arrow[dr] & \\ A \arrow[r] & T \end{tikzcd}$$

commutes. The abelian variety $A = \mathrm{Alb}(X)$ *is called the* Albanese variety *of* X *and* α *is called* Albanese map. *Moreover the morphism* α *induces an isomorphism*

$$\alpha^* : H^0(\Omega^1_A) \to H^0(\Omega^1_X)$$

Let us see some consequences of Theorem 5.11.

1. If $H^0(\Omega^1_X) = 0$ (for example $X = \mathbb{P}^n$, rational surfaces or with $q = 0$) then each map $X \to T$ is constant.

2. $\forall F : X \to Y$ morphism of smooth projective varieties, there exists a unique morphism $\mathrm{Alb}(F) : \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\alpha_X"'] & Y \arrow[d, "\alpha_Y"] \\ \mathrm{Alb}(X) \arrow[r, "\mathrm{Alb}(F)"] & \mathrm{Alb}(Y) \end{tikzcd}$$

   commutes.

3. $\alpha(X)$ is not contained in any sub-torus $T' \subset \mathrm{Alb}(X)$, indeed, considering $X \to T'$ the uniqueness is contradicted. In particular, if $H^0(\Omega^1_X) \neq 0$ $\alpha(X)$ is not reduced to a point, that is α is not constant. This implies that α is constant$\Leftrightarrow$ $H^0(\Omega^1_X) = 0$, hence α is constant $\Leftrightarrow q = 0$.

4. If X is a curve, then α is the Abel-Jacobi map, $\mathrm{Alb}(X)$ is equal to the Jacobian JX and

$$\alpha : X \to JX$$

   is embedded as a smooth curve.

Proposition 5.12. *Let* S *be a projective smooth surface,* $\alpha : S \to \mathrm{Alb}(S)$ *the Albanese map. Suppose that* $\alpha(S) = C$ *is a curve. Then* C *is a smooth curve of genus* q.

The following lemma is given without proof.

Lemma 5.13. *Let S be a smooth projective surface and C an irreducible curve; let $\eta : \widetilde{C} \to C$ the* normalization *of C (that is $\widetilde{C}$ is a smooth curve and the map η is generically injective) then $\exists S \to \widetilde{C}$ such that*

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & \widetilde{C} \arrow[d] \\ & C \end{tikzcd}$$

commutes.

Proof of Proposition 5.12. We have a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, mapsto, "\alpha"'] & \widetilde{C} \arrow[dl] \arrow[drr] & & \\ C \subset \mathrm{Alb}(S) \arrow[rrr, shift left, "u"] & & & \mathrm{Alb}(\widetilde{C}) = J\widetilde{C} \arrow[lll, shift left, "v"] \end{tikzcd}$$

where both u and v exist by the universal property. Moreover, by uniqueness, $u = v^{-1}$. Hence $\alpha(S) = \widetilde{C} \subset JC = \mathrm{Alb}(C) = \mathrm{Alb}(S)$, hence it is smooth of genus $\dim JC = \dim \mathrm{Alb}(S) = q$. □

Hence for all S projective we have three possibilities:

- $q = 0$;
- $\dim \alpha(S) = 1$ and $\alpha(S) = C \hookrightarrow JC$ smooth curve of genus q;
- $\dim \alpha(S) = 2$.

Lemma 5.14. *If $p_g = 0$ and $q \geq 1$, then* $\dim \alpha(S) = 1$.

Proof. Since $q \neq 0$ $\dim \alpha(S) \neq 0$. If $\dim \alpha(S) = 2$, let then $p \in \mathrm{Alb}(S)$ be a smooth point, then there exists $T_p\alpha(S) \subset T_p \mathrm{Alb}(S)$; we shall remark that

$$p \in \alpha(S) \subset \mathrm{Alb}(S) = V/\Gamma$$
$$T_p\alpha(S) \subset T_p \mathrm{Alb}(S) \cong V.$$

Let $u_1, u_2 \in V$ independent directions in $T_p\alpha(S)$. Let

$$\omega := du_1 \wedge du_2 \in H^0(\Omega^2(\mathrm{Alb}(S)))$$

that does not vanish identically on $T_p\alpha(S) \subset V$. Then $\alpha^*\omega \in H^0(\Omega^2(S))$ does not identically vanish in $\alpha^{-1}(p)$. Hence $p_g(S) \neq 0$: a contradiction. □

Theorem 5.15. *Let $\Phi : S \dashrightarrow S'$ be a birational map among two non ruled projective surfaces. If S and S' are minimal, then Φ is biregular.*

Another way to state Theorem 5.15 is that non ruled surfaces have a unique minimal model.

Proof. The only thing to prove is that Φ is a morphism. Let us resolve Φ in the way outlined in the following diagram:

$$\begin{tikzcd} & & \hat{S} \arrow[dl, "\varepsilon_n"'] \arrow[ddr, "f"] & \\ & \cdots \arrow[dl, "\varepsilon_1"'] & & \\ S \arrow[rrr, dashed] & & & S' \end{tikzcd}$$

let n be the minimum number of blow ups necessary to resolve Φ. Arguing by contradiction, let us suppose that $n > 1$; let E be the exceptional curve of ε_n: $E \cong \mathbb{P}^1$, $E^2 = -1$ and $K \cdot E = -1$. $f|_E$ is not constant, since otherwise $\varepsilon_1 \circ \cdots \circ \varepsilon_{n-1}$ would have resolved Φ as well. $C = f(E)$ is an irreducible divisor in S', hence E is the strict transform of C. Recall that f is also a composition of blow ups. Then one of the following occurs:

- None of the blow ups composing f involves points of C;
- Some of those blow ups involves points of C.

In the first case $f|_E : E \to C$ is an isomorphism, $C \cong \mathbb{P}^1$, $C^2 = E^2 = -1$ and $K_{\hat{S}} \cdot E = K_{S'} \cdot C = -1$, contradicting the minimality of S'. Hence the second case holds; then E is the strict transform of C and by the formulas of the blow ups (see Exercise 2.10) we deduce that $C^2 > E^2$ and $K_{S'} \cdot C < K_{\hat{S}} \cdot E = -1$. Since $p_a(C) \geq 0$, by the genus formula, $C^2 \geq 0$. Note that these two inequalities imply

$$(nK'_S) \cdot C \leq -2n < 0 \qquad \forall n \geq 1,$$

hence $|nK_{S'}| = \emptyset$; indeed, if $|nK_{S'}|$ contains a divisor D, then $D \cdot C \geq 0$, since every curve with positive autointersection number intersects any effective divisor nonnegatively. This means that $P_n = 0$ $\forall n \geq 1$. We must now distinguish two cases:

- If $q = 0$, then, taking into account that $P_2 = 0$, Castelnuovo's Theorem implies that S' is rational, which is excluded by hypotheses.
- If $q \geq 1$, $p_g = 0$ $P_n = 0$ $\forall n$. Then by Theorem 5.11 and Proposition 5.12, there exists
$$\alpha : S' \to B \subset \mathrm{Alb}(S')$$

with B smooth curve of genus q and $\mathrm{Alb}(S')$ abelian variety of dimension q. By Lemma 3.11 every map from E to B is constant; considering the composition

$$E \xrightarrow{f|_E} C \xrightarrow{\alpha|_C} B$$

it follows that $\alpha|_C$ is constant. Therefore C is contained in a fiber of α. By Lemma 3.7 $C^2 \geq 0$, then $C^2 = 0$ and rC with $r \geq 1$ is a fiber of α. Moreover $K \cdot C \leq -2$, thus the genus formula implies

$$0 \leq p_a(C) = 1 + \frac{C^2 + K \cdot C}{2} \leq 0,$$

hence $p_a(C) = 0$, $C \cong \mathbb{P}^1$ and $K \cdot C = -2$. Albanese map has a regular value, hence for $p \in \mathrm{Reg}(\alpha)$ $F_p = \alpha^* p$ is a smooth fiber. Since it is connected, it is irreducible. By the genus formula

$$0 \leq p_a(F_p) = 1 + \frac{K \cdot F + F^2}{2} \overset{(\#)}{=} 1 + \frac{K(rC) + (rC)^2}{2} = 1 - r,$$

where equality $(\#)$ holds for Remark 3.4. Hence $r \leq 1$, thus $r = 1$. The Noether-Enriques Theorem implies that S' is ruled over B, a contradiction.

□

5.4 Surfaces with $p_g = 0$ and $q \geq 1$

Studying the Albanese map of these surfaces one can prove that they are quotients of the product of two curves by the action of a finite group. We give the precise statement of their classification without proof.

Theorem 5.16. *S is a minimal projective surface with $p_g = 0$ and $q \geq 1$ if and only if $S = (B \times F)/G$ where B, F are Riemann surfaces, G is a finite group, with*

$$G \hookrightarrow \mathrm{Aut}(B) \qquad G \hookrightarrow \mathrm{Aut}(F),$$

that acts over $B \times F$ in the natural way (that is $g(b, f) = (gb, gf)$ for $g \in G$, $(b, f) \in B \times F$) and

- *B/G is an elliptic curve;*
- *$F/G \cong \mathbb{P}^1$;*
- *one of the following holds:*

1. *B is elliptic and G is a translation group;*
2. *F is elliptic and G acts freely on $B \times F$.*

This implies that $q = 1$ and $K^2 = 0$.