Viscosity solutions for an optimal control problem with Preisach hysteresis nonlinearities

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Abstract

We study a finite horizon problem for a system whose evolution is governed by a controlled ordinary differential equation, which takes also account of a hysteretic component: namely, the output of a Preisach operator of hysteresis. We derive a discontinuous infinite dimensional Hamilton-Jacobi equation and prove that, under fairly general hypotheses, the value function is the unique bounded and uniformly continuous viscosity solution of the corresponding Cauchy problem.

1 Introduction

In this paper, we derive and study the infinite dimensional and discontinuous Hamilton-Jacobi-Bellman equation, for an optimal control problem of finite horizon type, given by the following controlled dynamical system

$$\begin{cases} y'(t) = f(y(t), w(t), \alpha(t)) & t > 0 \\ w(t) = \mathcal{H}_\mu[y, \xi_0](t) & t \geq 0 \\ y(0) = y_0, \end{cases} \tag{1.1}$$

and by the following cost functional and value function

$$J(y_0, \xi_0, t, \alpha) := \int_0^t e^{-\lambda s} \ell(y(s), w(s), \alpha(s)) ds + e^{-\lambda t} h(y(t), w(t));$$
$$V(y_0, \xi_0, t) := \inf_{\alpha \in \mathcal{A}} J(y_0, \xi_0, t, \alpha).$$

Here, $\mathcal{A}$ is the set of measurable controls $\alpha : [0, +\infty[\to A$, with A compact; the scalar functions $y(\cdot)$ and $w(\cdot)$ are, respectively, the input and the output of the Preisach hysteresis operator $\mathcal{H}_\mu$, with initial configuration ξ_0.

The Preisach operator consists of a superposition of (possibly infinite) rectangular hysteresis loops, the so-called delayed relays. Every delayed relay gives a switching rule between a continuous scalar input and a discontinuous output. The delayed relay is characterized by the thresholds $\rho_1 < \rho_2$, respectively for switching down (from 1 to -1) and for switching up (from -1 to 1) (see Figure 1). For an input $u \in C^0([0, +\infty[)$, for an initial value of the output $\xi \in \{1, -1\}$, and for thresholds $\rho = (\rho_1, \rho_2)$, we denote the output by

$$w(\cdot) =: h_\rho[u, \xi](\cdot).$$

Every delayed relay corresponds to a point of the so-called Preisach plane

$$\mathcal{P} := \left\{ \rho = (\rho_1, \rho_2) \Big| \rho_1 < \rho_2 \right\}.$$

Given a signed finite Borel measure μ on $\mathcal{P}$, for every continuous input $u \in C^0([0, +\infty[)$, and for every initial configuration $\xi : \mathcal{P} \to \{1, -1\}$, the output of the Preisach operator $\mathcal{H}_\mu$ is defined by

$$w(t) = \mathcal{H}_\mu[u, \xi](t) := \int_{\mathcal{P}} h_\rho[u, \xi(\rho)](t) d\mu(\rho) \quad \forall t \geq 0.$$

The system exhibits the evolution of the so-called internal variable

$$\xi_t : \mathcal{P} \to \{1, -1\}, \ \ \rho \mapsto h_\rho[u, \xi(\rho)](t). \tag{1.2}$$

Such evolution gives the evolution of two sets on the Preisach plane, namely the sets of points respectively corresponding to relays switched on 1 (i.e. $\xi_t \equiv 1$) and to relays switched on -1 (i.e. $\xi_t \equiv -1$), (see Figure 2). We can restricted ourselves to the case when those sets are separated by a maximal antimonotone graph on $\mathcal{P}$. We have the bijection

$$\xi \text{ internal variable } \longleftrightarrow \ B_\xi \text{ maximal antimonotone graph on } \mathcal{P}. \tag{1.3}$$

Under general hypotheses on μ, the output is continuous. Moreover, for a suitable measure ν on $\mathcal{P}$, the map $(u, \xi) \mapsto \mathcal{H}_\mu[u, \xi](\cdot)$ is Lipschitz continuous for the topologies of $C^0([0, T]) \times L^2(\mathcal{P}, \nu)$ and $C^0([0, T])$.

Let us come back to the optimal control problem. Note that we are working with two measures on $\mathcal{P}$: μ, which is signed and enters on the definition of the Preisach operator: it is a datum of the problem; ν, which is positive and gives the space $L^2(\mathcal{P},\nu)$ into which we are going to embed our problem: its choice is at our disposal. Of course, μ and ν should be somehow related.

By the regularity of $\mathcal{H}_\mu$, (1.1) has a unique solution, and, since $\mathcal{H}_\mu$ satisfies a semigroup property, we easily derive the dynamic programming principle for the value function V. From this, we deduce a Hamilton-Jacobi equation satisfied by V in the (adapted) viscosity sense. Such equation consists of a usual "finite dimensional part" (which involves the derivative with respect to t and to y), and of a "infinite dimensional" part (which involves the Fréchet differentials with respect to $\xi \in L^2(\mathcal{P},\nu)$).

The probably "natural" space for the functions ξ, is, after the correspondence (1.3) and after a change of variables, the space of the maximal antimonotone graphs on $\mathcal{P}$ regarded as continuous functions. However, such space is not reflexive. Hence, we regard ξ as element of $L^2(\mathcal{P},\nu)$.

This brings some problem for the Hamilton-Jacobi equation and its viscosity solution approach. One problem, which is also due to the Preisach hysteresis model, is the fact that the evolution $t \mapsto \xi_t$, given by (1.1) and (1.2), does not belong to $W^{1,1}((0,T),L^2(\mathcal{P},\nu))$. If the input is increasing (respectively, decreasing), then the right derivative of $t \mapsto \xi_t$ is related to the possible final horizontal segment (respectively, final vertical segment) which links B_{ξ_t} to the line $\rho_1=\rho_2$ (see Figure 2). Hence, we are lead to consider a quantity T^u (respectively, T^l), which in some sense is the trace from above (respectively, from the left) on the final horizontal segment (respectively, on the final vertical segment) of a suitable combination of the first and second Fréchet differentials of the test function. The presence of the second Fréchet differential is due to the fact that we have to control the difference of the values that a test function takes on two elements of $L^2(\mathcal{P},\nu)$, which differ only for the behavior on a thin rectangle. To this end, we need to work with the second order Taylor formula.

A second problem is the fact that the two quantities $T^u[\phi](y,\xi)$ and $T^l[\phi](y,\xi)$, which depend on the test function ϕ and on the state-space point (y,ξ), are not continuous for the topology of $(y,\xi) \in I\!R \times L^2(\mathcal{P},\nu)$. This fact leads us to consider a "lower" and a "upper" Hamilton-Jacobi equation, given by the two Hamiltonians

$$
\begin{aligned}
&H^-(y,\xi,\phi_t,D\phi,D^2\phi) := \sup_{a\in A}\Big\{-\phi_t f(y,w,a)\\
&-\Big(f^+(y,w,a)\,(T^u[\phi](y,\xi))^+ + f^-(y,w,a)\,\Big(T^l[\phi](y,\xi)\Big)^-\Big)\\
&-\ell(y,w,a)\Big\},\\
&H^+(y,\xi,\phi_t,D\phi,D^2\phi) := \sup_{a\in A}\Big\{-\phi_t f(y,w,a)\\
&+\Big(f^+(y,w,a)\,(T^u[\phi](y,\xi))^- + f^-(y,w,a)\,\Big(T^l[\phi](y,\xi)\Big)^+\Big)\\
&-\ell(y,w,a)\Big\},
\end{aligned}
$$

where $(\cdot)^+$ and $(\cdot)^-$ are the positive and the negative part, and $w = \int_{\mathcal{P}}\xi d\mu$.

The "infinite dimensional" components of the Hamiltonians, in some sense, play the role of a boundary condition, where the boundary is given by the couples (y,ξ) with final horizontal or final vertical segment (if such segments do not appear, then T^u and T^l vanish). The problem here is that such "boundary" is dense in $\mathbb{R}\times L^2(\mathcal{P},\nu)$, and moreover the topology of $L^2(\mathcal{P},\nu)$ is not strong enough in order to force the couple (y,ξ) to not belong to such "boundary". However, under fairly general hypotheses on the measure ν (without any pretension of sharpness), and working with suitable test functions, we are able to force the sign of T^u and T^l, in order make them vanishing inside the Hamiltonians. In this effort, we are helped by the fact that the state-space has the property that every bounded subset has compact closure in $\mathbb{R}\times L^2(\mathcal{P},\nu)$, and hence we do not need to use any variational principle.

Our main result gives, under general hypotheses, the uniqueness of the value function V as viscosity solution of the Cauchy problem

$$
\begin{cases}
V_t(y,\xi,t) + \lambda V(y,\xi,t) + H^-(y,\xi,DV(y,\xi,t),D^2(y,\xi,t)) \le 0\\
V_t(y,\xi,t) + \lambda V(y,\xi,t) + H^+(y,\xi,DV(y,\xi,t),D^2(y,\xi,t)) \ge 0\\
V(y,\xi,0) = h(y,w),
\end{cases}
$$

with test functions from a suitable subset of $C^1(\mathbb{R}\times]0,+\infty[)\times C^2(L^2(\mathcal{P},\nu))$.

The Preisach operator is a well-studied model for scalar hysteresis (see the monographs Krasnoselskii-Pokrovskii [8], Mayergoyz [11], Visintin [12], Brokate-Sprekels [6] and Krejci [9]). It is probably the most versatile model for representing hysteresis phenomena (i.e. rate independent memory effects) from physical and natural sciences, as for instance ferromagnetic hysteresis, hysteresis in filtration through porous media, phase transitions, shape memory alloys and smart structure materials.

An optimal control problem for ODE with the Preisach hysteresis model is studied by Brokate in [5], concerning necessary conditions for optimality. In that work the author does not apply the dynamic programming method. Recently, Belbas and Mayergoyz studied in [4] (see also the references therein) an optimal control problem for ODE with hysteresis of Madelung type. They derive the dynamic programming principle, but they do not study the corresponding Hamilton-Jacobi equation. The present author in [1], [2] studied two optimal control problems for ODE with hysteresis. In those papers, a Hamilton-Jacobi equation satisfied by the value function is derived and studied in the framework of viscosity solutions theory. However, since the used hysteresis models are different, the derived equations are also different from the present one (in particular, in [1] the case of the Play/Prandtl-Ishlinskii model and in [2] the case of a finite sum of delayed relays are, respectively, studied).

The viscosity solutions theory for Hamilton-Jacobi equations, and its connection with optimal control problems, is a well known framework. We refer the reader to the book by Bardi-Capuzzo Dolcetta [3] for the theory in finite dimension and to the works by Crandall-Lions [7] and Lions [10], for some results about the infinite dimensional case of first and second order (however, our situation is different).

The plan of the paper is as follows. In Section 2, we describe the delayed relay and the Preisach model. In Section 3, we state the control problem. In Section 4, we rigorously derive the Hamilton-Jacobi equation satisfied by the value function. In Section 5, we prove the uniqueness result.

2 The Preisach model

In this section we introduce the Preisach model of hysteresis. We essentially follow Visintin [12] (and we refer the reader to it for a more precise and rigorous description). However, we change somewhere definitions in order to fit better the control problem in the sequel.

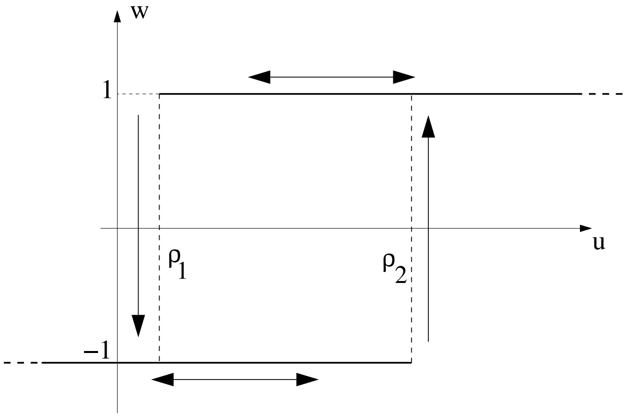

Figure 1: Delayed relay.

2.1 The discontinuous delayed relay

Let us fix two thresholds $\rho_1 < \rho_2$ and consider a graph as in Figure 1. It represents the relationship between a continuous input $u(t)$ and a discontinuous output $w(t)$ which switches between the values 1 and -1. The (delayed) switching rule is as follows. Let $u(0) \leq \rho_1$. Then $w(0) = -1$, and $w(t) = -1$ as long as $u(t) < \rho_2$; if at some instant u reaches ρ_2, then w switches up to 1, where it remains as long as $u(t) > \rho_1$; if later, u reaches ρ_1 then w switches down to -1; and so on. Otherwise, if $u(0) \geq \rho_2$, then $w(0) = 1$ and the evolution of w is similarly described. Finally, if $\rho_1 < u(0) < \rho_2$, then we have to give the initial value of w, because both 1 and -1 are admissible, and then the evolution follows the same rules as before. For any time interval $[0, T]$ and for every couple of thresholds $\rho = (\rho_1, \rho_2)$, we define the corresponding delayed relay operator in the following way

$$h_\rho : C^0([0,T]) \times \{-1, 1\} \to L^\infty(0,T), \quad (u, \xi) \mapsto h_\rho[u, \xi](\cdot) := w(\cdot),$$

where ξ (the initial value of the output) plays a role only if $\rho_1 < u(0) < \rho_2$.

2.2 The Preisach model

The thresholds of the delayed relay operators form the so called Preisach plane

$$\mathcal{P} := \left\{ \rho = (\rho_1, \rho_2) \in I\!R^2 \middle| \rho_1 < \rho_2 \right\}.$$

We denote by $\mathcal{R}$ the family of Borel measurable functions $\mathcal{P} \to \{-1, 1\}$, and by $\xi : \rho \mapsto \xi(\rho)$ a generic element of $\mathcal{R}$, which we intend to represent the initial configurations of all the delayed relays. We fix a finite signed Borel measure μ over $\mathcal{P}$, and introduce the corresponding Preisach operator

$$\begin{aligned} &\mathcal{H}_\mu : C^0([0,T]) \times \mathcal{R} \to L^\infty(0,T) \\ &w(t) = \mathcal{H}_\mu[u,\xi](t) := \int_{\mathcal{P}} h_\rho[u,\xi(\rho)](t) d\mu(\rho) \quad \forall t \in [0,T]. \end{aligned}$$

Setting $\xi_0(\rho) = \xi(\rho)$, the evolution $t \mapsto \xi_t$ with $\xi_t(\rho) = h_\rho[u, \xi_0](t)$ represents the evolution of the internal variables ξ. Note that, even if we know the evolution of the input u, the knowledge of the value of the output $w(t)$ is not enough for determining the evolution of w in $]t, T]$; we need to now the value of the internal variable ξ_t, namely the configuration of all relays.

Looking at the Preisach plane, we can see an interesting and useful geometric interpretation of the evolution of the internal variables ξ_t. First of all note that, setting $A^+(t)$ (respectively $A^-(t)$) the subset of $\mathcal{P}$ given by the points ρ corresponding to the relays which at the time t are switched on 1 (respectively -1), then by definition we have

$$\mathcal{H}_\mu[u,\xi](t) = \mu(A^+(t)) - \mu(A^-(t)).$$

Let us fix an initial configuration ξ. Let us suppose that there exists a maximal antimonotone graph B on $\overline{\mathcal{P}}$ such that all the relays corresponding to the points of $\mathcal{P}$ which are above (respectively below) B are switched on -1 (respectively on 1), see Figure 2(i). The "end point" of the graph B is the point $(u(0), u(0))$ on the line $\rho_1 = \rho_2$, where $u(0)$ is the initial value of the input. Let us suppose that u increases from $u(0)$ to $u(t')$. Then some relays switch from -1 to 1, and the new configuration is represented by Figure 2(ii): a horizontal segment is formed and is up-ward moving on the Preisach plane. Now, let us suppose that the input decreases from $u(t')$ to $u(t'')$. Then some relays switch from 1 to -1, and the new configuration is represented by Figure 2(iii): a vertical segment is formed and is left-ward moving on the Preisach plane. If later, the input increases again to a value $u(t^*)$, then the new configuration is represented by Figure 2(iv).

A rigorous proof of the following result can be found in Visintin [12].

Theorem 2.1 *Let us take an initial configuration ξ which corresponds to a maximal antimonotone graph B (as above explained). By applying an input*

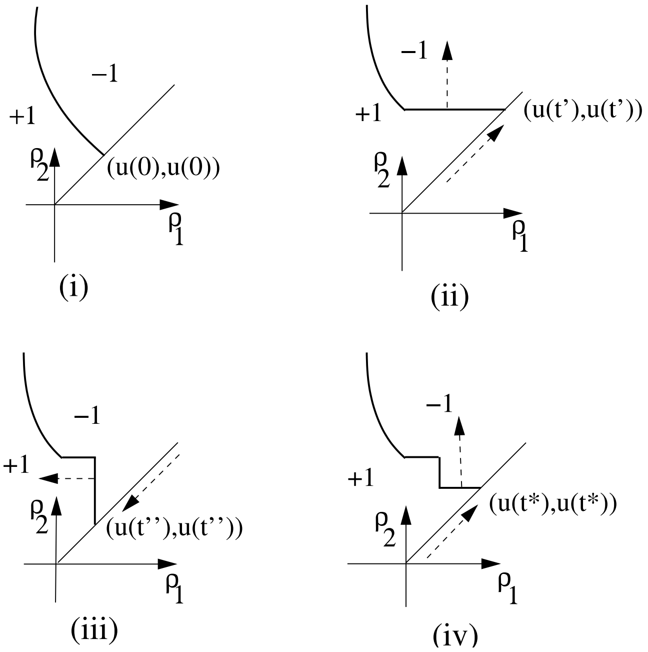

Figure 2: Evolution on the Preisach plane

$u \in C^0([0,T])$, at any time $t \in [0,T]$ we have a configuration on the Preisach plane given by a maximal antimonotone graph which equals B out of a compact set, and, near the line $\rho_1 = \rho_2$, consists of at most a countable family of horizontal and vertical segments.

Definition 2.1. *We denote by $\mathcal{B}$ the set of all maximal antimonotone graphs on $\overline{\mathcal{P}}$, and by $\mathcal{S}$ the set of all functions $\xi : \mathcal{P} \to \{-1, 1\}$ such that there exists $B \in \mathcal{B}$ with $\xi \equiv 1$ (respectively $\xi \equiv -1$) below (respectively above) B. Hence there exists a bijection between $\mathcal{S}$ and $\mathcal{B}$. For every $\xi \in \mathcal{S}$ we denote by $B_\xi \in \mathcal{B}$ the corresponding maximal antimonotone graph. Moreover, we denote by $\mathcal{O}$ the set given by the couples $(y, \xi) \in I\!R \times \mathcal{S}$ such that $(y, y) \in B_\xi$. Moreover, we say that a couple $(u, \xi) \in C^0([0,T]) \times \mathcal{S}$ is admissible if $(u(0), \xi) \in \mathcal{O}$.*

Note that we do not care about the value of ξ on the graph B_ξ itself, indeed we will always use measures on $\mathcal{P}$ which "do not see" such graphs.

Sometimes, it is useful to consider a new system of coordinates on $\mathcal{P}$:

$$\sigma_1 := \frac{\rho_1 - \rho_2}{\sqrt{2}}, \quad \sigma_2 := \frac{\rho_1 + \rho_2}{\sqrt{2}}. \tag{2.1}$$

With respect to those coordinates, every element of $\mathcal{B}$ is the graph of a Lipschitz continuous function $\sigma_1 \mapsto \sigma_2$, with Lipschitz constant not larger than 1. For $B \in \mathcal{B}$, $\varepsilon > 0$, we define the ε-neighborhood of B in $\mathcal{P}$

$$N(B,\varepsilon) := \left\{ (\sigma_1, \sigma_2 + \delta) \in I\!R^+ \times I\!R \middle| (\sigma_1,\sigma_2) \in B, |\delta| \le \varepsilon \right\}.$$

Theorem 2.2 *Let μ be a finite signed Borel measure on $\mathcal{P}$. Let us suppose that there exists $L > 0$ such that*

$$|\mu|(N(B,\varepsilon)) \le L\varepsilon \qquad \forall \varepsilon > 0, \forall B \in \mathcal{B}, \tag{2.2}$$

where $|\mu|$ is the total variation of μ. Then, for every admissible couple (u,ξ) we have $\mathcal{H}_\mu[u,\xi] \in C^0([0,T])$, and there exists $C > 0$ (depending only on L) such that for every admissible couples $(u,\xi),(v,\eta)$

$$\|\mathcal{H}_\mu[u,\xi] - \mathcal{H}_\mu[v,\eta]\|_{C^0([0,T])} \le C \left(\|u-v\|_{C^0([0,T])} + \|\xi - \eta\|_{L^1(\mathcal{P},|\mu|)} \right). \tag{2.3}$$

For the proof see Visintin [12] pages 114-115, after slight modifications due to our definition of admissible couples.

For the proof of the following theorem see Visintin [12], page 100.

Theorem 2.3 *For any signed Borel measure μ over $\mathcal{P}$, the corresponding Preisach operator $\mathcal{H}_\mu$ fulfills the following semigroup property*

$$\begin{array}{l} \text{if } [t_1,t_2] \subset [0,T], \text{then } \mathcal{H}_\mu[u,\xi](t_2) = \mathcal{H}_\mu[u(t_1+\cdot),\xi_{t_1}](t_2 - t_1) \\ \text{for every admissible } (u,\xi). \end{array} \tag{2.4}$$

Remark 2.1. Let ν be a finite positive Borel measure on $\mathcal{P}$. If

$$|\mu|(B) \le \nu(B) \;\; \forall \text{ Borel set } B \subseteq \mathcal{P}, \tag{2.5}$$

then $\mathcal{H}_\mu$ is also Lipschitz continuous when $\mathcal{S}$ has the topology of $L^1(\mathcal{P},\nu)$.

Let $(y,\xi) \in \mathcal{O}$. It is important to know how B_ξ is linked to the line $\rho_1 = \rho_2$: by a horizontal segment or by a vertical segment or in a "oblique" manner (the latter may hold only at the initial time). We define

$$\begin{array}{l} \hat\rho_1(y,\xi) := \inf\left\{ \rho_1 \le y | [\rho_1, y] \times \{y\} \subset B_\xi \right\} \\ \hat\rho_2(y,\xi) := \sup\left\{ \rho_2 \ge y | \{y\} \times [y,\rho_2] \subset B_\xi \right\}. \end{array} \tag{2.6}$$

For instance, when $\hat{\rho}_1(y,\xi) < y$, then B_ξ is linked by a horizontal segment. At least one from $\hat{\rho}_1(y,\xi), \hat{\rho}_2(y,\xi)$ always coincides with y.

Remark 2.2. When we regard $\mathcal{S}$ as equipped with the topology of $L^p(\mathcal{P},\nu)$, we have to identify functions which coincide ν-a.e. on $\mathcal{P}$. A more rigorous definition of $\mathcal{O}$ should be: $(y,\xi) \in I\!\!R \times L^p(\mathcal{P},\nu)$ belongs to $\mathcal{O}$ if and only if there exists a maximal antimonotone graph B on $\overline{\mathcal{P}}$, such that $(y,y) \in B$ and $\xi = 1$ ν-a.e. below B and $\xi = -1$ ν-a.e. above B. Hence, for any $(y,\xi) \in \mathcal{O} \subset I\!\!R \times L^p(\mathcal{P},\nu)$, we may have different choices of the representative element of the equivalence class, of the graph B_ξ, and consequently of the quantities $\hat{\rho}_1(y,\xi)$ and $\hat{\rho}_2(y,\xi)$. However, this will be not a problem for the sequel of the paper: we can just work with an arbitrary choice of the representatives (see also Remark 4.1). Note that B_ξ, $\hat{\rho}_1$ and $\hat{\rho}_2$ are uniquely defined if $\mathcal{S}$ is equipped with the norm of $L^p(\mathcal{P},\mu')$ with μ' satisfying

$$\mu'(U) > 0 \quad \forall U \subset \mathcal{P} \text{ open.}$$

However, even in that case, $\hat{\rho}_1(y,\xi)$ and $\hat{\rho}_2(y,\xi)$ are not continuous. We can approximate any couple (y,ξ) satisfying $y - \hat{\rho}_1(y,\xi) > 0$ by couples (y_n,ξ_n) satisfying $y_n - \hat{\rho}_1(y_n,\xi_n) \equiv 0$.

Proposition 2.4 *Let ν be a finite positive Borel measure on $\mathcal{P}$. Then the closure of any bounded set of $\mathcal{O} \subset I\!\!R \times L^2(\mathcal{P},\nu)$ is compact.*

Proof. First of all note that a subset of $\mathcal{O}$ is bounded if and only if its projection on the y-component is bounded in $I\!\!R$. Let (y_n,ξ_n) be a bounded sequence in $\mathcal{O}$. The boundedness of y_n implies that, on every compact set of $\mathcal{P}$, graphs B_{ξ_n} are equibounded and equilipschitzean with respect to the coordinates σ_1,σ_2. Note that the uniform convergence of B_{ξ_n} on a compact set $K \subset \mathcal{P}$ implies that the limit B is still a maximal antimonotone graph on K, and that if ξ is the corresponding function on K (i.e. $B = B_\xi$ on K), then $\xi_n \to \xi$ in $L^2(K,\nu)$. Hence we obtain the conclusion using the Ascoli-Arzelà Theorem and the fact that ν is finite on $\mathcal{P}$. $\square$

3 The control problem

We consider the following controlled dynamical system

$$
\begin{cases}
y'(t) = f(y(t), w(t), \alpha(t)) & t > 0 \\
w(t) = \mathcal{H}_\mu[y, \xi_0](t) & t \geq 0 \\
y(0) = y_0 & \\
(y_0, \xi_0) \in \mathcal{O}, &
\end{cases}
\tag{3.1}
$$

where the continuous function $f : I\!\!R \times I\!\!R \times A \to I\!\!R$, with $A \subset I\!\!R^q$ compact, satisfies

$$
\begin{array}{l}
\exists M > 0 \text{ such that } |f(y, w, a)| \leq M \ \forall (y, w, a) \in I\!\!R \times I\!\!R \times A \\
|f(y_1, w_1, a) - f(y_2, w_2, a)| \leq M(|y_1 - y_2| + |w_1 - w_2|) \\
\forall (y_1, w_1, a), (y_2, w_2, a) \in I\!\!R \times I\!\!R \times A;
\end{array}
\tag{3.2}
$$

and α is a measurable control, that is

$$
\alpha \in \mathcal{A} := \{\beta : [0, +\infty[\to A \text{ measurable}\}.
$$

The following result follows from the Lipschitz continuity of f, and from the Lipschitz continuity and the semigroup property of the Preisach operator (Theorems 2.2 and 2.3, see also Remark 2.1). The proof of existence can be given by the contraction principle or by a delayed approximation argument; the estimate (3.3) can be proved using also the Gronwall inequality (see for instance Brokate-Sprekels [6] and Bagagiolo [1]).

Proposition 3.1 *Let μ and ν be respectively a finite signed Borel measure and a finite positive Borel measure on $\mathcal{P}$ satisfying (2.2) and (2.5). For any choice of $(y_0, \xi_0) \in \mathcal{O}$ and $\alpha \in \mathcal{A}$, there exists a unique solution $(y(\cdot), w(\cdot))$ of the system (3.1), defined in $[0, +\infty[$. Moreover, we also denote by $t \mapsto \xi_t$ the evolution of the internal variables ξ. It is easy to see that $(y(t), \xi_t) \in \mathcal{O}$ for every $t \geq 0$. Moreover, we have the following regularities: $y(\cdot) \in C([0, +\infty[, I\!\!R)$, $w(\cdot) \in C([0, +\infty[, I\!\!R)$, and $\xi_{(\cdot)} \in C([0, +\infty[, L^1(\mathcal{P}, \nu))$. Finally, for every $T > 0$ there exists a modulus of continuity ω_T such that, for every initial values $(y_1, \xi_1), (y_2, \xi_2) \in \mathcal{O}$, and for every control $\alpha \in \mathcal{A}$ the following inequality holds (here $(y_i(\cdot), w_i(\cdot))$ is the solution corresponding to (y_i, ξ_i) as initial datum, and α as control)*

$$
\begin{array}{l}
|y_1(t) - y_2(t)| + |w_1(t) - w_2(t)| \\
\leq \omega_T \left(\max \left\{ |y_1 - y_2|, \|\xi_1 - \xi_2\|_{L^1(\mathcal{P}, \nu)} \right\} \right) \ \forall t \in [0, T].
\end{array}
\tag{3.3}
$$

Remark 3.1. Since $\|\xi_1 - \xi_2\|_{L^1(\mathcal{P},\nu)} = \|\xi_1 - \xi_2\|^2_{L^2(\mathcal{P},\nu)}/2$ for every $\xi_1, \xi_2 \in \mathcal{S}$, then the trajectory is also continuous for the topology of $L^2(\mathcal{P}, \nu)$.

Let us consider a nonnegative function (running cost)

$$\ell : I\!\!R \times I\!\!R \times A \to [0, +\infty[\text{ continuous}$$

such that there exist $M > 0$ and a modulus of continuity ω satisfying

$$\begin{array}{l} |\ell(y, w, a)| \leq M, \\ |\ell(y_1, w_1, a) - \ell(y_1, w_1, a)| \leq \omega(|y_1 - y_2| + |w_1 - w_2|) \\ \forall (y, w, a), (y_1, w_1, a), (y_2, w_2, a) \in I\!\!R \times I\!\!R \times A. \end{array} \tag{3.4}$$

Moreover let us consider a nonnegative function (final cost)

$$h : I\!\!R \times I\!\!R \to [0, +\infty[\text{ bounded and uniformly continuous.} \tag{3.5}$$

Finally, let $\lambda \geq 0$ be a discount factor. For any initial state $(y, \xi) \in \mathcal{O}$, any control $\alpha \in \mathcal{A}$, and any time $t \geq 0$ we consider the following cost functional

$$J(y, \xi, t, \alpha) := \int_0^t e^{-\lambda s} \ell(y(s), w(s), \alpha(s)) ds + e^{-\lambda t} h(y(t), w(t)), \tag{3.6}$$

where $(y(\cdot), w(\cdot))$ is the trajectory of (3.1) with initial state (y, ξ) and control α. We then define the value function

$$V(y, \xi, t) := \inf_{\alpha \in \mathcal{A}} J(y, \xi, t, \alpha) \quad \forall (y, \xi, t) \in \mathcal{O} \times [0, +\infty[. \tag{3.7}$$

4 Dynamic programming

Due to the semigroup property (2.4) of the Preisach operator, the Dynamic Programming Principle holds.

Theorem 4.1 *(DPP:Dynamic Programming Principle) For all $(y, \xi) \in \mathcal{O}$, for all $t \geq 0$ and for all $0 \leq \tau \leq t$*

$$\begin{aligned} V(y, \xi, t) &= \inf_{\alpha \in \mathcal{A}} \Big(\int_0^\tau e^{-\lambda s} \ell(y(s), w(s)\alpha(s)) ds \\ &\quad + e^{-\lambda \tau} V(y(\tau), \xi_\tau, t - \tau) \Big). \end{aligned} \tag{4.1}$$

4.1 Test functions

In this subsection, ν is a finite positive Borel measure on $\mathcal{P}$.

For every $s > 0$ and for every $(y, \xi) \in \mathcal{O}$ we define the sets on $\mathcal{P}$

$$\mathcal{P}_s^u(y,\xi) := \left\{ \rho \in \mathcal{P} \middle| \hat{\rho}_1(y,\xi) \leq \rho_1 \leq y, y \leq \rho_2 \leq y+s \right\},$$
$$\mathcal{P}_s^l(y,\xi) := \left\{ \rho \in \mathcal{P} \middle| y-s \leq \rho_1 \leq y, y \leq \rho_2 \leq \hat{\rho}_2(y,\xi) \right\},$$

The superscripts u and l respectively stay for "upper" and "left".

Let us consider a function $\phi \in C^2(L^2(\mathcal{P},\nu))$. We identify $L^2(\mathcal{P},\nu)$ with its dual and denote by $D\phi(\xi)$ and by $D^2\phi(\xi)$ respectively the Fréchet differential and the second Fréchet differential of ϕ in ξ. In particular, $D\phi(\xi)$ represents the linear form on $L^2(\mathcal{P},\nu)$ acting as

$$\langle D\phi(\xi), \psi \rangle = \int_{\mathcal{P}} D\phi(\xi)\psi d\nu,$$

and $D^2\phi(\xi)$ is a bounded symmetric bilinear form on $L^2(\mathcal{P},\nu)$ whose value on the couple (ψ,η) we write as

$$[D^2\phi(\xi)](\psi,\eta).$$

Let $\mathcal{L}$ be the space of the symmetric bounded bilinear forms on $L^2(\mathcal{P},\nu)$. For every $(y,\xi) \in \mathcal{O}$, $(\psi,\pi) \in L^2(\mathcal{P},\nu)\times\mathcal{L}$, let $T^u[\psi,\pi](y,\xi)$ and $T^l[\psi,\pi](y,\xi)$ be respectively the following limits (which may possibly not exist)

$$\begin{aligned} T^u[\psi,\pi](y,\xi) &:= \lim_{s\to 0^+} \frac{1}{s}\left(2\langle\psi,\chi_{\mathcal{P}_s^u(y,\xi)}\rangle + 2\pi(\chi_{\mathcal{P}_s^u(y,\xi)},\chi_{\mathcal{P}_s^u(y,\xi)})\right); \\ T^l[\psi,\pi](y,\xi) &:= \lim_{s\to 0^+} \frac{1}{s}\left(2\langle\psi,\chi_{\mathcal{P}_s^l(y,\xi)}\rangle - 2\pi(\chi_{\mathcal{P}_s^l(y,\xi)},\chi_{\mathcal{P}_s^l(y,\xi)})\right); \end{aligned} \tag{4.2}$$

where χ_C is the characteristic function of $C \subset \mathcal{P}$.

Remark 4.1. It is not hard to check that the definitions of T^u and T^l are independent from the particular representative of the equivalence class of $(y,\xi) \in \mathcal{O} \subset I\!R \times L^2(\mathcal{P},\nu)$, and from the corresponding graph B_ξ (see also Remark 2.2).

We define the following set of test functions

$$\begin{aligned} \mathcal{V} := \Big\{ &\phi \in C^2(L^2(\mathcal{P},\nu)) \text{ such that } D\phi(\xi) \in L^\infty(\mathcal{P},\nu)\ \forall\xi\in\mathcal{S}, \text{ and} \\ &T^u[D\phi(\xi),D^2\phi(\xi)](y,\xi), T^l[D\phi(\xi),D^2\phi(\xi)](y,\xi) \text{ exist } \forall(y,\xi)\in\mathcal{O}\Big\}. \end{aligned} \tag{4.3}$$

Under fairly general hypotheses, the set $\mathcal{V}$ is certainly not empty. Indeed, let us suppose that ν is absolutely continuous with respect to the Lebesgue measure, and that its density $g(\rho)$ (i.e. $d\nu(\rho) = g(\rho)d\rho$) is continuous (we need much less, but note that after the condition (2.5), the choice of ν is at our disposal). On every horizontal and vertical segment of the form $[a,b]\times\{y\}$ and $\{y\}\times[c,d]$, we then have respectively the measures $d\nu(\rho_1) := g(\rho_1, y)d\rho_1$ and $d\nu(\rho_2) := g(y,\rho_2)d\rho_2$ (dropping in the entry of ν the indication of y which may be obvious by the context). Moreover, for every $\xi \in \mathcal{S}$, and for every horizontal and vertical segment $[a,b]\times\{y\}$ and $\{y\}\times[c,d]$, let us define the "up-trace" $\xi_u \in L^2(a,b;\nu_1)$ and the "left-trace" $\xi_l \in L^2(c,d;\nu_2)$ respectively as the traces on $[a,b]\times\{y\}$ and on $\{y\}\times[c,d]$ of ξ regarded as function on the rectangles $(a,b)\times(y,y+s)$ and $(y-s,y)\times(c,d)$ with $s>0$. Note that, by the definition of $\mathcal{S}$, such traces are well defined, and they are independent on the equivalence class of ξ in $L^2(\mathcal{P},\nu)$.

Now, let us take $(\overline{y},\overline{\xi}) \in \mathcal{O}$, and consider the function

$$\phi : \xi \mapsto \|\xi - \overline{\xi}\|^2_{L^2(\mathcal{P},\nu)}. \tag{4.4}$$

For every $(y,\xi)\in\mathcal{O}$, we have (note that on $\mathcal{P}^u_\tau(y,\xi)$ it is $\xi \equiv -1$)

$$\begin{aligned}
&T^u[D\phi(\xi), D^2\phi(\xi)](y,\xi)\\
&= \lim_{s\to 0}\frac{1}{s}\left(2\langle D\varphi(\xi), \chi_{\mathcal{P}^u_s(y,\xi)}\rangle + 2\left[D^2\varphi(\xi)\right]\left(\chi_{\mathcal{P}^u_s(y,\xi)}, \chi_{\mathcal{P}^u_s(y,\xi)}\right)\right)\\
&= \lim_{s\to 0}\frac{4}{s}\int_{\mathcal{P}^u_s(y,\xi)}(\xi - \overline{\xi} + 1)d\nu = \lim_{s\to 0} -\frac{4}{s}\int_{\mathcal{P}^u_s(y,\xi)}\overline{\xi}d\nu\\
&= -4\int_{\hat{\rho}_1(y,\xi)}^{y}\overline{\xi}_u(\rho_1,y)d\nu_1(\rho_1).
\end{aligned} \tag{4.5}$$

In the same way we have

$$T^l[D\phi(\xi), D^2\phi(\xi)](y,\xi) = -4\int_y^{\hat{\rho}_2(y,\xi)}\overline{\xi}_l(y,\rho_2)d\nu_2(\rho_2).$$

hence, when $\overline{\xi}\in\mathcal{S}$, the function (4.4) belongs to $\mathcal{V}$. There are other examples of functions in $\mathcal{V}$, which will be useful in the sequel. In a similar way as before, we can see that for every open regular subset $C \subset \mathcal{P}$, considering the measure $d\nu_C := \chi_C d\nu$ on $\mathcal{P}$, the function

$$\varphi : \xi \mapsto \|\xi - \overline{\xi}\|^2_{L^2(\mathcal{P},\nu_C)} \tag{4.6}$$

belongs to $\mathcal{V}$. Finally, the following function also belongs to $\mathcal{V}$

$$\phi : \xi \mapsto e^{\int_{\mathcal{P}} \xi d\nu_C}. \tag{4.7}$$

4.2 The Hamilton-Jacobi equation

We will use the following hypothesis on the Borel finite measures μ (signed) and ν (positive) on $\mathcal{P}$ ($\mathcal{L}^2$ is the Lebesgue measure on $\mathcal{P}$)

$$\begin{array}{l} \nu \text{ has density } g \in L^\infty(\mathcal{P}, \mathcal{L}^2) \text{ with respect to } \mathcal{L}^2; \\ \text{the functions (4.4)–(4.7) belong to } \mathcal{V}; \text{ (2.2), (2.5) hold.} \end{array} \tag{4.8}$$

We will always consider $\mathcal{O}$ equipped with the topology of $I\!R \times L^2(\mathcal{P}, \nu)$.

We consider the following infinite dimensional Hamiltonian, defined on (a subset of) $(y, \xi, p, \psi, \pi) \in \mathcal{O} \times I\!R \times L^2(\mathcal{P}, \nu) \times \mathcal{L}$

$$\begin{array}{l} H(y, \xi, p, \psi, \pi) := \sup\limits_{a \in A} \Big\{ - pf(y, w, a) \\ - \Big(f^+(y, w, a) T^u[\psi, \pi](y, \xi) - f^-(y, w, a) T^l[\psi, \pi](y, \xi) \Big) \\ -\ell(y, w, a) \Big\}, \end{array} \tag{4.9}$$

where $f^+ = \max(0, f)$, $f^- = \max(0, -f)$ and $w = \int_{\mathcal{P}} \xi d\mu$.

The Hamiltonian (4.9) is not continuous for the topology of $\mathcal{O}$ (see Remark 2.2). Hence, in order to perform viscosity solution techniques, we consider the following "lower" and "upper" Hamiltonians H^- and H^+:

$$\begin{array}{l} H^-(y, \xi, p, \psi, \pi) := \sup\limits_{a \in A} \Big\{ - pf(y, w, a) \\ - \Big(f^+(y, w, a) \left(T^u[\psi, \pi](y, \xi)\right)^+ + f^-(y, w, a) \Big(T^l[\psi, \pi](y, \xi)\Big)^- \Big) \\ -\ell(y, w, a) \Big\}, \end{array} \tag{4.10}$$

$$\begin{array}{l} H^+(y, \xi, p, \psi, \pi) := \sup\limits_{a \in A} \Big\{ - pf(y, w, a) \\ + \Big(f^+(y, w, a) \left(T^u[\psi, \pi](y, \xi)\right)^- + f^-(y, w, a) \Big(T^l[\psi, \pi](y, \xi)\Big)^+ \Big) \\ -\ell(y, w, a) \Big\}, \end{array} \tag{4.11}$$

In the sequel $BUC(\mathcal{O} \times [0, T])$ is the space of bounded and uniformly continuous functions on $\mathcal{O} \times [0, T]$; V_t and V_y mean derivatives, and D is the differential with respect to $\xi \in L^2(\mathcal{P}, \nu)$.

Theorem 4.2 *Let (4.8) holds. For every $T > 0$ the following holds: the value function V belongs to $BUC(\mathcal{O} \times [0,T])$; moreover it is a viscosity solution of the following Cauchy problem for a Hamilton-Jacobi equation*

$$\begin{cases} V_t(y,\xi,t) + \lambda V(y,\xi,t) \\ +H\Big(y,\xi,V_y(y,\xi,t),DV(y,\xi,t),D^2V(y,\xi,t)\Big) = 0 \quad in\ \mathcal{O}\times]0,T] \\ V(y,\xi,0) = h(y,w), \end{cases} \tag{4.12}$$

with test functions taken in $C^1(I\!R\times]0,T]) \times \mathcal{V}$.

By viscosity solution of (4.12) with $C^1(I\!R\times]0,T]) \times \mathcal{V}$ as space of test functions we mean that V satisfies the initial condition and that:

(*subsolution*) for every $(\overline{y},\overline{\xi},\overline{t}) \in \mathcal{O}\times]0,T]$, every $\varphi \in C^1(I\!R\times]0,T])$, and every $\phi \in \mathcal{V}$ such that $(\overline{y},\overline{\xi},\overline{t})$ is a local maximum for $V(y,\xi,t)-\varphi(y,t)-\phi(\xi)$ with respect to $\mathcal{O}\times]0,T]$, the following holds

$$\begin{aligned} &\varphi_t(\overline{y},\overline{t}) + \lambda V(\overline{y},\overline{\xi},\overline{t}) \\ &+H^-\Big(\overline{y},\overline{\xi},\varphi_y(\overline{y},\overline{t}),D\phi(\overline{\xi}),D^2\phi(\overline{\xi})\Big) \le 0; \end{aligned} \tag{4.13}$$

(*supersolution*) for every $(\overline{y},\overline{\xi},\overline{t}) \in \mathcal{O}\times]0,T]$, every $\varphi \in C^1(I\!R\times]0,T])$, and every $\phi \in \mathcal{V}$ such that $(\overline{y},\overline{\xi},\overline{t})$ is a local minimum for $V(y,\xi,t)-\varphi(y,t)-\phi(\xi)$ with respect to $\mathcal{O}\times]0,T]$, the following holds

$$\begin{aligned} &\varphi_t(\overline{y},\overline{t}) + \lambda V(\overline{y},\overline{\xi},\overline{t}) \\ &+H^+\Big(\overline{y},\overline{\xi},\varphi_y(\overline{y},\overline{t}),D\phi(\overline{\xi}),D^2\phi(\overline{\xi})\Big) \ge 0. \end{aligned} \tag{4.14}$$

Proof of Theorem 4.2. The fact that $V \in BUC(\mathcal{O} \times [0,T])$ comes from (3.2)–(3.5) and Remark 3.1 (see Bardi-Capuzzo Dolcetta [3] page 148 and Bagagiolo [1]).

By the definition, it is evident that V satisfies the initial condition.

Let us prove that V is subsolution. Since the equation consists of a "finite dimensional part" and of an "infinite dimensional part" (i.e. the part involving T^u and T^l), and since the technique we are going to use is standard for the finite dimensional one, we will be mostly concerned with the treatment of the infinite dimensional part, referring the reader to Bardi-Capuzzo Dolcetta [3] for other details.

We prove that V satisfies (4.13) even replacing H^- by H itself, which a fortiori will give the conclusion. Let us take $(y,\xi,t) \in \mathcal{O}\times]0,T]$, $\varphi \in$

$C^1(I\!R\times]0,T])$ and $\phi \in \mathcal{V}$ such that $V-\varphi-\phi$ has a local maximum in (y,ξ,t) with respect to $\mathcal{O}\times]0,T]$. Recall that $\mathcal{O}$ is an invariant set for the trajectory $(y(t),\xi_t)$ given by the solution of the system (3.1). Fix any constant control $\alpha \equiv a \in A$, write $w = \int_{\mathcal{P}} \xi d\mu$ and consider the evolutions $y(\cdot)$, $w(\cdot)$, $\xi_{\cdot}$ given by the system (3.1) with α as control and (y,ξ) as initial datum. For $\delta > 0$ sufficiently small, for every $0 \le \tau \le \delta \le t$ we have

$$\varphi(y,t) + \phi(\xi) - \varphi(y(\tau), t-\tau) - \phi(\xi_\tau) \le V(y,\xi,t) - V(y(\tau),\xi_\tau,t-\tau).$$

By DPP (4.1), we get

$$\begin{aligned}&\varphi(y,t) + \phi(\xi) - \varphi(y(\tau), t-\tau) - \phi(\xi_\tau) \le \\ &\int_0^\tau e^{-\lambda s}\ell(y(s),w(s),a)ds + V(y(\tau),\xi_\tau,t-\tau)(e^{-\lambda\tau}-1).\end{aligned} \tag{4.15}$$

We divide (4.15) by τ and let $\tau \to 0^+$. Since $0 < t-\tau < T$, by the regularity of φ, even if $t = T$, we can perform such limit. We have three possibilities

a) $f(y_0,w_0,a) = 0$. Then $y(\cdot) \equiv y$, $\xi_{(\cdot)} \equiv \xi$, $w(\cdot) \equiv w$ and we get

$$\phi_t(y,\xi,t) + \lambda V(y,\xi,t) - \ell(y,w,a) \le 0 \tag{4.16}$$

b) $f(y,w,a) > 0$. For small $\delta > 0$, $y(\cdot)$ is increasing in $[0,\delta]$. In the Preisach plane $\mathcal{P}$ the horizontal segment $[\hat\rho_1(y(\tau),\xi_\tau), y(\tau)] \times \{y(\tau)\}$ is moving up, starting from the possibly degenerate (that is a point) initial one $[\hat\rho_1(y,\xi),y] \times \{y\}$. Hence, $\hat\rho_2(y(\tau),\xi_\tau) = y(\tau)$ for every $0 < \tau \le \delta$, and $\tau \to \hat\rho_1(y(\tau),\xi_\tau)$ is non increasing and right-continuous in $\tau = 0$.

We claim that

$$\xi_\tau - \xi = 2\chi_{\mathcal{P}^u_{f(y,w,a)\tau}(y,\xi)} + \sigma(\tau), \tag{4.17}$$

where $\sigma(\tau)$ is a function η_τ such that $\|\eta_\tau\|_{L^1(\mathcal{P},\nu)}/\tau \to 0$ as $\tau \to 0$. Indeed, $\xi_\tau - \xi = 2\chi_{\mathcal{P}(\tau)}$ where $\mathcal{P}(\tau)$ is a "trapezoidal" region given by the disjoint union of the following sets (see Figure 3): i) the rectangle $\mathcal{P}^u_{y(\tau)-y}(y,\xi)$ (which may be reduced to a vertical segment if $\hat\rho_1(y,\xi) = y$); ii) the triangle T_τ with vertices (y,y), $(y,y(\tau))$, $(y(\tau),y(\tau))$; iii) the "triangular region" $R_\tau = \{(\rho_1,\rho_2) \in \mathcal{P} | \rho_1 \le \hat\rho_1(y,\xi), B_\xi(\rho_1) \le \rho_2 \le y(\tau)\}$ (which, when B_ξ is linked to the line $\rho_1 = \rho_2$ by a vertical segment or when $\hat\rho_1(y,\xi) = -\infty$, at least for small τ, may be empty). Note that $\nu(T_\tau) = o(\tau)$ and $\nu(\mathcal{P}^u_{y(\tau)-y}(y,\xi)) = O(\tau)$.

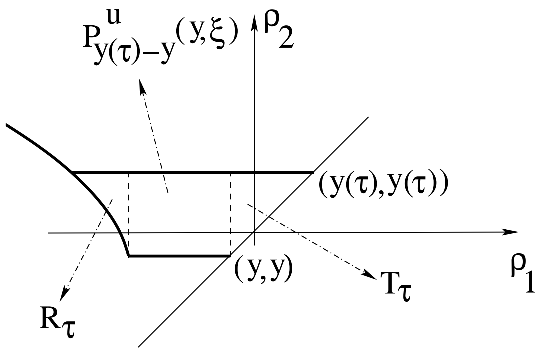

Figure 3: graphic explanation for the proof of Theorem 4.2.

We have also $\nu(R_\tau) = o(\tau)$, since $\lim_{\tau\to 0^+} \hat\rho_1(y(\tau),\xi_\tau) = \hat\rho_1(y,\xi)$ (see also (4.8)). Finally, since $y(\tau) = y + f(y,w,a)\tau + o(\tau)$, we get

$$\chi_{\mathcal{P}^u_{y(\tau)-y}(y,\xi)} = \chi_{\mathcal{P}^u_{f(y,w,a)\tau}(y,\xi)} + \sigma(\tau),$$

and the claim is proven. Hence, by (4.17), using the Taylor's second order formula for ϕ in ξ, and recalling that $D\phi(\xi) \in L^\infty(\mathcal{P},\nu)$, we conclude that

$$\begin{aligned}
&\lim_{\tau\to 0^+} \frac{\phi(\xi_\tau) - \phi(\xi)}{\tau} \\
&= f(y,w,a) \lim_{\tau\to 0^+} \frac{2\langle D\phi(\xi), \chi_{\mathcal{P}^u_{f(y,w,a)\tau}}\rangle + 2\,[D^2\phi(\xi)]\,(\chi_{\mathcal{P}^u_{f(y,w,a)\tau}}, \chi_{\mathcal{P}^u_{f(y,w,a)\tau}})}{f(y,w,a)\tau} \\
&= f(y,w,a) T^u[D\phi(\xi), D^2\phi(\xi)](y,\xi),
\end{aligned}$$

where we dropped the entry (y,ξ) in $\mathcal{P}^u_{f(y,w,a)\tau}(y,\xi)$. This concludes b).

c) $f(y,w,a) < 0$. We argue as in the case b). We have that $y(\cdot)$ is decreasing in $[0,\delta]$ for small δ. Hence, in the Preisach plane $\mathcal{P}$ the vertical segment $\{y(\tau)\} \times [y, \hat\rho_2(y(\tau),\xi_\tau)]$ is moving towards left, starting from the possibly degenerate (that is a point) initial one $\{y\} \times [y, \hat\rho_2(y,\xi)]$. Hence, $\hat\rho_1(y(\tau),\xi_\tau) = y(\tau)$ for every $0 < \tau \le \delta$, and $\tau \to \hat\rho_2(y(\tau),\xi_\tau)$ non decreasing and right-continuous in $\tau = 0$. In a similar way as above, we have that

$$\xi_\tau - \xi = -2\chi_{\mathcal{P}^l_{-f(y,w,a)\tau}(y,\xi)} + \sigma(\tau).$$

Hence, we conclude the step c) by (we drop again the entry (y,ξ))

$$
\begin{aligned}
&\lim_{\tau\to 0^+}\frac{\phi(\xi_\tau)-\phi(\xi)}{\tau}\\
&= f(y,w,a)\lim_{\tau\to 0^+}\frac{2\langle D\phi(\xi),\chi_{\mathcal{P}^l_{-f(y,w,a)\tau}}\rangle - 2\,[D^2\phi(\xi)]\,(\chi_{\mathcal{P}^l_{-f(y,w,a)\tau}},\chi_{\mathcal{P}^l_{-f(y,w,a)\tau}})}{-f(y,w,a)\tau}\\
&= f(y,w,a)T^l[D\phi(\xi),D^2\phi(\xi)](y,\xi).
\end{aligned}
$$

By the arbitrariness of $a\in A$, using standard techniques we can conclude.

Now, let us prove that V is supersolution. Let $\phi\in\mathcal{V}$, $\varphi\in C^1(I\!\!R\times]0,T])$, be such that $(y,\xi,t)\in\mathcal{O}\times]0,T]$ is of local minimum for $V-\varphi-\phi$ with respect to $\mathcal{O}\times]0.T]$. For any $\varepsilon>0$ and any $\tau>0$ sufficiently small, arguing as in the step b), by DPP we find a measurable control $\alpha(\cdot)$, depending on ε and τ, such that

$$
\begin{aligned}
&\varphi(y,t)+\phi(\xi)-\varphi(y(\tau),t-\tau)-\phi(\xi_\tau)\\
&\geq\int_0^\tau e^{-\lambda s}\ell(y(s),w(s),\alpha(s))ds+V(y(\tau),\xi_\tau,t-\tau)(1-e^{-\lambda\tau})-\tau\varepsilon.
\end{aligned}
\tag{4.18}
$$

Let us suppose that $\hat\rho_2(y,\xi)=y$. For any τ we define

$$
\begin{aligned}
&y_+(\tau):=\max_{s\in[0,\tau]}y(s),\quad \xi_+[\tau]:=\text{internal variable of }\mathcal{H}_\mu[y_+,\xi](\tau);\\
&y_-(\tau):=\min_{s\in[0,\tau]}y(s),\quad \xi_-[\tau]:=\text{internal variable of }\mathcal{H}_\mu[y_-,\xi](\tau).
\end{aligned}
$$

We claim that, at any instant $\tau\geq 0$, we have (σ is as in (4.17))

$$
\xi_\tau=\xi_+[\tau]+\sigma(\tau),\tag{4.19}
$$

We define $r(\tau):=\max(y_+(\tau),\hat\rho_2(y_-(\tau),\xi_-[\tau]))$. Since $\hat\rho_2(y,\xi)=y$, then $\lim_{\tau\to 0^+}\hat\rho_2(y_-(\tau),\xi_-[\tau])=y$ and hence $\lim_{\tau\to 0^+}r(\tau)=y$. Moreover, $|\xi_\tau-\xi_+[\tau]|\leq 2\chi_T$, where T is the triangle of vertices $(y_-(\tau),y_-(\tau))$, $(y_-(\tau),r(\tau))$, and $(r(\tau),r(\tau))$ (see the example in Figure 4). Hence (4.19) follows.

We suppose that (see also Lemma 4.3 below) for every $\tau>0$ and $\varepsilon>0$,

$$
y_+(\tau)-y=\int_0^\tau\chi_d(y(s),y_+(s))f^+(y_+(s),w(s),\alpha(s))ds>0.\tag{4.20}
$$

We use the notations $c_+(\tau):=y_+(\tau)-y$ and $\chi_\tau:=\chi_{\mathcal{P}^u_{c_+(\tau)}(y,\xi)}$. We get

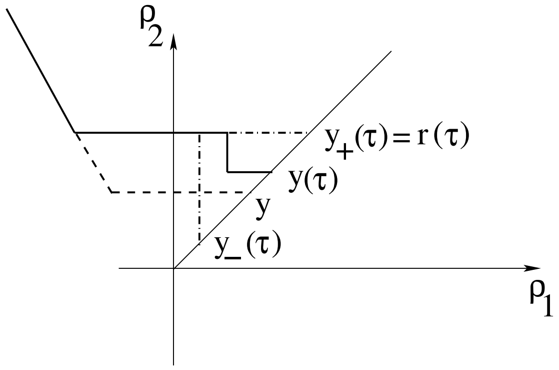

Figure 4: the filled graph is B_{ξ_τ}; the dashed graph is B_ξ; the pointed-dashed lines define the triangle T (see the proof of Theorem 4.2).

$$
\begin{aligned}
&\frac{\phi(\xi_\tau)-\phi(\xi)}{\tau} = \frac{c_+(\tau)}{\tau}\left(\frac{2\langle D\phi(\xi),\chi_\tau\rangle + 2\,[D^2\phi(\xi)]\,(\chi_\tau,\chi_\tau)}{c_+(\tau)}\right) + O(\tau)\\
&\geq -\frac{1}{\tau}\int_0^\tau f^+(y_+(s),w(s),\alpha(s))ds\\
&\times\left(\frac{2\langle D\phi(\xi),\chi_\tau\rangle + 2\,[D^2\phi(\xi)]\,(\chi_\tau,\chi_\tau)}{c_+(\tau)}\right)^- + O(\tau)\\
&= -\frac{1}{\tau}\int_0^\tau f^+(y_+(s),w(s),\alpha(s))\left(T^u[D\phi(\xi),D^2\phi(\xi)](y,\xi)\right)^- ds + O(\tau).
\end{aligned}
$$

On the other hand, if (4.20) does not hold, then $\xi_+[\tau]=\xi$ for all τ, and

$$
\begin{aligned}
&\lim_{\tau\to 0^+}\frac{\phi(\xi_\tau)-\phi(\xi)}{\tau} = 0\\
&\geq -\lim_{\tau\to 0^+}\frac{1}{\tau}\int_0^\tau f^+(y_+(s),w(s),\alpha(s))\left(T^u[D\phi(\xi),D^2\phi(\xi)](y,\xi)\right)^- ds.
\end{aligned}
$$

In both cases, we get

$$
\begin{aligned}
&\lim_{\tau\to 0^+}\frac{\varphi(y,t)+\phi(\xi)-\varphi(y(\tau),t-\tau)-\phi(\xi_\tau)}{\tau}\\
&\leq \lim_{\tau\to 0^+} -\frac{1}{\tau}\int_0^\tau \Big(\varphi_y(y(s),t-s)f(y(s),w(s),\alpha(s)) - \varphi_t(y(s),t-s) \qquad (4.21)\\
&-f^+(y_+(s),w(s),\alpha(s))\left(T^u[D\phi(\xi),D^2\phi(\xi)](y,\xi)\right)^-\Big)\,ds + O(\tau).
\end{aligned}
$$

Since $\hat{\rho}_2(y,\xi) = y$, we have $T^l[D\phi(\xi), D^2\phi(\xi)](y,\xi) = 0$. Hence, from (4.21), we conclude in a standard way (see Bardi-Capuzzo Dolcetta [3], page 151).

Finally, if $\hat{\rho}_1(y,\xi) = y$, we change the role between $y_+(\cdot)$ and $y_-(\cdot)$. $\square$

Lemma 4.3 *Let $y \in W^{1,1}(0,T)$ be given, and let us consider the functions*

$$y_+(t) := \max_{s\in[0,t]} y(s), \quad y_-(t) := \min_{s\in[0,t]} y(s) \quad t \in [0,T].$$

Then, for almost every $t \in [0,T]$, we have

$$y'_+(t) = \chi_d\Big(y(t), y_+(t)\Big)(y'(t))^+, \quad y'_-(t) = -\chi_d\Big(y(t), y_-(t)\Big)(y'(t))^-, \quad (4.22)$$

where $(x_1,x_2) \mapsto \chi_d(x_1,x_2)$ is the characteristic function of $\{x_1 = x_2\}$ in $\mathbb{R}^2$.

Proof. We sketch the proof for y_+, which is characterized as the unique element of $W^{1,1}(0,T)$ such that

$$\begin{cases} y'_+(t)\Big(y(t) - y_+(t) + r\Big) \geq 0 & \forall r \geq 0, \text{ a.e. } t \in (0,T) \\ y_+(t) \geq y(t) & \forall t \in [0,T] \\ y_+(0) = y(0). \end{cases}$$

Indeed, we can regard $y_+(\cdot)$ as the output of a particular case of the Play operator of hysteresis with input $y(\cdot)$ (see Visintin [12]). From this we get

$$|y'_+(t)| \leq |y'(t)| \quad \text{a.e. } t \in (0,T). \quad (4.23)$$

Let $t \in (0,T)$ be such that $y'(t)$ and $y'_+(t)$ exist. If $y_+(\tau) > y(\tau)$, then, by continuity, (4.22) holds in t. Otherwise, if $y_+(t) = y(t)$, we should have

$$y'(t) \geq 0. \quad (4.24)$$

Hence, by (4.23), (4.24) and the following, we conclude.

$$y'_+(t) = \lim_{h\to 0} \frac{y_+(t+h) - y_+(t)}{h} \geq \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} = y'(t).$$

$\square$

5 Uniqueness

We assume the following hypothesis

$$\nu(U) > 0 \quad \text{for all open set } U \subset \mathcal{P}. \tag{5.1}$$

Theorem 5.1 *Let (4.8) and (5.1) hold. Let u and v be bounded and uniformly continuous functions on $\mathcal{O} \times [0,T] \subset I\!R \times L^2(\mathcal{P},\nu) \times [0,T]$ for every $T > 0$, which respectively are viscosity subsolution and viscosity supersolution of the Cauchy problem (4.12) for every $T > 0$. Then*

$$u(y,\xi,t) \leq v(y,\xi,t) \quad \forall (y,\xi,t) \in \mathcal{O} \times [0,+\infty[. \tag{5.2}$$

As usual, from (5.2), the uniqueness of the viscosity solution follows. We prove Theorem 5.1 using Theorem 5.2 below, which gives the same result under the following hypothesis (recall (2.1)).

$$\begin{array}{l} \exists r > 0 \text{ such that, setting } Q_r := \left\{ (\sigma_1,\sigma_2) \in \mathcal{P} \middle| 0 < \sigma_1 < r \right\}, \\ \nu(Q_r) = 0, \quad \nu(U) > 0 \text{ for every open set } U \subset \mathcal{P} \setminus Q_r. \end{array} \tag{5.3}$$

The hypothesis (5.3) can be assumed for instance when the support of μ is confined on a compact set of $\mathcal{P}$ (which is open).

Theorem 5.2 *Let us suppose that the hypotheses of Theorem 5.1 are satisfied, but (5.1) is replaced by (5.3). Then the thesis of Theorem 5.1 holds.*

Proof of Theorem 5.1. For every $r > 0$ we consider the strip Q_r and define two measure μ_r and ν_r on $\mathcal{P}$ by

$$d\mu_r := \chi_{\mathcal{P}\setminus Q_r} d\mu, \quad d\nu_r := \chi_{\mathcal{P}\setminus Q_r} d\nu.$$

Note that μ_r and ν_r satisfy (4.8) and (5.3). Moreover, for every $r > 0$ we consider the Preisach operator $\mathcal{H}_{\mu_r}$ corresponding to the measure μ_r. Hence, we have a family of optimal control problems $\{P_r\}_{r>0}$, which are given by f ℓ, h, A and $\mathcal{H}_{\mu_r}$. Finally, we denote by the subscript r every formula, quantity, equation and set corresponding to the problem P_r and to the measure ν_r. However, we consider $\mathcal{O}$ as independent from r (see Remarks 2.1 and 4.1).

A similar argument as in Proposition 2.4 shows that, for every $(y,\xi) \in \mathcal{O}$,

$$S_r(y,\xi) := \left\{ \eta \in \mathcal{S} \middle| (y,\eta) \in \mathcal{O}, \eta = \xi \text{ in } \mathcal{P} \setminus Q_r \right\} \tag{5.4}$$

is compact for the topology of $L^2(\mathcal{P},\nu)$. Hence, for every $r > 0$ we define

$$\begin{aligned} u_r(y,\xi,t) &= \max \left\{ u(y,\eta,t) \middle| \eta \in \mathcal{S}_r(y,\xi) \right\}, \\ v_r(y,\xi,t) &= \min \left\{ u(y,\eta,t) \middle| \eta \in \mathcal{S}_r(y,\xi) \right\}. \end{aligned}$$

We are now going to prove three claims. First claim: for every $T > 0$, u_r, v_r are bounded and uniformly continuous in $\mathcal{O} \times [0,T] \subset I\!R \times L^2(\mathcal{P},\nu_r) \times [0,T]$. Let us prove the claim for v_r, the other case being analogous. For every (y,ξ,t) let us take an element $(y,\tilde{\xi},t)$ where the minimum in the definition is achieved. For every $(y_1,\xi_1),(y_2,\xi_2) \in \mathcal{O}$, we can construct a function $\xi_{1_{\xi_2}}$ such that $(y_1,\xi_{1_{\xi_2}}) \in \mathcal{O}$, $\xi_{1_{\xi_2}} = \xi_1$ in $\mathcal{P} \setminus Q_r$ and

$$\|\xi_{1_{\xi_2}} - \xi_2\|_{L^2(Q_r,\nu)} \le \omega_r(|y_1 - y_2| + \|\xi_1 - \xi_2\|_{L^2(\mathcal{P}\setminus Q_r,\nu_r)}),$$

where ω_r is a modulus of continuity depending only on r. Indeed, it is sufficient to note that, by (5.3), $\|\xi_1 - \xi_2\|_{L^2(\mathcal{P}\setminus Q_r,\nu_r)} \to 0$ implies that $|B_{\xi_1} \cap \{\sigma_1 = r\} - B_{\xi_2} \cap \{\sigma_1 = r\}| \to 0$. Then, we have ($\omega$ is a modulus of continuity)

$$\begin{aligned} & v_r(y_1,\xi_1,t_1) - v_r(y_2,\xi_2,t_2) \le v(y_1,\xi_{1_{\tilde{\xi}_2}},t_1) - v(y_2,\tilde{\xi}_2,t_2) \\ & \le \omega\left(|y_1 - y_2| + \|\xi_{1_{\tilde{\xi}_2}} - \tilde{\xi}_2\|_{L^2(\mathcal{P},\nu)} + |t_1 - t_2|\right) \\ & = \omega\left(|y_1 - y_2| + \left(\|\xi_1 - \xi_2\|^2_{L^2(\mathcal{P}\setminus Q_r,\nu_r)} + \|\xi_{1_{\tilde{\xi}_2}} - \tilde{\xi}_2\|^2_{L^2(Q_r,\nu)}\right)^{1/2} + |t_1 - t_2|\right), \end{aligned}$$

and the uniform continuity of v_r follows. The boundedness is obvious.

Second claim: u_r and v_r respectively pointwise converge to u and v on $\mathcal{O}\times]0,+\infty[$ as $r \to 0^+$. It follows from the uniform continuity of u, v and the fact that $\|\xi - \eta\|_{L^2(\mathcal{P},\nu)} = O(r)$ uniformly with respect to $\eta \in S_r(y,\xi)$.

Third claim: u_r and v_r are respectively sub- and super-solution of the Cauchy problem $(4.12)_r$. We prove that v_r is supersolution. Let $(y,\xi,t) \in \mathcal{O}\times]0,T]$ be a minimum point in $\mathcal{O}\times]0,T] \subset I\!R \times L^2(\mathcal{P},\nu_r)\times]0,T]$ for $v_r - \varphi - \phi_r$ with $\varphi \in C^1(I\!R\times]0,T])$ and $\phi_r \in \mathcal{V}_r$. First of all note that the function

$$\phi : L^2(\mathcal{P},\nu) \to I\!R, \quad \eta \mapsto \phi_r(\chi_{\mathcal{P}\setminus Q_r}\eta)$$

belongs to $\mathcal{V}$ and moreover for every $(y,\xi) \in \mathcal{O}$ we have

$$T^u_r[D_r\phi_r(\xi), D^2_r\phi_r(\xi)](y,\xi) = T^u[D\phi(\xi), D^2\phi(\xi)](y,\xi),$$
$$T^l_r[D_r\phi_r(\xi), D^2_r\phi_r(\xi)](y,\xi) = T^l[D\phi(\xi), D^2\phi(\xi)](y,\xi).$$

Moreover $(y,\tilde{\xi},t)$ is a point of local minimum for $v-\varphi-\phi$ with respect to $\mathcal{O}\times]0,T] \subset I\!R \times L^2(\mathcal{P},\nu)\times]0,T]$. Hence, the claim is proven.

By the claims above and by Theorem 5.2, we then have:

$$u(y,\xi,t) \leq v(y,\xi,t)\ \forall (y,\xi,t) \in \mathcal{O}\times[0,+\infty[.$$

□

Proof of Theorem 5.2. By contradiction, let $T>0$ be such that

$$m := \sup_{\mathcal{O}\times[0,T]} (u-v) > 0. \tag{5.5}$$

By virtue of Proposition 2.4, it is enough to consider two different cases: 1) the supremum in (5.5) is achieved on a point $(\overline{y},\overline{\xi},\overline{t}) \in \mathcal{O}\times[0,T]$; 2) there is a sequence $\{(y_\delta,\xi_\delta,t_\delta)\}_{\delta>0} \subseteq \mathcal{O}\times[0,T]$, with $|y_\delta| \to +\infty$ as $\delta\to 0^+$, and

$$u(y_\delta,\xi_\delta,t_\delta) - v(y_\delta,\xi_\delta,t_\delta) \geq m-\delta > 0, \tag{5.6}$$

Case 1). We consider two sub-cases: a) $\hat{\rho}_2(\overline{y},\overline{\xi}) = \overline{y}$; b) $\hat{\rho}_1(\overline{y},\overline{\xi}) = \overline{y}$.

Sub-case a). We use the following notations: $\|\cdot\| := \|\cdot\|_{L^2(\mathcal{P},\nu)}$, and $\{\xi=1\} := \{\rho\in\mathcal{P}|\xi(\rho)=1\}$ (similarly for $\{\xi=-1\}$). We define

$$\psi(\xi) := e^{-\int_{\{\overline{\xi}=-1\}}\overline{\xi}d\nu} - e^{-\int_{\{\overline{\xi}=-1\}}\xi d\nu} - 2\int_{\{\overline{\xi}=-1\}}(1+\xi)d\nu + \int_{\mathcal{P}}(\overline{\xi}-\xi)^2 d\nu.$$

Hence $\psi\in\mathcal{V}$. We claim that there exists a constant $c>0$ such that

$$\psi(\xi) \geq c\|\xi-\overline{\xi}\|^2 \quad \forall \xi\in\mathcal{S}. \tag{5.7}$$

Indeed, the claim follows from the following relations

$$\psi(\xi) \geq \left(e^{\int_{\{\overline{\xi}=-1\}}\overline{\xi}d\nu}\right)\int_{\{\overline{\xi}=-1\}}(\xi-\overline{\xi})d\nu = \frac{e^{\int_{\{\overline{\xi}=-1\}}\overline{\xi}d\nu}}{2}\int_{\{\overline{\xi}=-1\}\cap\{\xi=1\}}(\xi-\overline{\xi})^2;$$
$$-2\int_{\{\overline{\xi}=-1\}}(1+\xi)d\nu + \int_{\mathcal{P}}(\overline{\xi}-\xi)^2 d\nu = \int_{\{\overline{\xi}=1\}\cap\{\xi=-1\}}(\xi-\overline{\xi})^2 d\nu.$$

For every $\varepsilon>0$, on $(\mathcal{O}\times[0,T])\times(\mathcal{O}\times[0,T])$, we define the function

$$
\begin{aligned}
\Psi(y_1,\xi_1,t_1,y_2.\xi_2,t_2) &:= u(y_1,\xi_1,t_1) - v(y_2,\xi_2,t_2) - \frac{|y_1-y_2|^2}{\varepsilon} \\
-\frac{1}{\varepsilon}(\psi(\xi_1)+\psi(\xi_2)) &- \frac{|t_1-t_2|^2}{\varepsilon} - |y_1-\overline{y}|^2 - \beta(b(y_1)+b(y_2)),
\end{aligned} \tag{5.8}
$$

where $b : I\!R \to I\!R$ is nonnegative, differentiable with bounded derivative, $b(\overline{y}) = 0$, and $b(y) \to +\infty$ as $|y| \to +\infty$; β is a positive parameter. We have

$$
\Psi(\overline{y},\overline{\xi},\overline{t},\overline{y},\overline{\xi},\overline{t}) = u(\overline{y},\overline{\xi},\overline{t}) - v(\overline{y},\overline{\xi},\overline{t}) = m > 0,
$$

and, using the boundedness of u and v, for $|y_1|, |y_2|$ large enough, we have

$$
\Psi(y_1,\xi_1,t_1,y_2,\xi_2,t_2) < 0.
$$

Hence (see Proposition 2.4), Ψ has a positive maximum on $(\mathcal{O}\times[0,T])\times(\mathcal{O}\times[0,T])$. Let $(y_1^\varepsilon,\xi_1^\varepsilon,t_1^\varepsilon,y_2^\varepsilon,\xi_2^\varepsilon,t_2^\varepsilon)$ be such a maximum point. By (5.7), and by standard techniques, we obtain

$$
\begin{aligned}
&\frac{|y_1^\varepsilon-y_2^\varepsilon|^2}{\varepsilon} + c\frac{\|\xi_1^\varepsilon-\overline{\xi}\|^2}{\varepsilon} + c\frac{\|\xi_2^\varepsilon-\overline{\xi}\|^2}{\varepsilon} + \frac{|t_1^\varepsilon-t_2^\varepsilon|^2}{\varepsilon} + |y_1^\varepsilon-\overline{y}|^2 \\
&\le \omega(|y_1^\varepsilon-y_2^\varepsilon| + \|\xi_1^\varepsilon-\xi_2^\varepsilon\| + |t_1^\varepsilon-t_2^\varepsilon|) \le M,
\end{aligned} \tag{5.9}
$$

with ω modulus of continuity. Hence, for $\varepsilon \to 0^+$, we have

$$
\max\left(\frac{|y_1^\varepsilon-y_2^\varepsilon|^2}{\varepsilon}, \frac{\|\xi_1^\varepsilon-\overline{\xi}\|^2}{\varepsilon}, \frac{\|\xi_2^\varepsilon-\overline{\xi}\|^2}{\varepsilon}, \frac{|t_1^\varepsilon-t_2^\varepsilon|^2}{\varepsilon}, |y_1^\varepsilon-\overline{y}|^2\right) \to 0. \tag{5.10}
$$

By (5.10), y_1^ε and y_2^ε tend to y, and, since $\hat{\rho}_2(\overline{y},\overline{\xi}) = \overline{y}$, by (5.3), we have

$$
\hat{\rho}_2(y_i^\varepsilon,\xi_i^\varepsilon) = y_i^\varepsilon \text{ for } i = 1,2 \text{ and } \varepsilon \text{ sufficiently small.} \tag{5.11}
$$

Indeed, if (5.11) does not hold, then $\|\xi_i^\varepsilon - \overline{\xi}\| \not\to 0$: there is a triangle in $\mathcal{P}\setminus Q_r$ where $|\xi_i^\varepsilon - \overline{\xi}| = 2$ for all ε (the triangle T_2 in Figure 5).

As usual, $(y_1^\varepsilon,\xi_1^\varepsilon,t_1^\varepsilon)$ is of maximum in $\mathcal{O}\times]0,T]$ (note that $t_i^\varepsilon > 0$) for

$$
(y,\xi,t) \mapsto u(y,\xi,t) - \frac{|y-y_2^\varepsilon|^2}{\varepsilon} - \frac{\psi(\xi)}{\varepsilon} - \frac{|t-t_2^\varepsilon|^2}{\varepsilon} - |y-\overline{y}|^2 - \beta b(y),
$$

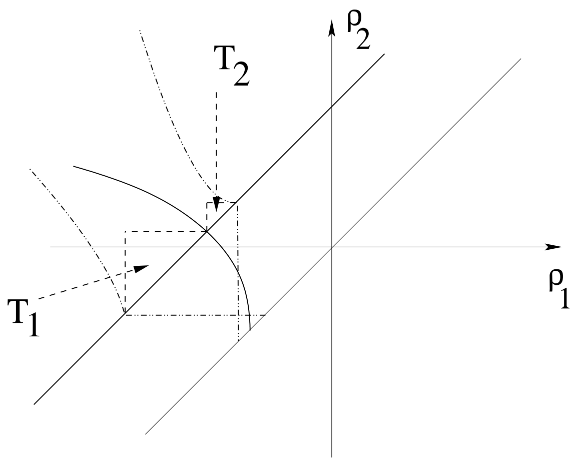

Figure 5: graphic explanation for the proof of Theorem 5.2. The filled graph $\overline{\xi}$ is such that $\hat{\rho}_1(\overline{y},\overline{\xi}) = \hat{\rho}_2(\overline{y},\overline{\xi}) = \overline{y}$.

as well as $(y_2^\varepsilon, \xi_2^\varepsilon, t_2^\varepsilon)$ is a minimum point in $\mathcal{O}\times]0,T]$ for

$$(y,\xi,t) \mapsto v(y,\xi,t) + \frac{|y-y_1^\varepsilon|^2}{\varepsilon} + \frac{\psi(\xi)}{\varepsilon} + \frac{|t-t_1^\varepsilon|^2}{\varepsilon} + \beta b(y).$$

By (5.11), we have $T^l[D\psi(\xi_i^\varepsilon), D^2(\xi_i^\varepsilon)](y_i^\varepsilon,\xi_i^\varepsilon) = 0$ for $i = 1,2$ and small ε. The exponential terms of ψ do not give contributions to T^u. Hence, see (4.5),

$$\begin{aligned}
&T^u[D\psi(\xi_i^\varepsilon), D^2(\xi_i^\varepsilon)](y_i^\varepsilon,\xi_i^\varepsilon) \\
&= \lim_{s\to 0^+} -\frac{4}{\varepsilon s}\left(\nu\left(P_s^u(y_i^\varepsilon,\xi_i^\varepsilon)\cap\{\overline{\xi}=-1\}\right) + \int_{P_s^u(y_i^\varepsilon,\xi_i^\varepsilon)} \overline{\xi}\,d\nu\right) \\
&= \lim_{s\to 0^+} -\frac{4}{\varepsilon s}\nu\left(P_s^u(y_i^\varepsilon,\xi_i^\varepsilon)\cap\{\overline{\xi}=1\}\right) \le 0.
\end{aligned}$$

Hence, in the Hamilton-Jacobi equation, the terms involving T^u and T^l vanish. We can then apply standard techniques to the "finite dimensional" part of the equation (see Bardi-Capuzzo Dolcetta [3]), and get the contradiction.

Sub-case b). In this sub-case we define the function $\gamma \in \mathcal{V}$

$$\gamma(\xi) := e^{\int_{\{\overline{\xi}=1\}}\overline{\xi}d\nu} - e^{\int_{\{\overline{\xi}=1\}}\xi d\nu} - 2\int_{\{\overline{\xi}=1\}}(1-\xi)d\nu + \int_{\mathcal{P}}(\xi-\overline{\xi})^2 d\nu.$$

There exists $c > 0$ such that (replacing ψ with γ) (5.7) holds. We then define

$$\Gamma(y_1,\xi_1,t_1,y_2.\xi_2,t_2) := u(y_1,\xi_1,t_1) - v(y_2,\xi_2,t_2) - \frac{|y_1-y_2|^2}{\varepsilon}$$
$$-\frac{1}{\varepsilon}(\gamma(\xi_1)+\gamma(\xi_2)) - \frac{|t_1-t_2|^2}{\varepsilon} - |y_1-\overline{y}|^2 - \beta(b(y_1)+b(y_2)).$$

As before, Γ has a local maximum point $(y_1^\varepsilon,\xi_1^\varepsilon,t_1^\varepsilon,y_2^\varepsilon,\xi_2^\varepsilon,t_2^\varepsilon)$ in $(\mathcal{O}\times]0,T])\times(\mathcal{O}\times]0,T])$, with the same convergence as in (5.10). Moreover, we have $\hat{\rho}_1(y_i,\xi_i) = y_i$, and $T^l[D\gamma(\xi_i^\varepsilon), D^2\gamma(\xi_i^\varepsilon)](y_i^\varepsilon,\xi_i^\varepsilon) \geq 0$ for $i=1,2$ and small ε. We then get the conclusion as in the sub-case a).

Case 2). For every $\varepsilon,\delta>0$, on $(\mathcal{O}\times[0,T])\times(\mathcal{O}\times[0,T])$, we define

$$\Lambda(y_1,\xi_1,t_1,y_2,\xi_2,t_2) := u(y_1,\xi_1,t_1) - v(y_2,\xi_2,t_2) - \frac{|y_1-y_2|^2}{\varepsilon}$$
$$-\frac{\|\xi_1-\xi_2\|^2}{\varepsilon} - \frac{|t_1-t_2|^2}{\varepsilon} - |y_1-y_\delta|^2 - \beta(b_\delta(y_1)+b_\delta(y_2)),$$

where $b_\delta(y)$ is positive with derivatives uniformly bounded with respect to δ, $b_\delta(y)\to+\infty$ as $|y|\to+\infty$ and $b_\delta(y_\delta)=0$. Again, Λ has a maximum point $(y_1^{\varepsilon,\delta},\xi_1^{\varepsilon,\delta},t_1^{\varepsilon,\delta},y_1^{\varepsilon,\delta},\xi_1^{\varepsilon,\delta},t_1^{\varepsilon,\delta})$ on $(\mathcal{O}\times]0,T])\times(\mathcal{O}\times]0,T])$, and as $\max\{\varepsilon,\delta\}\to 0$,

$$\max\left(\frac{|y_1^{\varepsilon,\delta}-y_2^{\varepsilon,\delta}|^2}{\varepsilon}, \frac{\|\xi_1^{\varepsilon,\delta}-\xi_2^{\varepsilon,\delta}\|^2}{\varepsilon}, \frac{|t_1^\varepsilon-t_2^\varepsilon|^2}{\varepsilon}, |y_1^{\varepsilon,\delta}-y_\delta|^2\right)\to 0.$$

Since $|y_\delta|\to+\infty$ as $\delta\to 0$, and $\nu(\mathcal{P})<+\infty$, for every fixed $\varepsilon>0$ we can find $\delta_\varepsilon>0$ such that for $\delta\leq\delta_\varepsilon$ we have $\nu(\{\rho\in\mathcal{P}|\rho_2\geq|y_\delta|\})\leq\varepsilon^2$ and $\nu(\{\rho\in\mathcal{P}|\rho_1\leq-|y_\delta|\})\leq\varepsilon^2$. Moreover we can suppose that $\delta_\varepsilon\to 0$ as $\varepsilon\to 0$. We then recover two test functions ζ_1 and ζ_2 (depending on ε) such that $(y_i^{\varepsilon,\delta},\xi_i^{\varepsilon,\delta},t_i^{\varepsilon,\delta})$ is, respectively for $i=1,2$, a point of local maximum and local minimum for $u-\zeta_1$ and $v-\zeta_2$, and

$$\delta\leq\delta_\varepsilon \Rightarrow \left|T^j[D\zeta_i(\xi_i^{\varepsilon,\delta}), D^2\zeta_i(\xi_i^{\varepsilon,\delta})](y_i^{\varepsilon,\delta},\xi_i^{\varepsilon,\delta})\right|\leq\varepsilon.\ \ \forall i=1,2\ \forall j=u,l.$$

Hence, the "infinite dimensional" terms are infinitesimal as $\varepsilon\to 0$. We then obtain the desired contradiction applying standard techniques to the remaining "finite dimensional" part of the equation. $\square$

Remark 5.1. The hypotheses (5.1) and (5.3) are not necessary for the uniqueness result. For instance, if μ has compact support in $\overline{\mathcal{P}}$, one would like to work with a measure ν with compact support too, e.g. the Lebesgue measure on a rectangular triangle. This is the typical situation considered in Mayergoyz [11]. Slight modifications of the previous proof still get the uniqueness result (note that, out of such triangle, the infinite dimensional part vanishes). Moreover, the hypothesis (5.3) can be weakened, considering not a strip Q_r but other geometrical situation. However, we point out again that ν is not a datum of the control problem, but, after (2.5), its choice is at our disposal, in order to fit the problem in a suitable functional space.

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