Ordinary Differential Equations

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Contents

1 Introduction

These are the notes of the 42-hours course "Ordinary Differential Equations" ("Equazioni Differenziali Ordinarie") that I am going to deliver for the "Laurea in Matematica" of the University of Trento. I have decided to write them in English (even if the course will be delivered in Italian), since in the future they may be also suitable for the "Laurea Magistralis in Mathematics", whose official language is English.

To profitably read these notes, some background on Analysis, Geometry and Linear Algebra is needed (roughly speaking: the subject of the Analysis and Geometry courses of the first two years of the degree in Mathematics). Other advanced concepts will be somewhere used: convergence of functions, topology for functions spaces, complete metric spaces, advanced linear algebra. A familiarity with such concepts is certainly welcome. However, in the Appendix some of the requested notions and results are briefly treated.

The mathematical notations in these notes are the standard ones. In particular $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{C}$ respectively stand for the sets of real, natural, rational and complex numbers. Moreover, if $n \in \mathbb{N} \setminus \{0\}$, then $\mathbb{R}^n$ is, as usual, the n-dimensional space $\mathbb{R} \times \ldots \times \mathbb{R}$ i.e. the cartesian product of $\mathbb{R}$ n-times. The points (or vectors) of $\mathbb{R}^n$, when thought as string of coordinates, will be usually written as a line, even if, when a $m \times n$ matrix is applying to them, they must be think as column. With the notation $[a, b]$ we will mean the interval of real numbers $\{x \in \mathbb{R} | a \leq x \leq b\}$, that is the closed interval containing its extreme points. In the same way $]a, b[$ will denote the open interval without extreme points $\{x \in \mathbb{R} | a < x < b\}$[1], $]a, b]$ the semi-open interval $\{x \in \mathbb{R} | a < x \leq b\}$ and $[a, b[$ the semi-open interval $\{x \in \mathbb{R} | a \leq x < b\}$[2].

The time-derivative will be usually denoted by "y' ", and very rarely by "$\dot{y}$".

The Euclidean norm in $\mathbb{R}^n$, with $n > 1$, will be denoted by $\| \cdot \|_{\mathbb{R}^n}$ or, if there is no ambiguity, simply by $\| \cdot \|$. The absolute value of $\mathbb{R}$ is denoted by $| \cdot |$. When $x \in \mathbb{R}^n$, with $n > 1$, and when $r > 0$, with the notation $B_{\mathbb{R}^n}(x, r)$ we will denote the open ball centered in x with radius r

$$B_{\mathbb{R}^n}(x, r) = \{z \in \mathbb{R}^n | \|z - x\|_{\mathbb{R}^n} < r\}.$$

Also in this case, if no ambiguity arises, we may use the simpler notation $B(x, r)$.

If A is a subset of $\mathbb{R}^n$, by $\overline{A}$ we will denote its closure.

In these notes the formulas will be enumerated by $(x.y)$ where x is the number of the section (independently from the number of the subsection) and y is the running number of the formula inside the section. Moreover, the statements will be labeled by "S $x.y$" where "S" is the type of the statement (Theorem, Proposition, Lemma, Corollary, Definition, Remark, Example), x is the number of the section (independently from the number of the subsection), and y is the running number of the statement inside the section (independently from the type of the statement).

The symbol "$\square$" will mean the end of a proof.

[1] Here, we of course permit $a = -\infty$ as well as $b = +\infty$.

[2] In the last two cases we, respectively, permit $a = -\infty$ and $b = +\infty$.

Some possible references are[3]

- G. De Marco: *Analisi Due -seconda parte*, Decibel-Zanichelli, Padova 1993.
- C.D. Pagani - S. Salsa: *Analisi Matematica, Volume 2*, Masson, Milano 1991.
- L.C. Piccinini, G. Stampacchia, G. Vidossich: *Ordinary Differential Equations in* $\mathbb{R}^n$, Springer-Verlag, New York 1984.
- W. Walter: *Ordinary Differential Equations*, Springer-Verlag, New York 1998.

Please feel free to point out to me the mathematical as well as the english mistakes, which are for sure present in the following pages.

Let's start!

1.1 Motivating examples

The theory of ordinary differential equations is one of the most powerful method that humans have invented/discovered[4] and continuously improved for describing the natural phenomena whose investigation is fundamental for the progress of humanity. But its power is not limited to the "natural phenomena" (physical, biological, chemical etc.), it is also fundamental for the study and the construction of mechanical systems (engineering) as well as for the study and the prediction of the economical/social behavior of our real world.

An ordinary differential equations is a functional equation which involves an unknown function and its derivatives. The term "ordinary" means that the unknown is a function of a single real variable and hence all the derivatives are "ordinary derivatives".

A natural extension of the theory of ordinary differential equations is the theory of partial differential equations[5], which is certainly more suitable for describing those phenomena whose space-dependence is not negligible. However, most of the results about partial differential equations were not be obtainable without a good theory for the ordinary differential equations.

Since the unknown function depends only on a real variable, it is natural to give it the meaning of <u>time</u>, denoting it by $t \in \mathbb{R}$, and to interpret the solution as the <u>evolution</u> of the system under study. Here are some examples in this sense.

[3] The first two have strongly inspired the present notes, the second two may be suggested for deeper readings.

[4] Is Mathematics invented or is it discovered? We do not enter in such a diatribe. We leave it to philosophers.

[5] *A partial differential equation* is a functional equation which involves an unknown function and its derivatives. The term "partial" means that the unknown depends on many real variables and hence the derivatives in the equation are "partial derivatives".

Example 1.1 Capital management. The evolution of the amount of a capital at disposal is represented by a time-dependent function $K(\cdot)$. At every instant[6], there is a fraction cK which is re-invested with an instantaneous interest rate given by i, and there is also a fraction dK which is spent without earning. Here c, d, i are all real numbers between 0 and 1 and $c + d \leq 1$. The evolution law for the capital is then given by the equation

$$K'(t) = (ic - d)K(t), \tag{1.1}$$

which means that K *instantaneously tends to increase for the capitalization of the re-invested quantity* cK*, and instantaneously tends to decrease for the spent quantity* dK. Even if this is a very simple model, it is obvious that the possibility of computing the capital evolution $K(\cdot)$, i.e of solving the equation (1.1), is extremely important for the management of the capital. For instance, one may be interested in suitably choosing the coefficients c and d[7] in order to get a desired performance of the capital without too much reducing the expendable amount. To this end, the capability of solving the equation is mandatory. Here, a solution is a one-time derivable real-valued function K, defined on a suitable interval $I \subseteq \mathbb{R}$, $K : I \to \mathbb{R}$.

Example 1.2 Falling with the parachute. A body is falling hanged to its parachute. Denoting by g the gravity acceleration and by $\beta > 0$ the viscosity coefficient produced by the parachute, the law of the motion is, in a upward oriented one-dimensional framework,

$$x''(t) = -g - \beta x'(t). \tag{1.2}$$

This means that the time-law of the fall (i.e. the time-dependent function $x(\cdot)$) must solve the functional equation (1.2), which says: *the acceleration of the falling body is given by the downward gravity acceleration plus a term which depends on the velocity of the fall and on a viscosity coefficient.* This last term is responsible of the fall's safety: bigger is β slower is the fall[8]. With air's viscosity fixed, the coefficient β depends only on the shape of the parachute. Hence, one may be interested in calculating a suitable[9] coefficient β and then construct a corresponding parachute. It is obvious that the "suitableness" of β may be tested only if we know the corresponding evolution x, that is only if we can solve the equation (1.2) for all fixed value of β. A solution is then a two-times derivable real-valued function x, defined on a suitable interval $I \subseteq \mathbb{R}$, $x : I \to \mathbb{R}$.

Example 1.3 Filling a reservoir. A natural water reservoir is modeled by a bidimensional rectangular box $[a, b] \times [0, H]$. Let us suppose that the reservoir is filled in by a water source whose rate of introduction of water is constantly equal to $c > 0$ (volume of

[6] Let us suppose that the model is based on a "continuous time" instead of a "discrete time" (for instance:day by day, week by week...) as may be more natural to assume. The "continuity" of the time may be reasonable if we are looking to the evolution of the capital in a long period: ten, twenty years.

[7] Unfortunately, the most important coefficient in (1.1), that is i, is not at disposal of the investor, but it is decided by the bank. And also changing bank is not helpful.

[8] Note that, when the body is falling, x' is negative and hence $-\beta x'$ is positive

[9] "Suitable" could mean: safely reach the ground without spending too much time.

water per unit time). Let us suppose that the vertical layers of the reservoir have some degree of porosity. This means that, at every instant t, an amount of water exits through the points (a, h), (b, h) with a rate that is proportional to the porosity and to the quantity of water over the point (the pressure). In particular, let us denote by $u(t)$ the level of water inside the reservoir at the time t and let us suppose that the porosity depends on the height[10]. This means that there exists a function $g : [0, h] \to [0, 1]$ such that the rate of exit through (a, h) and (b, h) at time t is equal to zero if $u(t) < h$, otherwise it is given by $g(h)(u(t) - h)$.[11] Hence, the rate at the time t of the total volume of water that exits through the vertical layers is given by

$$\int_0^{u(t)} g(h)(u(t) - h)dh.$$

Hence, the instantaneous variation of the level u, that is its time derivative, is given by

$$u'(t) = \frac{c}{b-a} - \frac{1}{b-a}\int_0^{u(t)} g(h)(u(t) - h)dh.$$

For instance, if

$$g(h) = \frac{h}{H},$$

which means that the more permeable soils are on the surface, we then get the equation

$$u'(t) = \frac{c}{b-a} - \frac{u^3(t)}{6(b-a)H}. \tag{1.3}$$

If the source supplies water with a rate which is not constant but it depends on time, let us say $c(t)$, then the equation is

$$u'(t) = \frac{c(t)}{b-a} - \frac{u^3(t)}{6(b-a)H}. \tag{1.4}$$

Being able to calculate the solution u of (1.3) (or of (1.4)) permits to predict whether (and possibly at which time) the reservoir will become empty, or filled up, or whether its level will converge to a stable value. Here a solution is a one-time derivable real-valued function.

Example 1.4 The prey-predator model of Lotka-Volterra. Two species of animals, X and Y, occupy a certain region and interact. Let us denote by $x(t) \geq 0$ and by $y(t) \geq 0$ the number of present individuals at time t for both species respectively. Let us suppose that

[10]Such a situation is common in the application where, for deep reservoirs, different types of soils are stratified along the wall of the reservoir.

[11]Note that the point is not permeable if $g(h) = 0$, and hence nothing exit through that point; it is completely permeable if $g(h) = 1$, and hence if $u(t) > h$, then through that point the water exits with rate given by $(u(t) - h)$. If $0 < g(h) < 1$ all the intermediate cases may hold.

the species Y is given by predators whose preys are exactly the individuals X. Moreover, we suppose that the relative rate of increasing for the preys (i.e x'/x) is constant and positive when there are no predators ($y = 0$), and instead linearly decreases as function of y in case of presence of predators ($y > 0$). On the other hand, we suppose that the relative rate of increasing for the predators (i.e. y'/y) is constant and negative when there are no preys ($x = 0$)[12], and instead is linearly increasing as function of x in case of presence of preys ($x > 0$). Hence we have the following system

$$\left\{ \begin{array}{l} \dfrac{x'}{x} = \alpha - \beta y \\ \dfrac{y'}{y} = -\gamma + \delta x \end{array} \right. ,$$

where $\alpha, \beta, \gamma, \delta$ are positive constants. The system can also be written as

$$\left\{ \begin{array}{l} x' = x(\alpha - \beta y) \\ y' = y(-\gamma + \delta x) \end{array} \right. \tag{1.5}$$

Solving (1.5) may permit to study the evolution of the two species, which is certainly important from many points of view[13]. A solution of (1.5) is a one-time derivable vectorial function $t \mapsto (x(t), y(t)) \in \mathbb{R}^2$.

Up to now, we have considered model problems where the variable of the unknown function has the meaning of time, and the solution the meaning of evolution. However, this not the only case (even if it is a natural framework). Next two examples show cases where the real variable of the unknown function does not have the meaning of time, but rather the meaning of space[14] [15].

Example 1.5 The catenary curve. A homogeneous chain is hanged to a vertical wall by its extremes on two points, not on the same vertical line, and it is subject only to the gravity force. Which is the shape attained by the chain? Let us suppose that the shape of the chain is given by the graph of a function $y : [a, b] \to \mathbb{R}$, $x \mapsto y(x)$, where a and b are the abscissas of the two hanging points. On every piece of arch of the chain, the resultant of all the applied forces must be zero, since the chain is in equilibrium. Such forces are: 1) the total weight of the piece of arch, 2) the force exerted on the right extremum by the remaining right part of the chain, 3) the force exerted on the left extremum by the remaining left part of the chain. The first force is vertical and downward, the other two are tangential. Let our piece of arch be the

[12] They have nothing to eat.

[13] Nowadays we can in particular say "from an ecological point of view".

[14] However, in the sequel, we will often adopt the point of view of "time" and "evolution". It is obvious that, from an analytical point of view, the meaning given to the variable (time, space, what else...) and the name given to it (t, x, p...) is completely meaningless.

[15] The second of the next examples (Optimal control) starts from a problem of "evolution" but in the ordinary differential equation (1.7) the variable p has the meaning of space: the starting point of the evolution.

part of the graph over the subinterval $[x_l, x_r] \subseteq [a, b]$. We write our three forces by their horizontal and vertical components in the following way: 1) $\mathbf{P} = (0, -p)$ distributed on all the piece of arch, 2) $\mathbf{T}^r(x_r, y(x_r)) = (T_1^r(x_r, y(x_r)), T_2^r(x_r, y(x_r)))$, 3) $\mathbf{T}^l(x_l, y(x_l)) = (T_1^l(x_l, y(x_l)), T_2^l(x_l, y(x_l)))$. Since we must have

$$\mathbf{P} + \mathbf{T}^r(x_r, y(x_r)) + \mathbf{T}^l(x_l, y(x_l)) = (0, 0),$$

we then deduce that the modules of the horizontal component of $\mathbf{T}^r(x_r, y(x_r))$ and of $\mathbf{T}^l(x_l, y(x_l))$ are equal. By the arbitrariness of x_l and x_r, we deduce that such a modulus is constant, let us denote it by $c > 0$. Now, let $x_v \in [a, b]$ be a point of minimum for y (i.e. a point of minimum height for the chain). On $(x_v, y(x_v))$ the tangent is then horizontal, and, if we repeat the previous argument on the interval $[x_v, x_r]$, we have that $\mathbf{T}^l(x_v, y(x_v))$ has null second component (since it is tangent). Hence, the vertical weight must be balanced only by the vertical component of $\mathbf{T}^r(x_r, y(x_r))$. Let us denote by g the modulus of the gravity acceleration, and by μ the constant linear mass-density of the chain. Hence the weight of the arch over the interval $[x_v, x_r]$ is given by[16]

$$p = \int_{x_v}^{x_r} g\mu\sqrt{1 + (y'(x))^2}dx.$$

We then get

$$T_2^r(x_r, y_(r)) = \int_{x_v}^{x_r} g\mu\sqrt{1 + (y'(x))^2}dx.$$

Since the ratio $T_2^r(x_r, y_(r))/T_1^r(x_r, y_(r)) = T_2^r(x_r, y_(r))/c$ is the angular coefficient of the graph of y in $(x_r, y(x_r))$, that is $y'(x_r)$, we also get, for the arbitrariness of $x_r \geq x_v$ and repeating similar consideration for points to the left of x_v,

$$y'(x) = \frac{g\mu}{c}\int_{x_v}^{x} \sqrt{1 + (y'(\xi))^2}d\xi, \quad \forall x \in]a, b[.$$

Differentiating, we finally obtain

$$y''(x) = \frac{g\mu}{c}\sqrt{1 + (y'(x))^2} \quad \forall x \in]a, b[. \tag{1.6}$$

Being able to calculate the solution of (1.6) permits to know the shape of the chain[17].

[16] It is the curvilinear integral of the infinitesimal weight $g\mu ds$, where $ds = \sqrt{1 + (y'(x))^2}dx$ is the infinitesimal element of length of the arch.

[17] Actually, to exactly know the shape of the chain, we need some other information: the heights of the hanging points and the total length of the chain, otherwise many solutions are possible. But this is a common feature. Also in previous examples we usually need some other information as, for instance, the value of the solution in a fixed instant.

Example 1.6 Optimal control. A material point is constrained to move, without friction, on a one-dimensional guide. On the guide there is a system of coordinates, let us say $p \in \mathbb{R}$. The material point has to reach a fixed point positioned on $\overline{p}$, and it has several choices for moving: one per every value of the parameters $a \in [-1, 1]$. That is, at every instant $t \geq 0$, for every choice of a, it moves with instantaneous velocity equal to a. However, every such a choice has a cost, which depends on the actual position p of the point on the guide and on the parameter a, via an instantaneous cost function $\ell(p, a)$. The goal is to reach the target point $\overline{p}$ using a suitable moving strategy $a(t) \in [-1, 1]$ in order to minimize the following quantity

$$t^* + \int_0^{t^*} \ell(p(t), a(t))dt,$$

where t^* is the reaching time of the target point and $p(\cdot)$ is the evolution of our point. In other words, using a suitable moving strategy $a(\cdot)$, we want to reach the target point trying to minimize a "combination" of the spent time and the total cost given by ℓ[18]. For every starting point p, we can consider the "optimal function" $V(p)$ which is the optimum (i.e. the minimum cost) that we can get starting from p. Under suitable hypotheses[19], V solves the following equation

$$\sup_{a \in [-1,1]} \{-V'(p)a - \ell(p, a)\} = 1. \tag{1.7}$$

Being able to calculate V from (1.7) may permit to get useful information on the minimization problem, for instance on how to construct an optimal strategy $a(\cdot)$[20].

1.2 Notations, definitions and further considerations

An ordinary differential equation is an expression of the following type:

$$F(t, y^{(n)}, y^{(n-1)}, \dots, y', y) = 0, \tag{1.8}$$

where $F : \mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^k \times \cdots \times \mathbb{R}^k = \mathbb{R} \times \left(\mathbb{R}^k\right)^{n+1} \to \mathbb{R}$ is a function; $t \in \mathbb{R}$ is a scalar parameter; y is the unknown function, which is a function of t and takes values in $\mathbb{R}^k$ with $k \in \mathbb{N} \setminus \{0\}$; y' is the first derivative of y and, for every i, $y^{(i)}$ is the i-th derivative y. The highest order of derivation which occurs in the equation is said the order of the equation. If $k = 1$, then the equation is said a scalar equation.

Solving (1.8) means to find an interval $I \subseteq \mathbb{R}$ and a n-times derivable function $y : I \to \mathbb{R}^k$ such that for every $t \in I$

[18] The idea is that using values of a which give high velocities, and then reduce the spent time a lot, is probably not convenient from the point of view of ℓ: such velocities may be expensive. Hence a suitable combination of high and cheap velocities is needed.

[19] Actually, such "suitable hypotheses" are very delicate.

[20] However, in the real application, the evolution of the point p is not one-dimensional, but it is an evolution in $\mathbb{R}^n$, and hence the equation (1.7) satisfied by the optimum V is a partial differential equation, with V' replaced by the gradient ∇V.

$$(t, y^{(n)}(t), y^{(n-1)}(t), \dots, y'(t), y(t)) \in \mathcal{D},$$

and also that

$$F(t, y^{(n)}(t), y^{(n-1)}(t), \dots, y'(t), y(t)) = 0.$$

The function y is said a <u>solution of the ordinary differential equation</u>.

Let us note that a solution is a function from an interval of the real line to $\mathbb{R}^k$. Then it is a parametrization of a curve in $\mathbb{R}^k$, and the law $t \mapsto y(t)$ is the time-law of running such a curve. For this reason, a solution of an ordinary differential equation is sometimes called a <u>trajectory</u>.

An ordinary differential equation may have infinitely many solutions, or finitely many solutions or even no solutions at all. We define the <u>general integral of the equation</u> as the following set (which may be infinite, finite, a singleton or empty)

$$\mathcal{I} := \left\{ y : I \to \mathbb{R}^k \middle| I \subseteq \mathbb{R} \text{ is an interval and } y \text{ is solution of the equation} \right\} \tag{1.9}$$

In other words the general integral is the set of all solutions of the equation, everyone defined on its interval of definition.

The equation (1.8) is in the so-called <u>non-normal form</u>, that is the highest order derivative (the one of order n, in our case) is not "privileged", that is it is not "isolated", it is not "outside" from the function F. On the contrary we say that an ordinary differential equation is <u>in normal form</u> if it is of the following type

$$y^{(n)} = f(t, y^{(n-1)}, \dots, y', y), \tag{1.10}$$

where $f : \mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^k \cdots \times \mathbb{R}^k \to \mathbb{R}^k$. In particular, let us note that the co-domain of f is now $\mathbb{R}^k$ (and not anymore $\mathbb{R}$), and that the domain is contained in $\mathbb{R} \times \left(\mathbb{R}^k\right)^n$ (and not anymore $\mathbb{R} \times \left(\mathbb{R}^k\right)^{n+1}$). Hence, we have a <u>system of k scalar differential equations of n order</u>[21]. Indeed, denoting $y = (y_1, \dots, y_k)$ and $f = (f_1, \dots, f_k)$ by their components, we get

$$\begin{cases} y_1^{(n)} = f_1\left(t, (y_1^{(n-1)}, \dots, y_k^{(n-1)}), \dots, (y_1', \dots, y_k'), (y_1, \dots, y_k)\right) \\ y_2^{(n)} = f_2\left(t, (y_1^{(n-1)}, \dots, y_k^{(n-1)}), \dots, (y_1', \dots, y_k'), (y_1, \dots, y_k)\right) \\ \dots \qquad \dots \\ y_k^{(n)} = f_k\left(t, (y_1^{(n-1)}, \dots, y_k^{(n-1)}), \dots, (y_1', \dots, y_k'), (y_1, \dots, y_k)\right) \end{cases}$$

It is evident that all equations (systems) in normal form may be written in a non-normal form, for instance by

[21] Actually, systems of s non-normal equations as (1.8) in m unknowns may also be considered. However, we will not discuss such a situation.

$$F(t, y^{(n)}, \dots, y', y) := \left\| y^{(n)} - f(t, y^{(n-1)}, \dots, y', y) \right\|_{\mathbb{R}^k}.$$

On the contrary not all equations in non-normal form may be written in normal form. This depends on the solvability of the algebraic equation $F = 0$ with respect to its second entry $y^{(n)}$. For instance, the second order scalar equation

$$F(t, y'', y', y) = (y'')^2 - 1 = 0,$$

where F has domain $\mathcal{D} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ cannot be "globally" written in normal form, that is there is not a function $f(t, y', y)$ such that

$$F(t, y'', y', y) = 0 \Leftrightarrow y'' = f(t, y', y),$$

for every values $(t, y''(t), y'(t), y(t)) \in \mathcal{D}$. For example, the functions

$$y_1 : t \to \frac{t^2}{2}, \qquad y_2 : t \to -\frac{t^2}{2}$$

satisfy the equation $F(t, y'', y', y) = 0$, but they cannot satisfy $y'' = f(t, y', y)$ with the same f because otherwise, for $t = 0$, we should have

$$1 = f(0, 0, 0) = -1.$$

Hence, the property of being in normal form or in non-normal form does not depend on how we write the equation, but it is an intrinsic feature of the equation itself.

In general, the normal form equations are simpler to study.

An ordinary differential equation of the general form (1.8)[22] is linear homogeneous if it is linear in the unknown function y and its derivatives. That is if F is linear with respect to its second $n+1$ components. In other words if for every n-times differentiable functions $u, v : I \to \mathbb{R}^k$, for every $t \in I$, and for every scalars α, β, we have

$$\begin{aligned} &F(t, (\alpha u + \beta v)^{(n)}(t), \dots, (\alpha u + \beta v)'(t), (\alpha u + \beta v)(t)) \\ &= \alpha F(t, u^{(n)}(t), \dots, u'(t), u(t)) + \beta F(t, v^{(n)}(t), \dots v'(t), v(t)). \end{aligned}$$

An ordinary differential equation is said to be a linear nonhomogeneous equation if it is of the form

$$F(t, y^{(n)}, \dots, y', y) = g(t)$$

with F linear as before.

An ordinary differential equation is said to be a autonomous equation if it does not explicitly depend on the scalar variable $t \in \mathbb{R}$. Again referring to (1.8), we must have, for the non-normal form

[22] A similar definition obviously holds for equation in normal form, when we look to the linearity of the function f with respect to its second n components.

$$F(y^{(n)}, \dots, y', y) = 0,$$

with $F : \mathcal{D} \subseteq \left(\mathbb{R}^k\right)^{n+1} \to \mathbb{R}$, and, for the normal form

$$y^{(n)} = f(y^{(n-1)}, \dots, y', y),$$

with $f : \mathcal{D} \subseteq \left(\mathbb{R}^k\right)^n \to \mathbb{R}^k$. If the equation explicitly depends on t, we then speak of nonautonomous equation.

The systems of first-order equations in normal form, $y' = f(t, y)$, have a particular importance. Indeed, they are suitable for describing many evolutionary applied models. Moreover, every n-order scalar equation in normal form may be written as a first-order system of n scalar equations. Indeed, if we have the equation

$$y^{(n)} = g(t, y^{(n-1)}, \dots, y', y), \tag{1.11}$$

with $g : \mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, then we can define $Y_0 = y, Y_1 = y', \dots, Y_{n-1} = y^{(n-1)}$, $Y = (Y_0, \dots, Y_{n-1}) \in \mathbb{R}^n$, and write the system

$$\begin{cases} Y_0' = Y_1 \\ Y_1' = Y_2 \\ \dots \\ Y_{n-2}' = Y_{n-1} \\ Y_{n-1}' = g(t, Y_{n-1}, \dots, Y_1, Y_0) \end{cases} \tag{1.12}$$

It is evident that $y : I \to \mathbb{R}$ is a solution of (1.11) if and only if y is the first component of $Y : I \to \mathbb{R}^n$ with Y solution of (1.12). If we define

$$f(t, Y_0, \dots, Y_{n-1}) = \left(Y_1, Y_2, \dots, Y_{n-1}, g(t, Y_{n-1}, \dots, Y_1, Y_0)\right),$$

we then may write the system (1.12) as $Y' = f(t, Y)$.

Concerning the "evolutionary" feature of first-order systems, we will often use the interpretation of the solutions $y : I \to \mathbb{R}^k$ as trajectories (or curves) in $\mathbb{R}^k$, where y is the parametrization and I is the set of parameters (thought as "time"). For the particular case of first-order systems, the equality $y'(t) = f(t, y(t))$ means that, for every $t \in I$ and for every point $x = y(t)$ of the trajectory, the tangent vector to the trajectory itself is exactly given by $f(t, x) = f(t, y(t))$. In other words, if a particle is moving around $\mathbb{R}^k$ with the condition that, at any instant t, its vectorial velocity is given by $f(t, x)$, where x is the position of the particle at the time t, then the particle is necessarily moving along a trajectory given by a solution of the system. That is, if, for every time t and every position x, we assign the vectorial velocity of a motion by the law $v = f(t, x)$, then the motion must be along a trajectory solution of the system. In this setting, the function f is sometimes called dynamics and $\mathbb{R}^k$ the phase-space.

From the previous considerations, it is naturally to observe that, in order to uniquely determine the motion of the particle, we need to know, at least, its position at a fixed (initial) instant. That is we have to assign the following initial condition

$$y(t_0) = x_0$$

where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^k$ are fixed and such that $(t_0, x_0) \in \mathcal{D}$, the domain of f. Hence, we are assigning the value of the solution y at a fixed instant t_0. Then we have the following initial value first order system, more frequently called Cauchy problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = x_0. \end{cases}$$

Solving such a problem means to find a solution/trajectory which "passes" through x_0 at time t_0.

We end this subsection by applying all the definitions here given, to the equations of the examples of the previous subsection.

Equation (1.1) is: scalar, first-order, autonomous, linear homogeneous, in normal form.

Equation (1.2) is: scalar, second-order, autonomous, linear nonhomogeneous, in normal form.

Equation (1.3) is: scalar, first-order, autonomous, nonlinear, in normal form.

Equation (1.4) is: scalar, first-order, nonautonomous, nonlinear, in normal form.

Equation (1.5) is: a system of two first-order scalar equations, autonomous, nonlinear, in normal form.

Equation (1.6) is: scalar, second-order, autonomous, nonlinear, in normal form

Equation (1.7) is: scalar, first-order, autonomous, nonlinear, in non-normal form

1.3 Solving by hands and the need of a general theory

Let us consider the first order homogeneous nonautonomous linear equation

$$y'(t) = c(t)y(t), \quad t \in \mathbb{R}, \tag{1.13}$$

where $c : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $C : \mathbb{R} \to \mathbb{R}$ be a primitive of c. It is easy to see that, for every constant $k \in \mathbb{R}$, the function

$$y(t) = ke^{C(t)} \tag{1.14}$$

is a solution of (1.13).

The question is: are the functions of the form (1.14) all the solutions of (1.13)?

To answer the question, we are going to "work by hands" directly on the equation and then to get information on the solutions. First of all, we see that the null function $y \equiv 0$ is solution and it is of the form (1.14) with $k = 0$. Now, let y be a solution and let us suppose that there exists $t_0 \in \mathbb{R}$ such that $y(t_0) > 0$ (the case $y(t_0) < 0$ is similarly treated). Let $]a, b[\subseteq \mathbb{R}$ be the maximal interval such that

$$t_0 \in]a,b[\ \text{ and } \ y(t) > 0 \ \forall \ t \in]a,b[.$$

Starting from the equation (1.13) we then get

$$\frac{y'(t)}{y(t)} = c(t) \ \ \forall \ t \in]a,b[,$$

from which

$$\int_{t_0}^{t} \frac{y'(s)}{y(s)} ds = \int_{t_0}^{t} c(s) ds \ \ \forall \ t \in]a,b[.$$

Integrating and passing to the exponential, we finally get

$$y(t) = \left(y(t_0)e^{-C(t_0)}\right) e^{C(t)} \ \ \forall \ t \in]a,b[.$$

Hence, in the interval $]a,b[$, the solution y is of the form (1.14), with

$$k = \left(y(t_0)e^{-C(t_0)}\right) > 0. \tag{1.15}$$

Now, we observe that

$$a \in \mathbb{R} \implies y(a) = 0 \implies ke^{C(a)} = 0 \implies k = 0 \text{ contradiction!},$$

and similarly for b. Hence we must have $a = -\infty$ and $b = +\infty$, and we conclude that the solutions of (1.13) are exactly all the functions of the form (1.14)[23].

Now, the question is: which further conditions on the solution should we request, in order to uniquely fix the value of the constant k?

The answer is suggested by (1.15): we have to impose the value of the solution at a fixed time t_0, that is we have to consider the Cauchy problem

$$\begin{cases} y'(t) = c(t)y(t), \ t \in \mathbb{R}, \\ y(t_0) = y_0, \end{cases} \tag{1.16}$$

where $y_0 \in \mathbb{R}$ is the imposed value to the solution at $t = t_0$. It is now immediate to see that there is only one solution of the Cauchy problem (1.16): indeed we know that the solution is necessarily of the form (1.14) for some $k \in \mathbb{R}$, hence we get

$$y(t_0) = y_0 \implies ke^{C(t_0)} = y_0 \implies k = y_0 e^{-C(t_0)}.$$

Hence we have a unique solution to the problem (1.16), that is there exists a unique function y which solves (1.13) and, at the time t_0, passes through y_0. Such a function is[24]

[23]In particular, if $c(t) \equiv \tilde{c}$ is a constant, then the solutions are exactly all the functions of the form $y(t) = ke^{\tilde{c}t}$

[24]Again, if $c(\cdot)$ is the constant $\tilde{c}$, then we have $y(t) = y_0 e^{\tilde{c}(t-t_0)}$.

$$y(t) = \left(y_0 e^{-C(t_0)}\right) e^{C(t)} = y_0 e^{C(t)-C(t_0)}.$$

Let us summarize what we have discovered about the solutions of the equation (1.13) and of the Cauchy problem (1.16):

1) the solutions of (1.13) are the functions of the form $y_k(t) = ke^{C(t)}$ and they are defined (and solution) for all the times $t \in \mathbb{R}$;

2) for every $t_0, y_0 \in \mathbb{R}$ fixed, the solution of (1.16) is unique, and it is the function $y(t) = y_0 e^{C(t)-C(t_0)}$ (in particular, if $y_0 = 0$ then the unique solution is the null function $y \equiv 0$).

From 1) and 2) we can also get the following consideration

3) the general integral of (1.13), i.e. the set of all solutions, is a one-parameter family of functions

$$\tilde{I} = \{y_k : \mathbb{R} \to \mathbb{R} | k \in \mathbb{R}\},$$

and the correspondence $k \mapsto y_k$ between $\tilde{I}$ and $\mathbb{R}$ is a bijection. In particular, it is an injection because $k_1 \neq k_2$ implies $y_{k_1} \neq y_{k_2}$, since for instance they are different on t_0.

Now, we consider the following first order nonhomogeneous nonautonomous linear equation

$$y' = c(t)y + g(t), \tag{1.17}$$

where the function $c(\cdot)$ is as before and $g : \mathbb{R} \to \mathbb{R}$ is also continuous. Inspired by (1.14) in the previous case, we look for solutions of the form

$$y(t) = \alpha(t)e^{C(t)}, \tag{1.18}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a continuously derivable function. Differentiating and inserting in (1.17), we get

$$\alpha'(t) = g(t)e^{-C(t)}$$

and hence we must have

$$\alpha(t) = \int_{t_0}^{t} e^{-C(s)} g(s)ds + k,$$

where $t_0 \in \mathbb{R}$ is a fixed instant and $k \in \mathbb{R}$ is the integration constant. Hence, once t_0 is fixed, we have that, for all $k \in \mathbb{R}$, the function

$$y(t) = e^{C(t)} \left(k + \int_{t_0}^{t} e^{-C(s)} g(s)ds\right), \tag{1.19}$$

which is of the form (1.18), is a solution of (1.17).

Again, the question is: are the functions of the form (1.18) all the solutions of (1.17)?

Here, arguing as in the previous case is not completely immediate, due to presence of the term $g(t)$. However, using the already obtained existence and uniqueness results for the Cauchy problem (1.16), we immediate obtain a similar results for

$$\begin{cases} y'(t) = c(t)y(t) + g(t), & t \in \mathbb{R}, \\ y(t_0) = y_0, \end{cases} \tag{1.20}$$

where $t_0, y_0 \in \mathbb{R}$ are fixed values. Indeed, we easily get $k \in \mathbb{R}$ such that the function in (1.19) is a solution of (1.20):

$$y(t_0) = y_0 \implies k = y_0 e^{-C(t_0)}.$$

Now, using the linearity of (1.17), we have that, if $y(\cdot)$ and $z(\cdot)$ are any two solutions of (1.20), then the difference function $\psi = y - z$ is solution of (1.16) with condition $\psi(t_0) = 0$. But we already know that such a problem has a unique solution $\psi \equiv 0$. Hence we certainly have $y = z$, that is (1.20) has a unique solution, which of course is

$$y(t) = e^{C(t)} \left(y_0 e^{-C(t_0)} + \int_{t_0}^{t} e^{-C(s)} g(s) ds \right). \tag{1.21}$$

From such uniqueness result for (1.20), we can answer to the question whether all the solution of (1.17) are of the form (1.18). The answer is of course positive since, given any solution $y(\cdot)$ of (1.17) and denoted by y_0 its value in t_0, then such a function solves (1.20) and hence it is the function (1.21) which is of the form (1.18). Moreover, also in this case we get that the general integral of (1.17) is a one-parameter family of functions, one per every value of $k \in \mathbb{R}$.

What have we learned from the study of (1.17) and (1.20)? We have learned that, even if it is not obvious how to answer to our questions[25] via a direct hand-management of the equation, however we get a satisfactory answer using the already obtained uniqueness result for (1.16). In this, we are certainly helped by the fact that the equation is linear. Thus obtaining existence and uniqueness results for ordinary differential equations seems very important, even before making direct calculations for searching solutions. And what happens if the equation is not linear? Making the difference of two solutions is certainly not helpful. Hence, we are still more lead to think that a general and abstract theory concerning existence, uniqueness, comparison etc. of solutions is certainly useful and important. This is the subject of the next sections.

1.4 Frequently asked questions

Before starting with the general study of the ordinary differential equations, we make a list of those questions which are natural and common to formulate when we face an ordinary differential equation or a Cauchy problem.

Such questions are

[25] Are all the solutions of the form (1.18)? and is the solution of (1.20) unique?

(i) Does a solution exist?

(ii) If it exists, is it unique?

(iii) How long does a solution last?[26]

(iv) How much a solution is sensible with respect to parameters and coefficients which are present in the equation?

(v) How to calculate a solution?

Questions (i), (ii), (iii) and (iv) are of *qualitative* type. The question (v) is of *quantitative* type.

Another question is

(vi) Are there some types of equations which are easier to study than others, both from a qualitative and a quantitative point of view?

Another qualitative question is

(vii) If a solution exists for all times, what can we say about its behavior as time goes to infinity?

In the next sections, we will give some satisfactory answers to these questions. In particular questions (i) and (ii) are treated in Section 2; question (iii) is treated in Section 2 and Section 5; question (iv) is treated in Section 6 and Section 7; question (v) is treated in Section 4 and in Section 7; question (vi) is treated in Section 3; question (vii) is treated in Section 7.

[26] For instance, if we a priori fix a time interval $[t_1, t_2]$, does the solution exist for all those times?

2 Existence and uniqueness theorems for the Cauchy problem

2.1 Definitions: local and global existence

Let us consider the following Cauchy problem for a nonlinear nonautonomous system[27]

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = x_0, \end{cases} \tag{2.1}$$

where $(t_0, x_0) \in A \subseteq \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$ is fixed, with A a given open set and $f : A \to \mathbb{R}^n$ continuous.

Definition 2.1 *We say that a function $y : I \to \mathbb{R}^n$ is a solution of (2.1) on I if $I \subseteq \mathbb{R}$ is an open interval, $t_0 \in I$, $y \in C^1(I, \mathbb{R}^n)$, $y'(t) = f(t, y(t))$ for all $t \in I$ and $y(t_0) = x_0$.*

Note that Definition 2.1 also implies that $(t, y(t)) \in A$ for all $t \in I$, otherwise we cannot apply f to the couple $(t, y(t))$. Moreover, it is evident that, if $y : I \to \mathbb{R}^n$ is a solution on I, then it is also a solution on every open interval $J \subseteq I$ containing t_0.

Definition 2.2 *We say that the problem (2.1) admits a local solution (or equivalently is locally solvable) if there exist an open interval $I \subseteq \mathbb{R}$ and a function $y : I \to \mathbb{R}^n$ such that y is a solution of (2.1) on I.*

Definition 2.3 *We say that the problem (2.1) admits global solutions (or equivalently is globally solvable) if for every open interval $I \subseteq \mathbb{R}$ such that*

$$t_0 \in I, \qquad \{x \in \mathbb{R}^n | (t, x) \in A\} \neq \emptyset \ \ \forall\, t \in I, \tag{2.2}$$

there exists a function $y : I \to \mathbb{R}^n$ which is solution of (2.1) on I.

The difference between local and global solvability is that in the first case we cannot a-priori fix the time interval, but we only know that a (possibly very small) time interval exists. On the contrary, in the second case, for any (suitable) fixed time interval, we find a solution on it.

Of course, the global solvability implies the local one, but the contrary is false.

Example 2.4 We consider the problem

$$\begin{cases} y'(t) = y^2(t), \\ y(0) = 1. \end{cases}$$

Here, $n = 1$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(t, x) = x^2$, $A = \mathbb{R}^2$, $t_0 = 0$, $x_0 = 1$. Applying a uniqueness result from next subsections, we see that such a problem has the unique solution

[27]Here "nonlinear system" means that we do not require that the system is linear. But, of course, all the results can also be applied to the linear case. The same remark applies to "nonautonomous system".

$$y(t) = \frac{1}{1-t},$$

which is defined only in the time interval $]-\infty, 1[$. For example, if we a-priori fix the time interval $J =]-1, 2[$ (which is an admissible time interval for our problem, that is satisfies (2.2)), and we look for a solution on J, then we do not find anything. Hence the problem admits only a local solution.

In general, if A is arch-connected, we may consider the maximal interval satisfying (2.2)

$$I_{\max} = \{t \in \mathbb{R} | \exists x \in \mathbb{R}^n, (t, x) \in A\} =:]a, b[$$

Moreover, let us suppose that A is the maximal set of definition for f and that the problem is only locally solvable. Then there exist $a < \xi_1 < \xi_2 < b$ such that every solution exits from A[28] in the time interval $]\xi_1, \xi_2[$. Hence, looking for solution defined up to times $t \in]a, \xi_1[\cup]\xi_2, b[$ is meaningless since we certainly cannot apply f to the possible couple $(t, y(t)) \notin A$.

2.2 The integral representation

Proposition 2.5 *Let $f : A \to \mathbb{R}^n$ be continuous and $I \subseteq \mathbb{R}$ be an open interval. Then a function $y : I \to \mathbb{R}^n$ is a solution of (2.1) on I, if and only if $(t, y(t)) \in A$ for all $t \in I$, and, more important, y is continuous (i.e. $y \in C(I, \mathbb{R}^n)$) and*[29]

$$y(t) = x_0 + \int_{t_0}^{t} f(s, y(s))ds \qquad \forall t \in I \tag{2.3}$$

Proof. First of all let us note that, since y and f are both continuous, then the function of time $s \mapsto f(s, y(s))$ is also continuous and the integral in (2.3) exists for all $t \in I$. Let us now prove the equivalence.

$\Longrightarrow$) If y is a solution then, by definition, $y \in C^1(I, \mathbb{R}^n)$ and so, for every $t \in I$, integrating the equation from t_0 to t, we get

$$\int_{t_0}^{t} y'(s)ds = \int_{t_0}^{t} f(s, y(s))ds \implies y(t) = x_0 + \int_{t_0}^{t} f(s, y(s))ds.$$

$\Longleftarrow$) If y is continuous and satisfies (2.3), then we also have $y \in C^1(I, \mathbb{R}^n)$, and, by the fundamental theorem of calculus, its derivative is exactly $y'(t) = f(t, y(t))$. Since by (2.3) we obviously have $y(t_0) = x_0$, we then conclude. $\square$

[28] By an abuse of definition, this means that the couple $(t, y(t)) \in \mathbb{R}^{n+1}$ exits from A.

[29] The integral of a vectorial function f is the vector whose components are the integral of the components of f.

Proposition 2.5 gives an integral representation of the solution, as well as of the ordinary differential equation. It is a powerful tool for studying the Cauchy problem, since integrals are often more manageable than derivatives. Already starting from the next subsection we are going to make an intensive use of the integral representation (2.3).

Remark 2.6 *Note that the formula (2.3) is an "implicit formula". That is we cannot calculate the solution y just integrating in (2.3), because the second member itself depends on y.*

2.3 The Peano existence theorem under continuity hypothesis

Theorem 2.7 *If $f : A \to \mathbb{R}^n$ is continuous, then the Cauchy problem (2.1) is locally solvable for any choice of the datum $(t_0, x_0) \in A$. This means that, for any $(t_0, x_0) \in A$, there exist $\delta > 0$ (depending on (t_0, x_0)) and a function $y :]t_0 - \delta, t_0 + \delta[\to \mathbb{R}^n$ which is a solution of (2.1) on $]t_0 - \delta, t_0 + \delta[$.*

In the next two subsections we are going to give proofs of this theorem for the particular case where

$$A = J \times \mathbb{R}^n, \ J =]a, b[\subseteq \mathbb{R} \text{ open interval }, \quad f \text{ bounded on } A, \tag{2.4}$$

and we prove the existence of a solution in a right neighborhood of t_0: $[t_0, c[$. This is indeed a (right-) global existence result (for the arbitrariness of $t_0 < c < b$). The general case of local (both sides-) existence with arbitrary A, and f not bounded, may be obtained restricting the problem to a rectangular inside A[30].

In both following proofs, the idea is to approximate a solution by constructing a suitable sequence of quasi-solutions, and then passing to the limit using the Ascoli-Arzelà theorem 8.10.

2.3.1 A first proof: a delayed argument

If we look to the equation $y'(t) = f(t, y(t))$, we have an instantaneous relation (i.e. at the same instant t) between the left- and right-hand sides. Hence, we decide to little change the equation by introducing a time-delay $\alpha > 0$. The idea is to consider the delayed equation

$$y'(t) = f(t, y(t - \alpha)),$$

where the unknown term $y(t - \alpha)$ in the right-hand side may be supposed already known, indeed our goal is to solve the equation step-by-step, that is first in the interval $[t_0, t_0 + \alpha]$, then in $[t_0 + \alpha, t_0 + 2\alpha]$ and so on.

We define the following function in $[t_0 - \alpha, c]$

[30] In a similar way as we are going to do in the next Section under Lipschitz continuity hypothesis.

$$z_\alpha(t) = \begin{cases} x_0 & \text{if } t_0 - \alpha \le t \le t_0, \\ x_0 + \displaystyle\int_{t_0}^t f(s, z_\alpha(s-\alpha))ds & \text{if } t_0 \le t \le c. \end{cases}$$

Note that, for every $\alpha > 0$, the function z_α is well defined in $[t_0, c]$. For instance: if $t \in [t_0, t_0 + \alpha]$ then $z_\alpha(t) = x_0 + \int_{t_0}^t f(s, x_0)ds$, which is computable, and once we know z_α in $[t_0, t_0 + \alpha]$, we then can compute it in $[t_0 + \alpha, t_0 + 2\alpha]$, and so on. Since f is bounded, we have that, denoting by $M > 0$ its bound,

$$\|z_\alpha\|_{C([t_0,c];\mathbb{R}^n)} \le \|x_0\|_{\mathbb{R}^n} + M(c - t_0), \quad \|z_\alpha(t) - z_\alpha(s)\|_{\mathbb{R}^n} \le M|t-s| \ \forall t, s \in [t_0, c]. \quad (2.5)$$

Hence, the functions $\{z_\alpha\}_{\alpha>0}$ are equibounded and equicontinuous on the compact set $[t_0, c]$. By the Ascoli-Arzelà theorem, this implies that there exist a subsequence z_k (i.e. a subsequence $\alpha_k \to 0$ as $k \to +\infty$) and a continuous function $y : [t_0, c] \to \mathbb{R}^n$ such that z_k uniformly converges to y on $[t_0, c]$.

Note that also the functions $t \mapsto z_\alpha(t - \alpha)$ are well defined on $[t_0, c]$, and they also uniformly converge to y. This is true since we have

$$\|z_k(t - \alpha_k) - y(t)\| \le \|z_k(t - \alpha_k) - z_k(t)\| + \|z_k(t) - y(t)\|,$$

and both terms in the right-hand side are infinitesimal as $k \to +\infty$ independently on $t \in [t_0, c]$ (for the uniform convergence and the equicontinuity of z_k (2.5)).

Moreover, the functions $t \mapsto f(t, z_k(t - \alpha_k))$ uniformly converges to the function $t \mapsto f(t, y(t))$, on $[t_0, c]$. This is true since the couples $(t, z_k(t-\alpha_k)), (t, y(t))$ all belong to the compact set $[t_0, c] \times \overline{B}_{\mathbb{R}^n}(O, (\|x_0\|_{\mathbb{R}^n} + M(c - t_0)))$, where O is the origin of $\mathbb{R}^n$, and f, being continuous, is uniformly continuous on such a set.

Finally, passing to the limit, as $k \to +\infty$, in the definition of z_k, we get

$$y(t) = x_0 + \int_{t_0}^t f(s, y(s))ds,$$

which concludes the proof, proving that y is a solution.

2.3.2 A (sketched) second proof: the Euler method

Here, our goal is to construct an approximate solution y_δ by dividing the time interval $[t_0, c]$ into a series of a finite number of small subintervals $[t_k, t_{k+1}]$ and, on every such subintervals, defining y_δ as the segment starting from $y_\delta(t_k)$ and parallel to $f(t_k, y_\delta(t_k))$[31] (which may be considered as known if we apply such a procedure step by step (i.e. subinterval by subinterval)). The idea is then to make the limit as δ, the length of the subintervals, goes to zero.

[31]i.e. the linear function with slope given by $f(t_k, y_\delta(t_k))$

Let us take $\delta > 0$ and a finite set of instants

$$\mathcal{T}_\delta = \{t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = c\},$$

such that $t_{k+1} - t_k = \delta$ for all $k = 0, \ldots, m$ and $c - t_m \le \delta$.

We define the following function $y_\delta : [t_0, c[\to \mathbb{R}^n$

$$y_\delta(t) = y_\delta(t_{k-1}) + f(t_{k-1}, y_\delta(t_{k-1}))(t - t_{k-1}) \quad t \in [t_{k-1,t_k}] \ \forall k = 1, \ldots, m, m+1$$

This is a good definition and y_δ is a polygonal. Note that, in every subinterval $[t_{k-1}, t_k]$, $y'_\delta(t) = f(t_{k-1}, y_\delta(t_{k-1}))$, and hence, also in this case, we have a sort of "delay".

It can be (sufficiently easily) proved that, for every $\varepsilon > 0$, there exists $0 < \delta_\varepsilon < \varepsilon$ (i.e. a sufficiently fine partition of $[t_0, c]$), such that

$$\left\| y_{\delta_\varepsilon}(t) - x_0 + \int_{t_0}^t f(\tau, y_{\delta_\varepsilon}(\tau)) d\tau \right\|_{\mathbb{R}^n} \le \varepsilon \quad \forall t \in [t_0, c], \tag{2.6}$$

which, recalling the integral representation of the solution, may be seen as a criterium of goodness of the approximate solution. Moreover, in the same way as in the previous paragraph, we can prove that the sequence of functions

$$\varphi_\delta : t \mapsto x_0 + \int_{t_0}^t f(\tau, y_\delta(\tau)) d\tau$$

is equibounded and equicontinuous in $[t_0, c]$. Hence, also the sequence $\varphi_{\delta_\varepsilon}$ has the same properties and so, by the Ascoli-Arzelà theorem, there exists a subsequence $\varphi_{\delta_{\varepsilon(i)}}$ which, as $i \to +\infty$ (i.e. $\varepsilon(i) \to 0$), uniformly converges in $[t_0, c]$ to a continuous function φ. But then, $y_{\delta_{\varepsilon(i)}}$ also uniformly converges to φ. Indeed, for every $t \in [t_0, c]$, we have[32]

$$\|y_{\delta_{\varepsilon(i)}}(t) - \varphi(t)\| \le \|y_{\delta_{\varepsilon(i)}}(t) - \varphi_{\delta_{\varepsilon(i)}}(t)\| + \|\varphi_{\delta_{\varepsilon(i)}}(t) - \varphi(t)\| \le \varepsilon(i) + O(\varepsilon(i))$$

From this we deduce that, at the limit (passing to the limit inside the integral, noting that y_δ are equibounded by $\tilde{M}$ (since f is bounded), and that f is uniformly continuous on $[t_0, c] \times B_{\mathbb{R}^n}(0, \tilde{M})$), we have

$$\varphi(t) = x_0 + \int_{t_0}^t f(\tau, \varphi(t)) d\tau \quad \forall\ t \in [t_0, c],$$

and the proof is concluded: we have found a solution (i.e. φ) of the Cauchy problem.

[32]Here O means an infinitesimal with respect to its argument.

2.3.3 An example of non existence

Here, we give an example where the dynamics is discontinuous and the Cauchy problem has no solution at all. Let us consider $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -1 & \text{if } x > 0, \\ \dfrac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x < 0, \end{cases}$$

and consider the Cauchy problem

$$\begin{cases} y'(x) = f(y(x)), \\ y(0) = 0. \end{cases}$$

It is evident that such a problem cannot have a solution, even locally. Indeed, if a solution y exists (remember that we require $y \in C^1$) then $y'(0) = f(y(0)) = f(0) = 1/2 > 0$, and hence y is strictly increasing at $x = 0$. Since $y(0) = 0$, this means that $y > 0$ immediately to the right of $x = 0$, but this is impossible since when $y > 0$ the derivative y' must be negative.

Of course, if we change the value of f at $x = 0$, defining $f(0) = 0$, then f is still discontinuous but the Cauchy problem has a solution, even unique: the null function $y(x) \equiv 0$.

2.3.4 An example of non uniqueness

Let us consider the following scalar Cauchy problem

$$\begin{cases} y'(t) = (y(t))^{\frac{2}{3}}, \\ y(0) = 0. \end{cases}$$

Here, $f(t, x) = x^{2/3}$ which is defined on $A = \mathbb{R} \times \mathbb{R}$ and is continuous. Hence we have, at least, one local solution. Indeed, we have infinitely many local solutions.

First of all, let us note that the null function $y(t) \equiv 0$ is a solution. But, for every $\alpha, \beta \geq 0$, we also have the solution

$$y_{\alpha,\beta}(t) = \begin{cases} 0 & \text{if } -\beta \leq t \leq \alpha, \\ \left(\dfrac{t-\alpha}{3}\right)^3 & \text{if } t \geq \alpha, \\ \left(\dfrac{t+\beta}{3}\right)^3 & \text{if } t \leq -\beta. \end{cases}$$

2.4 Local existence and uniqueness under Lipschitz continuity hypothesis

Definition 2.8 *We say that the Cauchy problem (2.1) has a unique local solution if there exists an interval $\tilde{I}$ and a solution $\tilde{y} : \tilde{I} \to \mathbb{R}^n$ on $\tilde{I}$ such that, for every other solution $y : I \to \mathbb{R}^n$, we have $y = \tilde{y}$ on $I \cap \tilde{I}$*[33]*.*

Note that in the Definition 2.8 we suppose that a local solution exists and we say that it is unique in the sense that, in a suitable neighborhood of t_0, all possible solutions must coincide with $\tilde{y}$.

Definition 2.9 *The continuous function $f : A \to \mathbb{R}^n$ of the problem (2.1) is said to be locally Lipschitz continuous in $x \in \mathbb{R}^n$ uniformly with respect to $t \in \mathbb{R}$ if for every $(t_0, x_0) \in A$ there exist a neighborhood of (t_0, x_0), $U \subseteq A$, and a constant $L > 0$ such that*

$$(t, x), (t, z) \in U \implies \|f(t, x) - f(t, z)\|_{\mathbb{R}^n} \leq L\|x - z\|_{\mathbb{R}^n}. \tag{2.7}$$

Theorem 2.10 *(Local existence and uniqueness) If $f : A \to \mathbb{R}^n$ is continuous and satisfies the local uniform Lipschitz condition (2.7), then, for every datum $(t_0, x_0) \in A$, the Cauchy problem (2.1) has a unique local solution in the sense of Definition 2.8.*

Proof. Let $(t_0, x_0) \in A$ be fixed. Our goal is to find $\delta_1 > 0$ and $\delta_2 > 0$ such that, for every $0 < \delta' \leq \delta_1$, the operator

$$\begin{aligned} &T_{\delta'} : C\left([t_0 - \delta', t_0 + \delta']; \overline{B}(x_0, \delta_2)\right) \to C\left([t_0 - \delta', t_0 + \delta']; \overline{B}(x_0, \delta_2)\right), \\ &v \mapsto T_{\delta'}[v] : t \mapsto x_0 + \int_{t_0}^{t} f(s, v(s))ds \end{aligned} \tag{2.8}$$

is well defined and has a unique fixed point. In this way, via the integral representation of solutions (2.3), for every $0 < \delta' \leq \delta_1$, we simultaneously have existence of a solution on $]t_0 - \delta', t_0 + \delta'[$ and uniqueness on such intervals.[34]. Hence, the local uniqueness in the sense of Definition 2.8 comes from the following observation: let us take $0 < \tilde{\delta} \leq \delta_1$, define $\tilde{I} =]t_0 - \tilde{\delta}, t_0 + \tilde{\delta}[$ and define $\tilde{y}$ as the unique solution on $\tilde{I}$. Note that every solution is also a solution on every subinterval. Then, if $y : I \to \mathbb{R}^n$ is a solution, we must have $\tilde{y} = y$ on $\tilde{I} \cap I$ ($\subseteq]t_0 - \delta', t_0 + \delta'[$ for some $0 < \delta' < \tilde{\delta}$).

Hence, we are going to analyze the following four steps.

1) Find $\delta_1 > 0$ and $\delta_2 > 0$ such that, for every $0 < \delta' \leq \delta_1$, $T_{\delta'}$ is well defined, that is $T_{\delta'}(v) \in C\left([t_0 - \delta', t_0 + \delta']; \overline{B}(x_0, \delta_2)\right)$ for all $v \in C\left([t_0 - \delta', t_0 + \delta']; \overline{B}(x_0, \delta_2)\right)$.

2) Recognize that $y :]t_0 - \delta', t_0 + \delta'[\to \mathbb{R}^n$ is a solution of (2.1) on $]t_0 - \delta', t_0 + \delta'[$ if and only if it is a fixed point of $T_{\delta'}$.

3) Take δ_1 as in 1) and also such that, for every $0 < \delta' \leq \delta_1$, $T_{\delta'}$ is a contraction with respect to the infinity norm.

[33] Note that, since I and $\tilde{I}$ are both open intervals containing t_0, then $I \cap \tilde{I}$ is also an open interval containing t_0.

[34] This means that there exists only one solution on $]t_0 - \delta', t_0 + \delta'[$, for every δ'.

4) Apply the Contraction Lemma to $T_{\delta'}$ to observe that it has one and only one fixed point. Conclude by the point 2).

1) Let U and L be as in (2.7). Let us take $\overline{\delta}_1, \delta_2 > 0$ such that $[t_0 - \overline{\delta}_1, t_0 + \overline{\delta}_1] \times \overline{B}(x_0, \delta_2) \subseteq U$, and take $M > 0$ such that[35]

$$\|f(t,x)\| \leq M, \quad \forall\ (t,x) \in [t_0 - \overline{\delta}_1, t_0 + \overline{\delta}_1] \times \overline{B}(x_0, \delta_2).$$

For every $0 < \delta' \leq \overline{\delta}_1$, for every $v \in C\left([t_0 - \delta', t_0 + \delta'], \overline{B}(x_0, \delta_2)\right)$, and for every $t \in [t_0 - \delta', t_0 + \delta']$, we have the following estimate

$$\|T_{\delta'}[v](t) - x_0\| = \left\|\int_{t_0}^{t} f(s, v(s))ds\right\| \leq \left|\int_{t_0}^{t} \|f(s, v(s))\|ds\right| \leq \left|\int_{t_0}^{t} M ds\right| \leq M\delta' \leq M\overline{\delta}_1. \tag{2.9}$$

Hence, if we take $\delta_1 \leq \overline{\delta}_1$ such that $M\delta_1 \leq \delta_2$, then the point 1) is done[36].

2) Obvious by Proposition 2.5.

3) Let us take $0 < \delta' \leq \delta_1$. For every $u, v \in C\left([t_0 - \delta', t_0 + \delta'], \overline{B}(x_0, \delta_2)\right)$, and for every $t \in [t_0 - \delta', t_0 + \delta']$ we have

$$\begin{aligned}\|T_{\delta'}[u](t) - T_{\delta'}[v](t)\| &\leq \left|\int_{t_0}^{t} \|f(s, u(s)) - f(s, v(s))ds\right| \leq L\left|\int_{t_0}^{t} \|u(s) - v(s)\|ds\right| \\ &\leq L\left|\int_{t_0}^{t} \|u - v\|_\infty ds\right| \leq L\delta'\|u - v\|_\infty \leq L\delta_1\|u - v\|_\infty.\end{aligned}$$

By the arbitrariness of $t \in [t_0 - \delta', t_0 + \delta']$, and also because the last term in the right is independent from t, we then get

$$\|T_{\delta'}[u] - T_{\delta'}[v]\|_\infty \leq L\tilde{\delta}_1\|u - v\|_\infty,$$

where $\|\cdot - \cdot\|_\infty$ is the metrics in $C\left([t_0 - \delta', t_0 + \delta'], \overline{B}(x_0, \delta_2)\right)$.
Hence, if we take δ_1 as in point 1) and also such that $L\delta_1 < 1$, then point 3) is also done.

4) Immediate. $\square$

Remark 2.11 *A sufficient condition for satisfying the local Lipschitz hypothesis (2.7) is to be a C^1 function. Hence, every Cauchy problem with C^1 dynamics f has a unique local solution for every initial datum.*

[35] Such a M exists for the continuity of f and the compactness of $[t_0 - \overline{\delta}_1, t_0 + \overline{\delta}_1] \times \overline{B}(x_0, \delta_2)$.

[36] Also observe that, just by definition, $T_{\delta'}[v]$ is continuous.

2.5 Global existence and uniqueness under Lipschitz continuity hypothesis

Definition 2.12 *We say that the Cauchy problem (2.1) has a unique global solution if it has exactly one solution, in the sense that there exists a solution $\tilde{y} : \tilde{I} \to \mathbb{R}^n$, such that for every other solution $y : I \to \mathbb{R}^n$ we have: $I \subseteq \tilde{I}$ and $y = \tilde{y}$ on I.*

Note that Definition 2.12 also requires that a local solution exists. However, the global uniqueness is different from the local uniqueness of Definition 2.8. Indeed, in the second case we require that a local solution $\tilde{y}$ exists in a suitable interval $\tilde{I}$ and any other solution is equal to $\tilde{y}$ in the common instants of existence, but they may be different outside the common instants. On the other hand, the global uniqueness requires that a solution $\tilde{y}$ exists on an interval $\tilde{I}$, that no other solution may be defined for instants $t \notin \tilde{I}$, and that on the common instants of existence (necessarily contained in $\tilde{I}$) any other solution must be equal to $\tilde{y}$.

Of course, the global uniqueness of the solution implies the local one, but the contrary is false.

Example 2.13 The following Cauchy problem has a unique local solution, but not a unique global solution:

$$\begin{cases} y'(t) = (y(t))^{\frac{2}{3}}, \\ y(-1) = -\dfrac{1}{27}. \end{cases}$$

The following two functions are both solution[37]

$$y_1(t) = \begin{cases} \left(\dfrac{t}{3}\right)^3 & t \le 0, \\ 0 & t \ge 0 \end{cases}, \qquad y_2(t) = \left(\frac{t}{3}\right)^3 \ \forall\, t \in \mathbb{R}.$$

Hence, such a problem has no global uniqueness since y_1 and y_2 are both solution on $I = \mathbb{R}$ but they are different. On the contrary, it has a unique local solution $y(t) = (t/3)^3$, as can be seen by Theorem 2.10.

Remark 2.14 *The global uniqueness in the sense of Definition 2.12 does not imply that a global solution exists, that is that the problem is globally solvable in the sense of Definition 2.3. Indeed, the interval $\tilde{I}$ is not necessarily the maximal interval satisfying the condition (2.2). For instance, the problem in Example 2.4 has a unique global solution in the sense of Definition 2.12, but it is not globally solvable in the sense of Definition 2.3.*

Theorem 2.15 *(Global existence and uniqueness on a strip) Let us consider the Cauchy problem (2.1), with A given by the strip*

[37] See the example in Subsection 2.3.4.

$$A =]a, b[\times \mathbb{R}^n, \tag{2.10}$$

where $-\infty \le a < b \le +\infty$. If f satisfies the global uniform Lipschitz condition

$$\begin{aligned} &\exists\ L > 0 \text{ such that } \forall\ t \in]a, b[,\ \forall\ x, y \in \mathbb{R}^n : \\ &\|f(t, x) - f(t, y)\|_{\mathbb{R}^n} \le L\|x - y\|_{\mathbb{R}^n}. \end{aligned} \tag{2.11}$$

then, for every datum $(t_0, x_0) \in A$, the problem has a unique global solution, in the sense of Definition 2.12. Moreover the solution is defined on the whole interval $]a, b[$, that is the problem is globally solvable.

Proof. We are going to prove that on every subinterval of $]a, b[$ there exists one and only one solution, from which the conclusion will follow.

Let us refer to the proof of Theorem 2.10. Over there, given (t_0, x_0), we have taken the neighborhood U where we have found the local Lipschitz constant L (both depending on (t_0, x_0)); we have taken $\overline{\delta}_1$ and δ_2 depending on (t_0, x_0), and moreover we have taken M also depending on (t_0, x_0). All those choices have leaded us to choose

$$\delta_1 = \min\left\{\overline{\delta}_1, \frac{\delta_2}{M}, \frac{1}{L}\right\},$$

in order to prove the statement. Of course, δ_1 was also depending on (t_0, x_0), and recall that it was the semi-amplitude of the existence interval for the unique local solution. Now, our goal is to construct δ_1 independently from (t_0, x_0).

Let us fix $\varepsilon > 0$ sufficiently small and note that, in the present situation (if $a, b \in \mathbb{R}$), we can just take $U = [a+\varepsilon, b-\varepsilon] \times \mathbb{R}^n$ and L independently from $(t_0, x_0) \in [a+\varepsilon, b-\varepsilon] \times \mathbb{R}^n$[38]. In this way, once we have fixed $(t_0, x_0) \in U$, in order to remain inside U, $\overline{\delta}_1$ has only to satisfy, independently from x_0,

$$\overline{\delta}_1(t_0) = \min\{t_0 - a - \varepsilon, b - \varepsilon - t_0\},$$

and δ_2 can be any positive value. Moreover, let us not that, by (2.11), there exists a constant $c > 0$ (depending only on ε) such that

$$\|f(t, x)\| \le L\|x\| + c \quad \forall (t, x) \in [a + \varepsilon, b - \varepsilon] \times \mathbb{R}^n. \tag{2.12}$$

Indeed, we have

$$\begin{aligned} &\|f(t, x)\| = \|f(t, x) - f(t, 0) + f(t, 0)\| \le \|f(t, x) - f(t, 0)\| + \|f(t, 0)\| \\ &\le L\|x\| + \max_{s \in [a+\varepsilon, b-\varepsilon]} \|f(s, 0)\| = L\|x\| + c. \end{aligned}$$

Hence, for any $\delta_2 > 0$ and for any $(t_0, x_0) \in [a + \varepsilon, b - \varepsilon] \times \overline{B}(x_0, \delta_2)$, we have

[38] If $a = -\infty$ or $b = +\infty$, we have to replace $a + \varepsilon$ with an arbitrary $K < 0$ and $b - \varepsilon$ with an arbitrary $H > 0$, respectively.

$$\max_{(t,x)\in[t_0-\overline{\delta}_1(t_0),t_0+\overline{\delta}_1(t_0)]\times\overline{B}(x_0,\delta_2)}\|f(t,x)\|\le L\delta_2+L\|x_0\|+c.$$

We can then take $M=L\delta_2+L\|x_0\|+c$, which depends on x_0 and on ε (via c), but not on t_0. Hence, if we take

$$\delta_2=L\|x_0\|+c,$$

which is independent from t_0, we then get

$$\frac{\delta_2}{M}=\frac{1}{1+L},$$

which is definitely independent from $(t_0,x_0)\in[a+\varepsilon,b-\varepsilon]\times\mathbb{R}^n$ and from ε. In this way we finally get

$$0<\delta_1(t_0)\le\min\left\{t_0-a-\varepsilon,b-\varepsilon-t_0,\frac{1}{1+L}\right\},$$

depending on $t_0\in]a+\varepsilon,b-\varepsilon[$, such that, for every initial value $(t_0,x_0)\in]a+\varepsilon,b-\varepsilon[\times\mathbb{R}^n$, the unique local solution (uniqueness in the sense of Definition 2.8) exists in the interval $]t_0-\delta_1(t_0),t_0+\delta_1(t_0)[$. Hence, for $0<\varepsilon\le\frac{1}{1+L}$, and for all initial value $(t_0,x_0)\in]a+2\varepsilon,b-2\varepsilon[\times\mathbb{R}^n$ we can take constantly

$$\delta_1=\varepsilon.$$

This means that, for all possible initial value $(t_0,x_0)\in]a+2\varepsilon,b-2\varepsilon[\times\mathbb{R}^n$, the unique local solution exists, at least, in the whole interval $]t_0-\varepsilon,t_0+\varepsilon[$.

Now, let us definitely fix the initial value $(t_0,x_0)\in]a+2\varepsilon,b-2\varepsilon[\times\mathbb{R}^n$, and take

$$\begin{array}{l} t_1\in]t_0,t_0+\varepsilon[,\ t_2\in]t_1,t_1+\varepsilon[,\ \dots,\ t_k\in]t_{k-1},t_{k-1}+\varepsilon[,\ t_{k+1}\in]t_k,t_k+\varepsilon[\\ t^1\in]t_0-\varepsilon,t_0[,\ t^2\in]t^1-\varepsilon,t^1[,\ \dots,\ t^h\in]t^{h-1}-\varepsilon,t^{h-1}[,\ t^{h+1}\in]t^h-\varepsilon,t^h[\end{array}$$

It is evident that we can choose $k,h\in\mathbb{N}$ and $t_1,\dots,t_k,t_{k+1}$, $t^1,\dots,t^h,t^{h+1}$ such that $t_1,\dots,t_k\in]t_0,b-2\varepsilon[$, $t_{k+1}\ge b-2\varepsilon$, and that $t^1,\dots,t^h\in]a+2\varepsilon,t_0[$, $t^{h+1}\le a+2\varepsilon$.

Let us denote $I_i=]t_i-\varepsilon,t_i+\varepsilon[$ for all $i=0,\dots,k$, and $I^j=]t^j-\varepsilon,t^j+\varepsilon[$ for all $j=1,\dots,h$.

Hence we define the function $y:]a+2\varepsilon,b-2\varepsilon[\to\mathbb{R}^n$ as

$$y(t)=\begin{cases} y_0(t) & t\in I_0 \quad y_0 \text{ is the unique solution with datum } (t_0,x_0) \\ y_i(t) & t\in I_i \quad y_i \text{ is the unique solution with datum } (t_i,y_{i-1}(t_i)) \quad i=1,\dots,k \\ y^j(t) & t\in I_j \quad y^j \text{ is the unique solution with datum } (t^j,y^{j-1}(t^j)) \quad j=1,\dots,h. \end{cases}$$

By the local uniqueness on every intervals I_i and I^j, y is well defined and it turns out to be the unique global solution in $]a+2\varepsilon,b-2\varepsilon[$. By the arbitrariness of $\varepsilon>0$, we then conclude. □

Remark 2.16 *If in Theorem 2.15 we change the global uniform Lipschitz condition (2.11) in the weaker condition*

$$\begin{array}{l} \forall\ \delta > 0 \text{ sufficiently small } \exists\ L_\delta > 0 \text{ such that} \\ \|f(t,x) - f(t,y)\| \leq L_\delta \|x - y\|\ \forall\ (t,x),(t,y) \in]a+\delta, b-\delta[\times\mathbb{R}^n, \end{array} \tag{2.13}$$

then, the same thesis holds true. Indeed, it is sufficient to apply Theorem 2.15 to $]a+\delta, b-\delta[\times\mathbb{R}^n$ *for all* $\delta > 0$*, and then to construct a unique solution in* $]a,b[$*.*

Exercise. Show that the global existence and uniqueness defined by Definition 2.12 is equivalent to the following

i) a local solution exists and

ii) if $y_1 : I_1 \to \mathbb{R}^n$ and $y_2 : I_2 \to \mathbb{R}^n$ are two solutions on I_1 and I_2 respectively, then $y_1 = y_2$ on $I_1 \cap I_2$.

Exercise. Show, with a counterexample, that the following "natural" conditions do not imply the uniqueness of the local solution in the sense of Definition 2.8:

i) a local solution exists and

ii) if $y_1 : I_1 \to \mathbb{R}^n$ and $y_2 : I_2 \to \mathbb{R}^n$ are two solutions on I_1 and I_2 respectively, then there exists an open interval $J \subseteq I_1 \cap I_2$, containing t_0, such that $y_1 = y_2$ on J.

(*Hint.* Consider, for $t, x \geq 0$ the dynamics[39]

$$f(t,x) = \begin{cases} x^{\frac{2}{3}} & \text{if } 0 \leq x \leq t^4, \\ 0 & \text{if } x \geq t^4, \end{cases}$$

and the corresponding Cauchy problem with $y(0) = 0$.)

Exercise. Show, with a counterexample, that the uniqueness of a solution in an open interval I does not imply the uniqueness of the solution in every open subinterval $J \subseteq I$.

(*Hint.* For instance consider the function f such that, for $(t,x) \in [0,+\infty[\times[0,+\infty[$, is given by

$$f(t,x) = \begin{cases} x^{\frac{2}{3}} & \text{if } 0 \leq x \leq \sqrt{t}, \\ x^{\frac{4x+2t-6\sqrt{t}+2}{3(t+1-\sqrt{t})}} & \text{if } \sqrt{t} \leq x \leq t+1, \\ x^2 & \text{if } x \geq t+1. \end{cases}$$

Consider the Cauchy problem with initial datum $y(0) = 0$, and show that in $[0,+\infty[$ there is only the null solution $y \equiv 0$ but that on every proper subinterval $[0,k[$ there are infinitely many solutions.

Hint of the hint. Note that, apart from the zero solution, every solution starts as one of the solutions in the Subsection 2.3.4 and, outside of an "intermediate linking region", it goes on as the solution in Example 2.4, which does not exist for all times.)

[39] Such a dynamics is discontinuous, but for our purpose here is not a problem.

3 The linear case

3.1 Linear systems of first order equations

Let us consider a matrix-valued function and a vectorial-valued function

$$A : I \to \mathcal{M}_n,\ t \mapsto A(t); \qquad g : I \to \mathbb{R}^n$$

where $\mathcal{M}_n$ is the set of $n \times n$ real matrices, and $I \subseteq \mathbb{R}$ is in open interval.

A linear system of first order equations is a system of the following type

$$y'(t) = A(t)y(t) + g(t)\ t \in I, \tag{3.1}$$

which, if we write the vectors as $y(t) = (y_1(t), \dots, y_n(t))$, $g(t) = (g_1(t), \dots, g_n(t))$, and the matrix as $A(t) = (a_{ij}(t))_{i,j=1,\dots,n}$, is

$$\begin{cases} y_1'(t) = a_{11}(t)y_1(t) + \ldots + a_{1n}(t)y_n(t) + g_1(t) \\ \ldots \qquad \ldots \\ y_n'(t) = a_{n1}(t)y_1(t) + \ldots + a_{nn}(t)y_n(t) + g_n(t). \end{cases}$$

Here, $a_{ij} : I \to \mathbb{R}$ is the ij coefficient of A, and A is continuous (as matrix-valued function) if and only if a_{ij} is continuous (as real valued function) for every $i, j = 1, \dots, n$.

The system (3.1) is, of course, linear, nonhomogeneous and nonautonomous. It corresponds to the system $y' = f(t, y)$ with dynamics given by

$$f(t, x) = A(t)x + g(t) \ \ \forall\ (t, x) \in I \times \mathbb{R}^n. \tag{3.2}$$

We now consider the corresponding Cauchy problem

$$\begin{cases} y'(t) = A(t)y(t) + g(t) \ \ t \in I, \\ y(t_0) = x_0, \end{cases} \tag{3.3}$$

where $(t_0, x_0) \in I \times \mathbb{R}^n$ is fixed.

Theorem 3.1 *If A and g are continuous on I, then, for every $(t_0, x_0) \in I \times \mathbb{R}^n$, the linear Cauchy problem (3.3) is globally solvable in the sense of Definition 2.3, and it has a unique global solution in the sense of Definition 2.12.*

Proof. We apply Theorem 2.15 and Remark 2.16 to (3.3). By hypothesis, the dynamics f given in (3.2) is continuous. Hence, we have only to prove the uniform Lipschitz property in every compact subinterval $J \subseteq I$. Using the Cauchy-Schwarz inequality, for every $(t, x), (t, y) \in J \times \mathbb{R}^n$, we have

$$\|A(t)x + g(t) - (A(t)y + g(t))\|_{\mathbb{R}^n} = \|A(t)(x - y)\|_{\mathbb{R}^n} \le \|A(t)\|\|x - y\|_{\mathbb{R}^n},$$

where $\|A(t)\|$ is the matrix-norm of the matrix $A(t)$. Since A is continuous on I, then there exists $L_J > 0$ such that

$$\max_{t \in J} \|A(t)\| \le L_J,$$

and we conclude. $\square$

3.2 Homogeneous linear systems of first order equations

A homogeneous linear system of first order equations is of the form

$$y'(t) = A(t)y(t), \tag{3.4}$$

which differs from (3.1) only for $g \equiv 0$. We also have the corresponding Cauchy problem

$$\begin{cases} y'(t) = A(t)y(t), & t \in I, \\ y(t_0) = x_0. \end{cases} \tag{3.5}$$

Of course, if $A : I \to \mathcal{M}_n$ is continuous, then for (3.5) we have global solvability on I and uniqueness of the global solution. In particular, if $x_0 = 0 \in \mathbb{R}^n$, then the unique solution of (3.4) is the null function $y \equiv 0$.

3.2.1 The fundamental set of solutions and the Wronskian

Proposition 3.2 *Let $\mathcal{I}$ be the general integral of (3.4), i.e. the set of all solutions $y : I \to \mathbb{R}^n$. Then $\mathcal{I}$ is a vectorial space of dimension n. In particular, there exist n solutions $y_1, \dots, y_n$ which are linearly independent i.e.*

$$(c_1, \dots, c_n) \in \mathbb{R}^n, \; c_1 y_1(t) + \ldots + c_n y_n(t) = 0 \; \forall \; t \in I \implies (c_1, \dots, c_n) = (0, \dots, 0),$$

and such that, for any other solution $y : I \to \mathbb{R}^n$, there exist n constants $c_1, \dots, c_n \in \mathbb{R}$ such that

$$y(t) = c_1 y_1(t) + \ldots + c_n y_n(t) \quad \forall \; t \in I.$$

Such a set of solutions $y_1, \dots, y_n$ is called a fundamental set of solutions.

Proof. First of all, let us prove that $\mathcal{I}$ is a vectorial space. We have to prove that, for every two solutions y_1, y_2 and for every $\alpha_1, \alpha_2 \in \mathbb{R}$, the function $y = \alpha_1 y_1 + \alpha_2 y_2$ is also a solution[40]. This is immediate by linearity

$$\begin{aligned} & y'(t) = (\alpha_1 y_1 + \alpha_2 y_2)'(t) = \alpha_1 y_1'(t) + \alpha_2 y_2'(t) = \alpha_1 A(t) y_1(t) + \alpha_2 A(t) y_2(t) \\ & = A(t)(\alpha_1 y(t) + \alpha_2 y_2(t)) = A(t)y(t). \end{aligned}$$

Now, we prove that it has dimension n by exhibiting an isomorphism between $\mathbb{R}^n$ and $\mathcal{I}$. Let us fix $t_0 \in I$. We define the function

$$\psi : \mathbb{R}^n \to \mathcal{I}, \quad x \mapsto \psi(x) = y_x,$$

where y_x is the unique solution of the Cauchy problem (3.5) with initial datum $(t_0, x) \in I \times \mathbb{R}^n$. Linearity: by uniqueness we have

[40] The general integral is a subset of $C^1(I, \mathbb{R}^n)$ which is already a vectorial space. Hence, we have only to prove that $\mathcal{I}$ is closed with respect to the sum and to the multiplication by a scalar.

$$\alpha_1\psi(x_1) + \alpha_2\psi(x_2) = \alpha_1 y_{x_1} + \alpha_2 y_{x_2} = y_{\alpha_1 x_1 + \alpha_2 x_2} = \psi(\alpha_1 x_1 + \alpha_2 x_2),$$

since $(\alpha_1 y_{x_1} + \alpha_2 y_{x_2})(t_0) = \alpha_1 y_{x_1}(t_0) + \alpha_2 y_{x_2}(t_0) = \alpha_1 x_1 + \alpha_2 x_2$.
Injectivity:

$$\psi(x_1) = \psi(x_2) \implies y_{x_1} = y_{x_2} \implies y_{x_1}(t_0) = y_{x_2}(t_0) \implies x_1 = x_2.$$

Surjectivity:

$$\overline{y} \in \mathcal{I} \implies \overline{y} = \psi(\overline{y}(t_0)).$$

$\square$

Definition 3.3 *Given n functions $\varphi_1, \ldots, \varphi_n : I \to \mathbb{R}^n$ we define the* <u>*wronskian matrix*</u> *associated to them, as the time-dependent $n \times n$ matrix*

$$\mathbf{W}(t) = \begin{pmatrix} \varphi_{11}(t) & \varphi_{12}(t) & \ldots & \varphi_{1n}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) & \ldots & \varphi_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \ldots & \varphi_{nn}(t), \end{pmatrix} \tag{3.6}$$

where φ_{ij} is the i-th component of the function φ_j[41]*.*

The determinant

$$W(t) = \det \mathbf{W}(t) \tag{3.7}$$

is said the <u>*Wronskian*</u> *of the set of n functions.*

Proposition 3.4 *Given the n functions $\varphi_1, \ldots, \varphi_n : I \to \mathbb{R}^n$, if there exists $t_0 \in I$ such that*

$$W(t_0) \neq 0,$$

then the n functions are linearly independent.

Proof. We just observe that, for any $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ we have

$$c_1\varphi_1(t) + \ldots + c_n\varphi_n(t) = \mathbf{W}(t)c \ \forall\, t \in I.$$

Hence, let $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ be such that

$$c_1\varphi_1 + \ldots + c_n\varphi_n \equiv 0 \ \text{ in } I.$$

[41] In other words, the functions $\varphi_1, \ldots, \varphi_n$ are the columns of the matrix.

Hence, since by hypothesis $\mathbf{W}(t_0)$ is not singular, we have

$$\mathbf{W}(t_0)c = 0 \implies c = 0 \in \mathbb{R}^n.$$

$\square$

The condition $W(t_0) \neq 0$ is however not necessary for the linearly independence of the functions φ_i. Just consider the following example

$$n = 2,\ I =]-\frac{\pi}{2}, \frac{\pi}{2}[,\ \varphi_1(t) = (\sin t, \tan t),\ \varphi_2(t) = (\cos t, 1).$$

Hence, we have

$$W(t) = \det \begin{pmatrix} \sin t & \cos t \\ \tan t & 1 \end{pmatrix} = 0\ \forall t \in I,$$

but φ_1, φ_2 are linearly independent because

$$c_1 \sin t + c_2 \cos t = 0\ \forall\ t \in I \implies c_1 = c_2 = 0^{42}.$$

Theorem 3.5 *If the n functions, as in Proposition 3.4, are all solutions of the same homogeneous linear system of the form (3.4), then*

$$\exists\ t_0 \in I \text{ such that } W(t_0) \neq 0 \iff \varphi_1, \dots, \varphi_n \text{ are linearly independent.} \tag{3.8}$$

Moreover, in the case of linear independence, $W(t) \neq 0$ for all $t \in I$.

Proof. Let $\varphi_1, \dots, \varphi_n : I \to \mathbb{R}^n$ be solutions of (3.4) linearly independent, that is a fundamental system of solution. Let us prove that there exists $t_0 \in I$ such that $W(t_0) \neq 0$. Any other solution is of the form

$$\varphi(t) = \mathbf{W}(t)c,$$

with any $c \in \mathbb{R}^n$. If, by absurd, $W(t) = 0$ for all $t \in I$, then, fixed any $\tilde{t} \in I$, there exists $\tilde{c} \neq 0 \in \mathbb{R}^n$ such that $\mathbf{W}(\tilde{t})\tilde{c} = 0$. This implies that the solution $\tilde{\varphi} = \tilde{c}_1\varphi_1 + \ldots + \tilde{c}_n\varphi_n$ is zero in $\tilde{t}$. But, by the uniqueness result for the Cauchy problem, the unique solution of (3.4) which passes through zero at $\tilde{t}$ is the null function $\varphi \equiv 0$. By the linear independence of $\varphi_1, \dots, \varphi_n$ we then get $\tilde{c} = 0$, which is a contradiction. By the arbitrariness of $\tilde{t} \in I$, we have that, if $\varphi_1, \dots, \varphi_n$ is a fundamental set of solutions, $W(t) \neq 0$ for all $t \in I$. $\square$

Proposition 3.6 *If $\varphi_1, \dots, \varphi_n : I \to \mathbb{R}^n$ are solutions of (3.4), then the wronskian matrix satisfies the equation*

$$\mathbf{W}'(t) = A(t)\mathbf{W}(t) \quad \forall t \in I. \tag{3.9}$$

[42] Just check for $t = 0$ and for $t = \frac{\pi}{4}$.

Proof. Just a calculation. □

Remark 3.7 *Summarizing, we have found that:*

i) the general integral of the homogeneous linear system (3.4) is a vectorial space of dimension n;

ii) given a fundamental set of solutions (i.e. a basis) $y_1, \dots, y_n$, any other solution is, obviously, written in the form (which is called <u>*the general solution*</u>*)*

$$y(\cdot) = c_1 y_1(\cdot) + \ldots + c_n y_n(\cdot) = \mathbf{W}(\cdot)c, \quad c = (c_1, \dots, c_n) \in \mathbb{R}^n,$$

where $\mathbf{W}$ is the wronskian matrix of the fundamental set of solutions;

iii) The wronskian matrix of n solutions is either everywhere singular (i.e. $W(t) = 0$ for all t) or everywhere nonsingular (i.e. $W(t) \neq 0$ for all t). In particular it is everywhere nonsingular if and only if the n solutions are linearly independent (i.e. they are a fundamental set of solutions).

3.3 Homogeneous systems of first order linear equations with constant coefficients

In this subsection we suppose that the $n \times n$ matrix $A(\cdot)$ in (3.4) is constant: $A(\cdot) \equiv A \in \mathcal{M}_n$. That is the system has constant coefficients (i.e. it is autonomous). In this case, we can consider the system and the corresponding Cauchy problem as defined in the whole $\mathbb{R}$:

$$y'(t) = Ay(t) \quad t \in \mathbb{R}, \tag{3.10}$$

and

$$\begin{cases} y' = Ay \ \text{ in } \mathbb{R}, \\ y(t_0) = x_0 \end{cases} \tag{3.11}$$

with $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ fixed.

Proposition 3.8 *Every solution of (3.10) is defined in the whole $\mathbb{R}$. In particular, the Cauchy problem (3.11) is globally solvable in $\mathbb{R}$ and it has a unique global solution for every initial datum $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Moreover, such a unique solution is given by*

$$y(t) = e^{(t-t_0)A} x_0, \quad t \in \mathbb{R}, \tag{3.12}$$

and a fundamental set of solutions for the system is for instance given by the columns of the exponential matrix e^{tA}, that is the exponential matrix is the Wronskian matrix.

Proof. All the theses are immediate consequences of the results of the previous subsection and of the definition of exponential matrix and its differentiability properties. □

Remark 3.9 *Note that $t \mapsto (t-t_0)A$ is a primitive of the constant matrix valued function $t \mapsto A$, and hence (3.12) is coherent with the one-dimensional solution $y(t) = ke^{C(t)}$ as in (1.14). One may be then induced to think that a solution for the general non-constant homogeneous system is of the form*

$$y(t) = e^{\mathcal{A}(t)}x_0,$$

where $\mathcal{A}(\cdot)$ is a primitive of the matrix valued function $t \mapsto A(t)$. This is unfortunately not true since it is not true that

$$\frac{d}{dt}e^{\mathcal{A}(t)} = A(t)e^{\mathcal{A}(t)},$$

and the main problem for this fact is that in general $A(t)$ and $\mathcal{A}(t)$ do not commute.

Example 3.10 Find the solution of the Cauchy problem

$$\begin{cases} y_1'(t) = 2y_1(t) - y_2(t) + y_3(t) \\ y_2'(t) = y_1(t) + y_3(t) \\ y_3'(t) = y_1(t) - y_2(t) + 2y_3(t) \\ y_1(0) = 0,\ y_2(0) = 0,\ y_3(0) = 1. \end{cases}$$

This is the Cauchy problem for a linear system of three equations with constant coefficients. It is given by the 3×3 matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

and the initial datum is $(t_0, x_0) = (0, (0, 0, 1))$. The eigenvalues of A are 1 and 2, and A is diagonalizable since there exists a basis of $\mathbb{R}^n$ given by eigenvectors: $\{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$ (where the first two are corresponding to the eigenvalue 1). Hence, the diagonal matrix D, the passage matrix B and its inverse are

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Hence we get, for every $t \in \mathbb{R}$,

$$e^{tA} = Be^{tD}B^{-1} = \begin{pmatrix} e^{2t} & e^t - e^{2t} & -e^t + e^{2t} \\ -e^t + e^{2t} & 2e^t - e^{2t} & -e^t + e^{2t} \\ -e^t + e^{2t} & e^t - e^{2t} & e^{2t} \end{pmatrix}.$$

The solution is then (written as column)

$$y(t) = e^{tA}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -e^t + e^{2t} \\ -e^t + e^{2t} \\ e^{2t} \end{pmatrix}.$$

3.3.1 The homogeneous linear equation of n-th order with constant coefficients

We consider the following homogeneous linear equation of order n with constant coefficients (i.e. autonomous):

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \quad \text{in } \mathbb{R}. \tag{3.13}$$

We already know that it is equivalent to the linear system of n equation with constant coefficients

$$Y' = AY, \tag{3.14}$$

where $Y = (Y_0, Y_1, \ldots, Y_{n-1}) = (y, y', \ldots, y^{(n-1)}) \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & \ldots & \ldots & \ldots & -a_{n-2} & -a_{n-1}. \end{pmatrix}$$

For the linear system associated to the matrix A, we know what is a Cauchy problem. What does it correspond to, for the n order equation (3.13)? By our interpretation, fixing the values of $Y_0(t_0), \ldots, Y_{n-1}(t_0)$ corresponds to fixing the values of y and of its first $n-1$ derivatives: $y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0)$. Hence, the Cauchy problem for (3.13) is given by

$$\begin{cases} y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_1 y'(t) + a_0 y(t) = 0 \quad \text{in } \mathbb{R}, \\ y(t_0) = x_0, \\ y'(t_0) = x_1, \\ \ldots \quad \ldots \\ y^{(n-1)}(t_0) = x_{n-1}, \end{cases} \tag{3.15}$$

where the initial values $(t_0, x_0), (t_0, x_1), \ldots, (t_0, x_{n-1}) \in \mathbb{R} \times \mathbb{R}$ are fixed.

The proof of the following proposition is now immediate[43].

Proposition 3.11 *All the solutions of the equation (3.13) are defined on the whole $\mathbb{R}$. The general integral is a vectorial space of dimension n. For all choices of the initial values, the Cauchy problem (3.15) is globally solvable in $\mathbb{R}$ and it has a unique global solution (that is there exists a unique function $y \in C^n(\mathbb{R};\mathbb{R})$ which satisfies the equation and which, at a fixed instant $t_0 \in \mathbb{R}$, has preassigned values for the derivatives from zero order[44] up to the order $n-1$).*

[43] By the interpretation as a linear system, a similar proposition of course holds for a homogeneous linear equation of n order with non-constant coefficients. But here, we concentrate only on the case of constant coefficients

[44] The derivative of zero order is the function itself.

Also for the singular scalar equation of n order, for a set of n solutions, we can consider the wronskian matrix and the Wronskian. Still considering the interpretation as linear system, we immediately get the following proposition.

Proposition 3.12 *Given n solutions of (3.13), $\varphi_1, \ldots, \varphi_n : \mathbb{R} \to \mathbb{R}$, we define the wronskian matrix*

$$\mathbf{W}(t) = \begin{pmatrix} \varphi_1(t) & \ldots & \varphi_n(t) \\ \varphi_1'(t) & \ldots & \varphi_n'(t) \\ \ldots & \ldots & \ldots \\ \varphi_1^{(n-1)}(t) & \ldots & \varphi_n^{(n-1)}(t) \end{pmatrix}, \tag{3.16}$$

and the Wronskian

$$W(t) = \det \mathbf{W}(t).$$

Then, $W(t) \neq 0$ for all $t \in \mathbb{R}$ if and only if the n functions are a fundamental set of solutions (i.e. they are linearly independent). On the contrary (i.e. if the n functions are not linearly independent), $W(t) = 0$ for all $t \in \mathbb{R}$.

If $\varphi_1, \ldots, \varphi_n$ is a fundamental set of solutions, then the general solution is

$$\varphi(\cdot) = c_1\varphi_1(\cdot) + \ldots + c_n\varphi_n(\cdot) = (\mathbf{W}(\cdot)c)_1,$$

where $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ and $(\cdot)_1$ means the first component of the vector.

Now, we want to search for a fundamental system of solutions for the equation (3.13). Inspired by the solutions for the linear system as expressed in (3.12), we look for solutions of the form

$$y(t) = e^{\lambda t},$$

where $\lambda \in \mathbb{R}$ is a fixed parameter. Just inserting such a function in the equation, and imposing the equality to zero, we get

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0, \tag{3.17}$$

which is called the characteristic equation of (3.13). We have the following proposition.

Proposition 3.13 *Given $\lambda \in \mathbb{R}$, the function $y(t) = e^{\lambda t}$ is a solution of (3.13) if and only if λ is a solution of the characteristic equation (3.17).*

Moreover, let $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$, $r \leq n$, be all the distinct (complex) solutions of (3.17), together with their multiplicity $n_1, \ldots, n_r$, $\sum_{i=1}^r n_i = n$. Then, we can construct a fundamental system of solutions, by associating to any solutions λ_i a set of functions in the following way[45]*:*

[45] Here, $\overline{z}$ means the conjugate $\alpha - i\beta$ of $z = \alpha + i\beta \in \mathbb{C}$. Moreover, recall that, being the characteristic equation and algebraic equation with real coefficients, then if $\lambda \in \mathbb{C}$ is solution, $\overline{\lambda}$ is solution too.

$$
\begin{aligned}
&1)\lambda_i \in \mathbb{R} \implies \left\{e^{\lambda_i t}, te^{\lambda_i t}, \dots, t^{n_i-1}e^{\lambda_i t}\right\};\\
&2)\lambda_i = \overline{\lambda}_j = \alpha + i\beta \implies \left\{e^{\alpha t}\cos(\beta t), te^{\alpha t}\cos(\beta t), \dots, t^{n_i-1}e^{\alpha t}\cos(\beta t),\right.\\
&\qquad\qquad \left. e^{\alpha t}\sin(\beta t), te^{\alpha t}\sin(\beta t), \dots, t^{n_i-1}e^{\alpha t}\sin(\beta t)\right\}.
\end{aligned}
\tag{3.18}
$$

Before proving Proposition 3.13, we need the following lemma.

Lemma 3.14 *Given a linear homogeneous n order differential equation with constant coefficients as (3.13), we can consider the linear differential operator of n order*

$$
D : C^n(\mathbb{R};\mathbb{R}) \to C(\mathbb{R},\mathbb{R}), \qquad y \mapsto D(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y, \tag{3.19}
$$

that is

$$
D = \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \dots + a_1\frac{d}{dt} + a_0(id),
$$

where $id : C(\mathbb{R},\mathbb{R}) \to C(\mathbb{R};\mathbb{R})$ is the identity map. Moreover we can consider the corresponding polynomial

$$
p(\lambda) = \lambda^n + a_{n-1}y^{n-1} + \dots + a_1\lambda + a_0,
$$

which is called the characteristic polynomial of the differential equation. We write the characteristic polynomial p as decomposed in its irreducible (on $\mathbb{R}$) factors[46]

$$
p(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_r)^{n_r}(\lambda^2 + a_1\lambda + b_1)^{m_1} \cdots (\lambda^2 + a_s\lambda + b_s)^{m_s}
$$

Then we can decompose the differential operator D in its irreducible (on $\mathbb{R}$) factors of first and second order

$$
\begin{aligned}
D = &\left(\frac{d}{dt} - \lambda_1\right)^{n_1} \cdots \left(\frac{d}{dt} - \lambda_r\right)^{n_r}\\
&\cdot\left(\frac{d^2}{dt^2} + a_1\frac{d}{dt} + b_1\right)^{m_1} \cdots \left(\frac{d^2}{dt^2} + a_s\frac{d}{dt} + b_s\right)^{m_s},
\end{aligned}
\tag{3.20}
$$

where, for every $\lambda \in \mathbb{R}$, we have

$$
\begin{aligned}
&D_\lambda := \frac{d}{dt} - \lambda := \frac{d}{dt} - \lambda(id) : C^1(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R}),\\
&y \mapsto D_\lambda(y) = y' - \lambda y,
\end{aligned}
$$

[46]This means that $\lambda_i \in \mathbb{R}$, $1 \le i \le r \le n$, are all the real roots of p with their multiplicity $n_1, \dots, n_r$, and $\lambda^2 + a_j\lambda + b_j$, $a_j, b_j \in \mathbb{R}$, $1 \le j \le s \le n$, are all the second-degree irreducible factors, which means that they have two conjugate complex roots $\alpha_j + i\beta_j$, $\alpha_j - i\beta_j$, both with multiplicity m_j, where $\alpha_j = -(a_j/2)$, $\beta_j = \sqrt{b_j - (a_j^2/4)}$. Moreover, $n_1 + \dots + n_r + 2m_1 + \dots + 2m_s = n$.

for every $a, b \in \mathbb{R}$ we have

$$D^2_{a,b} := \frac{d^2}{dt^2} + a\frac{d}{dt} + b := \frac{d^2}{dt^2} + a\frac{d}{dt} + b(id) : C^2(\mathbb{R};\mathbb{R}) \to C(\mathbb{R},\mathbb{R}),$$
$$y \mapsto D^2_{a,b}(y) = y'' + ay' + by$$

and finally where multiplication means composition as operators. For instance

$$(D_\lambda D_\mu)(y) = \left(\frac{d}{dt} - \lambda\right)\left(\frac{d}{dt} - \mu\right)(y) = \left(\frac{d}{dt} - \lambda\right)(y' - \mu y)$$
$$= y'' - (\lambda + \mu)y' + \lambda\mu y = \left(\frac{d^2}{dt^2} - (\lambda+\mu)\frac{d}{dt} + \lambda\mu id\right)(y)$$

Proof of Lemma 3.14. Just a calculation. □

Proof of Proposition 3.13. The first assertion is obvious. Let us prove that the functions in (3.18) are all solution of (3.13). First of all note that in the decomposition (3.20), every factor may commute with the other ones. This is true since it is already true for the decomposition of the characteristic polynomial. Moreover, a function y is a solution if and only if $D(y) = 0$[47]. Let $\lambda_i \in \mathbb{R}$ be a root with multiplicity n_i, and let us consider the function $y(t) = t^\ell e^{\lambda_i t}$ with $0 \le \ell < n_i$ fixed. Then, among the factors of D, there is the operator

$$(D_{\lambda_i})^{\ell+1} = \left(\frac{d}{dt} - \lambda_i\right)^{\ell+1}.$$

A simple calculation shows that

$$(D_{\lambda_i})^{\ell+1}(y) = 0,$$

and then we conclude that $D(y) = 0$, that is y is a solution. On the other hand, for any irreducible factors $D^2_{a_j,b_j}$, we consider the following two functions

$$z(t) = t^\ell e^{-\frac{a_j}{2}t}\cos\left(\sqrt{b_j - \frac{a_j^2}{4}}t\right), \quad w(t) = t^\ell e^{-\frac{a_j}{2}t}\sin\left(\sqrt{b_j - \frac{a_j^2}{4}}t\right),$$

with $0 \le \ell < m_j$ fixed. Again, we have the factor $\left(D^2_{a_j,b_j}\right)^{\ell+1}$, for which another simple (but tedious) calculation shows that

$$\left(D^2_{a_j,b_j}\right)^{\ell+1}(z) = \left(D^2_{a_j,b_j}\right)^{\ell+1}(w) = 0,$$

and then we conclude (see also the footnote in Lemma 3.14).

[47]The general integral is exactly the kernel of the linear operator D.

Now, we have to prove that they are linearly independent (which is sufficient for proving that they are a fundamental set of solutions since they are exactly n functions). First of all let us prove that every block of solutions, i.e. solutions referring to the same irreducible factor, are linearly independent. Indeed, let λ_i be a real root with multiplicity n_i. Then the n_i functions

$$y_\ell(t) = t^\ell e^{\lambda_i t}, \quad 0 \le \ell < n_i$$

generate the wronskian matrix $n_i \times n_i$

$$\mathbf{W}_i(t) = \begin{pmatrix} y_1(t) & \cdots & y_{n_i}(t) \\ y_1'(t) & \cdots & y_{n_i}'(t) \\ \cdots & \cdots & \cdots \\ y_1^{(n_i-1)}(t) & \cdots & y_{n_i}^{(n_i-1)}(t) \end{pmatrix},$$

and one can easily see that $\mathbf{W}_i(0)$ is a triangular matrix with non-zero coefficients on the principal diagonal. Hence $W_i(0) \neq 0$ and the functions are linearly independent[48]. Now, for any irreducible factor, indexed by $1 \le j \le s$, we have to prove the linear independence of the functions

$$z_\ell(t) = t^\ell e^{\alpha_j t} \cos(\beta_j t), \quad 0 \le \ell < m_j,$$

as well as of the functions

$$w_\ell(t) = t^\ell e^{\alpha_j t} \sin(\beta_j t), \quad 0 \le \ell < m_j,$$

where α and β are as in the footnote in Lemma 3.14. The idea is to separately prove the linear independence of the functions $z_\ell + w_\ell$ and of the functions $z_\ell - w_\ell$. Just sketching the proof, let us consider the corresponding $m_j \times m_j$ wronskian matrices, a similar analysis as before, leads to the desired conclusion. Moreover, by the linear independence of cos and sin, we also get the linear independence of the whole set of functions $z_1, \ldots, z_{m_j}, w_1, \ldots, w_{m_j}$.

Now, for any $1 \le i \le r$ we have the set $\mathcal{I}_i$ which is the general integral of the linear equation of n_i order $(D_{\lambda_i})^{n_i}(y) = 0$, and for every $1 \le j \le s$ we consider the set $\mathcal{I}^j$, which is the general integral of the $2m_j$ order differential equation $\left(D^2_{a_j,b_j}\right)^{m_j}(y) = 0$. A basis of $\mathcal{I}_i$ is

$$\left\{e^{\lambda_i t}, te^{\lambda_i t}, \ldots, t^{n_i-1}e^{\lambda_i t}\right\},$$

whereas a basis of $\mathcal{I}^j$ is

$$\left\{e^{\alpha t}\cos(\beta t), te^{\alpha t}\cos(\beta t), \ldots, t^{m_j-1}e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t), te^{\alpha t}\sin(\beta t), \ldots, t^{m_j-1}e^{\alpha t}\sin(\beta t)\right\}.$$

Hence, denoted by $\mathcal{I}$ the general integral of (3.13), we have

[48]Since they are n_i functions which are solutions of the linear homogeneous equation of n_i order $D^{n_i}_{\lambda_1} y = 0$.

$$\mathcal{I}_1 + \cdots + \mathcal{I}_r + \mathcal{I}^1 + \cdots + \mathcal{I}^s \subseteq \mathcal{I}.$$

If the left hand-side is a direct sum, then we are done because, in such a case, collecting the bases of the subspaces we get a basis of the whole space. We see that it is really a direct sum since every non-null element of an addendum is not an element of another different addendum since it does not solve the corresponding differential equation. $\square$

Remark 3.15 *For searching a fundamental set of solutions, it is obviously more convenient to look for the (complex) roots of the characteristic polynomial, instead of looking for its irreducible factors, and then to apply the rule given in Proposition 3.13.*

Actually, all the theory of linear n order differential equations with constant real coefficients can be made for solutions $t \mapsto y(t)$ taking value in $\mathbb{C}$, and then all the solutions are of the exponential form $t \mapsto t^\ell e^{\lambda t}$, with λ (complex) root. In particular, if $\lambda = \alpha + i\beta$ is really complex (non-zero imaginary part), then, together with its conjugate, it gives the (complex) solutions

$$t^\ell e^{\lambda t} = t^\ell e^{\alpha t}(\cos(\beta t) + i \sin(\beta t)), \quad t^\ell e^{\overline{\lambda} t} = t^\ell e^{\alpha t}(\cos(\beta t) - i \sin(\beta t)),$$

which, combined, give the two real solutions $t^\ell e^{\alpha t}\cos(\beta t)$ and $t^\ell e^{\alpha t}\sin(\beta t)$

Example 3.16 Given the scalar linear homogeneous equation

$$y^{iv} - 2y''' + 2y'' - 2y' + y = 0,$$

i) find the general integral,
ii) solve the Cauchy problem with data $y(\pi) = -3, y'(\pi) = 0, y''(\pi) = 1, y'''(\pi) = 0$,
iii) find all solutions y such that $y(0) = 0$.
i) The characteristic equation is

$$\lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1 = 0,$$

whose roots are $\lambda_1 = 1$ (with multiplictity 2), $\lambda_2 = i$, $\lambda_3 = -i$. Hence, the general integral is given by the functions of the form

$$y(t) = c_1 e^t + c_2 t e^t + c_3 \cos t + c_4 \sin t, \quad c_1, \ldots, c_4 \in \mathbb{R}. \tag{3.21}$$

ii) We have to impose the initial data to the general form. First we have to compute the general form of the derivatives:

$$\begin{aligned}
y'(t) &= (c_1 + c_2)e^t + c_2 t e^t - c_3 \sin t + c_4 \cos t, \\
y''(t) &= (c_1 + 2c_2)e^t + c_2 t e^t - c_3 \cos t - c_4 \sin t, \\
y'''(t) &= (c_1 + 3c_2)e^t + c_2 t e^t + c_3 \sin t - c_4 \cos t,
\end{aligned}$$

from which we get

$$\begin{cases} c_1 e^{\pi} + c_2 \pi e^{\pi} - c_3 = -3 \\ (c_1 + c_2) e^{\pi} + c_2 \pi e^{\pi} - c_4 = 0 \\ (c_1 + 2c_2) e^{\pi} + c_2 \pi e^{\pi} + c_3 = 1 \\ (c_1 + 3c_2) e^{\pi} + c_2 \pi e^{\pi} + c_4 = 0, \end{cases}$$

whose solution is $(c_1, c_2, c_3, c_4) = (-(2+\pi)e^{-\pi}, e^{-\pi}, 1, -1)$. We finally have the solution of the Cauchy problem

$$y(t) = -(2+\pi)e^{t-\pi} + te^{t-\pi} + \cos t - \sin t.$$

iii) Using the general form (3.21), we immeditely get the necessary and sufficient condition $c_1 = -c_3$. Hence the requested solutions are all the functions of the form

$$y(t) = ae^t + bte^t - a\cos t + c\sin t, \quad a, b, c \in \mathbb{R}.$$

3.4 Nonhomogeneous systems of first order linear equations

We consider the nonhomogeneous linear system as in (3.1). In this case, if g is not the null function, the general integral is not a vectorial space anymore. To see this, just take a solution y and consider the function $\tilde{y} = 2y$. Then $\tilde{y}$ is solution of

$$\tilde{y}' = A\tilde{y} + 2g,$$

which is different from the originary system. However, the general integral is not so different from a vectorial space. Indeed it is a so-called affine space, that is the shift of a vectorial space[49]. The homogeneous linear system (3.4) (with the same matrix $A(t)$) is said <u>the associated homogeneous system</u> of the nonhomogeneous system (3.1).

Proposition 3.17 *Let us denote by $\overline{\mathcal{I}}$ and by $\mathcal{I}$ the general integral of the nonhomogeneous system and the general integral of the associated homogeneous system, respectively. Moreover, let $\overline{y}$ be a solution of the nonhomogeneous (called a <u>particular solution</u>[50]). Then*

$$\overline{\mathcal{I}} = \overline{y} + \mathcal{I}. \tag{3.22}$$

Proof. By the linearity, if $\tilde{y} \in \overline{\mathcal{I}}$, then $\tilde{y} - \overline{y} \in \mathcal{I}$. Hence we have $\tilde{y} = \overline{y} + (\tilde{y} - \overline{y}) \in \overline{y} + \mathcal{I}$. On the contrary, if $w \in \mathcal{I}$, then, again by linearity, $\overline{y} + w \in \overline{\mathcal{I}}$. □

[49] Exactly as it happens for the algebraic linear systems.

[50] Here, "particular" means that it is a specific solution, not written in "general" form. Of course, any (specific) solution is suitable.

Remark 3.18 *Proposition 3.17 says that the general integral of the nonhomogeneous linear system is the general integral of the associated homogeneous system (which is a vectorial space) shifted by a vector given by any particular solution of the nonhomogeneous one. Hence, if we denote by $\overline{y}$ such a particular solution, and if we have a fundamental set of solution for the associated homogeneous system, let us say $y_1, \ldots, y_n$, then the general solution of the nonhomogeneous is*

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + \overline{y}(t), \quad (c_1, \ldots, c_n) \in \mathbb{R}^n. \tag{3.23}$$

3.4.1 The constants variation method

As pointed out by Remark 3.18, if we know a particular solution of the nonhomogeneous system and a fundamental set of the associated homogeneous system, then we immediately have the general form of the solutions of the nonhomogeneous. The problem here is then given by the calculation of a particular solution and of a fundamental set. In general these are two hard problems. However, if we already know a fundamental system of the homogeneous, then the calculation of a particular solution of the nonhomogeneous is quite easy. The method is called the constants variation method, and it is based on the following argument. Let $y_1, \ldots, y_n$ be a (known) fundamental set for the homogeneous. Then, for every n constants $c_1, \ldots, c_n$, the function $c_1 y_1 + \cdots + c_n y_n$ is still a solution of the homogeneous. Then the idea is to look for a solution of the nonhomogeneous in the following form

$$\psi(t) = c_1(t) y_1(t) + \cdots + c_n(t) y_n(t) = \mathbf{W}(\mathbf{t}) c(t), \tag{3.24}$$

where $c_1(\cdot), \ldots, c_n(\cdot)$ are derivable functions, $c(t) = (c_1(t), \ldots, c_n(t))$ and $\mathbf{W}$ is the wronskian matrix of the fundamental set of solutions $y_1, \ldots, y_n$. Hence, the idea is: "let the constants be not constant, but variate on time".

The problem is now to find suitable n functions $c_1, \ldots, c_n$. By imposing that the function ψ defined in (3.24) is a solution of the nonhomogeneous system (3.1), recalling the equation (3.9) satisfied by the wronskian matrix, and also recalling its nonsingularity for every time, we have

$$\begin{aligned} &\psi \text{ is a solution } \iff \mathbf{W}'(t)c(t) + \mathbf{W}(t)c'(t) = A(t)\mathbf{W}(t)c(t) + g(t) \\ &\iff \mathbf{W}(t)c'(t) = g(t) \iff c'(t) = \mathbf{W}^{-1}(t)g(t) \end{aligned}$$

Hence, the problem is to calculate the vectorial integration (i.e. component by component)

$$c(t) = \int_\tau^t \mathbf{W}^{-1}(s) g(s) ds, \tag{3.25}$$

where τ is any instant in the time interval I (whose choice corresponds to the choice of the integration constant), and a particular solution is then of the form

$$\psi(t) = \mathbf{W}(t) \int_\tau^t \mathbf{W}^{-1}(s) g(s) ds.$$

The main difficulties in applying the constant variation method are of computational type: computing the inverse wronskian matrix and the integral in (3.25). Of course, we a-priori need to know a fundamental set of the associated homogeneous. This is another kind of problem, which is in general a hard question. In the next subsection we analyze a favorable case where we know how to compute a fundamental set of solutions: the nonhomogeneous linear equation with constant coefficients[51].

3.4.2 The nonhomogeneous linear system with constant coefficients

Let us suppose that the matrix A of system (3.1) has constant coefficients, that is the system is autonomous. Then, by the results of the previous section, we have an explicit formula for the solution of the Cauchy problem with datum (t_0, x_0), which involves the exponential matrix[52], and whose validity may also directly checked:

$$y(t) = e^{(t-t_0)A} x_0 + \int_{t_0}^t e^{(t-s)A} g(s) ds \tag{3.26}$$

Example 3.19 Let us consider the linear homogeneous system of Example 3.10, where we also insert the known term

$$g(t) = \begin{pmatrix} e^s \\ 0 \\ 0 \end{pmatrix},$$

that is we consider the Cauchy problem for the linear nonhomogeneous system

$$\begin{cases} y'(t) = Ay(t) + g(t) \\ y(0) = (0, 0, 1) \end{cases}$$

where A is the same as in Example 3.10. Hence, just applying (3.26), and recalling the exponential matrix and the solution as in Example 3.10, we get

$$y(t) = \begin{pmatrix} -e^t + e^{2t} \\ -e^t + e^{2t} \\ e^{2t} \end{pmatrix} + \int_0^t \begin{pmatrix} e^{2t-s} \\ -e^t + e^{2t-s} \\ -e^t + e^{2t-s} \end{pmatrix} ds = \begin{pmatrix} -2e^t + 2e^{2t} \\ -te^t - 2e^t + 2e^{2t} \\ -te^t - e^t + 2e^{2t} \end{pmatrix}.$$

[51] Of course, the constants variation methods works as well for the nonhomogeneous linear equation with non-constant coefficients. But in that case, the computation of a fundamental set of solutions is harder.

[52] Note that the column of the exponential matrix e^{tA} are a fundamental set of solutions and hence the exponential matrix is its Wronskian matrix. Indeed they are linearly independent since e^{tA} is nonsingular and also they are solution (the i-th column is the solution of the Cauchy problem with datum $y(0) = e_i$ where e_i is the i-th vector of the canonical basis of $\mathbb{R}^n$). Also note that e^{-tA} is the inverse of e^{tA}.

3.4.3 The nonhomogeneous linear equation of n-order with constants coefficients. A special case: the annihilators method

Let us consider the nonhomogeneous linear equation of n order with constant coefficients

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = g(t), \quad t \in \mathbb{R}, \tag{3.27}$$

where $a_0, \dots, a_{n-1} \in \mathbb{R}$ are constants, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. In this case, the associated homogeneous equation is given in (3.13). Using the usual interpretation of the equation as a linear system, we have that (3.27) is equivalent to the system

$$Y'(t) = AY(t) + G(t),$$

where the vector $Y(t) \in \mathbb{R}^n$ and the matrix $A \in \mathcal{M}_n$ are defined as in (3.14) and $G(t)$ is the vector $(0, \dots, 0, g(t)) \in \mathbb{R}^n$. Since a particular solution $\psi : \mathbb{R} \to \mathbb{R}$ of the equation is the first component of a particular solution of the associated system, we have (now we take $\tau = 0$)

$$\psi(t) = \left(\mathbf{W}(t) \int_0^t \mathbf{W}^{-1}(s) G(s) ds\right)_1 = \sum_{i=1}^n y_i(t) \int_0^t \left(\mathbf{W}^{-1}(s)\right)_{in} g(s) ds, \tag{3.28}$$

where $\mathbf{W}$ is the wronskian matrix of the fundamental set of solutions, $y_1, \dots, y_n$, of the associated homogeneous linear equation.

Example 3.20 We want to find the general integral of the equation

$$y'' - y = \frac{2}{1 + e^x}. \tag{3.29}$$

The associated homogeneous equation has the fundamental set of solutions $\{e^x, e^{-x}\}$, whose wronskian matrix and its inverse are

$$\mathbf{W}(x) = \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix}, \quad \mathbf{W}(x)^{-1} = -\frac{1}{2} \begin{pmatrix} -e^{-x} & -e^{-x} \\ -e^x & e^x \end{pmatrix}.$$

Applying formula (3.28) for the calculus of a particular solution, we then get the general integral[53]

$$y(t) = c_1 e^x + c_2 e^{-x} - 1 - xe^x + (e^x - e^{-x}) \log(1 + e^x).$$

(*The annihilators method).* Here we treat a favorable case where, for a nonhomogeneous linear equation with constant coefficients, we are able to easily calculate the general solution.

Let us consider the equation (3.27), and suppose that there exists a linear differential operator with constant coefficients D (see (3.19)) such that

[53]For the calculus of the integrals, note that $(e^{-s} - e^s)/(1 + e^s) = e^{-s} - 1$.

$$D(g)(t) = 0 \ \forall t \in \mathbb{R}.$$

We call such an operator an annihilator for g. Moreover, let $\overline{D}$ be the linear differential operator with constant coefficients corresponding to the homogeneous equation associated to (3.27). It is obvious that, if y is a solution of (3.27), then y is also a solution of the linear homogeneous equation with constant coefficients

$$D\overline{D}(y) = 0. \tag{3.30}$$

Once we have calculated the general integral of (3.30), then we can easily get the general solution of (3.27), just imposing to the general solution of (3.30) to also solve (3.27), and hence fixing some of the free parameters.

Of course, in general, a function g is not annihilated by any linear operator with constant coefficients, take for instance $g(t) = \log(t)$ or the right-hand side of (3.29), with x as t. However, here there is a list of (common) functions which are annihilated by linear operators with constant coefficients, and the annihilators are also reported[54]. Also note that, if f and g are annihilated by D_1 and D_2 respectively, then f, g and $f+g$ are annihilated by D_1D_2, and moreover μf is still annihilated by D_1, for every $\mu \in \mathbb{R}$. Hence, given the following list of functions with their annihilators, we can easily construct the annihilators for every linear combination of the functions. In the following, $m \in \mathbb{N}, \lambda, \alpha, \beta \in \mathbb{R}$ are arbitrary:

function	annihilator
$f(t) = t^m e^{\lambda t}$	$D = \left(\frac{d}{dt} - \lambda I\right)^{m+1}$,
$f(t) = t^m e^{\alpha t}\cos\beta t$	$D = \left(\frac{d^2}{dt^2} - 2\alpha\frac{d}{dt} + (\alpha^2+\beta^2)id\right)^{m+1}$,
$f(t) = t^m e^{\alpha t}\sin\beta t$	$D = \left(\frac{d^2}{dt^2} - 2\alpha\frac{d}{dt} + (\alpha^2+\beta^2)id\right)^{m+1}$.

Example 3.21 We want to find the general integral of the equation

$$y^{iv}(t) - 2y'''(t) + 2y''(t) - 2y'(t) + y(t) = t + t^2. \tag{3.31}$$

The associated homogeneous equation is the one studied in Example 3.16, which corresponds to the linear operator

$$\overline{D} = \left(\frac{d}{dt} - I\right)^2 \left(\frac{d^2}{dt^2} + I\right).$$

The right-hand side is annihilated by

$$\tilde{D} = \left(\frac{d}{dt}\right)^3,$$

[54]Note that, such functions are just all the functions which are solutions of some linear homogeneous equations with constant coefficients.

hence, the general integral of (3.31) is contained in the general integral of

$$\tilde{D}\overline{D}y = \left(\frac{d}{dt}\right)^3 \left(\frac{d}{dt} - I\right)^2 \left(\frac{d^2}{dt^2} + I\right) = 0,$$

which is

$$y(t) = c_1 e^t + c_2 t e^t + c_3 \cos t + c_4 \sin t + c_5 + c_6 t + c_7 t^2, \quad c_1, \dots, c_7 \in \mathbb{R}. \tag{3.32}$$

Now, we have to impose to (3.32) to satisfy (3.31). To this end, we have to fix the parameters c_5, c_6, c_7[55]. Inserting the function $c_5 + c_6 t + c_7 t^2$ in (3.31), we get $c_5 = 6, c_6 = 5, c_7 = 1$. Hence the general integral is

$$y(t) = c_1 e^t + c_2 t e^t + c_3 \cos t + c_4 \sin t + t^2 + 5t + 6, \quad c_1, \dots, c_4 \in \mathbb{R}.$$

3.5 Boundary value problems for second order linear equations

In this section we just sketch an important situation which frequently occurs in the applications[56].

Let us consider three continuous functions

$$p_0, p_1, p_2 : [a, b] \times \Lambda \to \mathbb{R},$$

where $[a, b]$ is a compact interval and $\Lambda \subseteq \mathbb{R}$ is a suitable set of parameters $\lambda \in \Lambda$. Moreover we also consider some real numbers

$$\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3, k_1, k_2,$$

and finally a continuous function $f : [a, b] \to \mathbb{R}$. The boundary value problem is the following: for any $\lambda \in \Lambda$ fixed, study the following linear problem in the unknown $y : [a, b] \to \mathbb{R}$, $x \mapsto y(x)$:

$$\begin{cases} p_2(x, \lambda) y''(x) + p_1(x, \lambda) y'(x) + p_0(x, \lambda) y(x) = f(x), \\ \alpha_0 y(a) + \alpha_1 y'(a) + \alpha_2 y(b) + \alpha_3 y'(b) = k_1, \\ \beta_0 y(a) + \beta_1 y'(a) + \beta_2 y(b) + \beta_3 y'(b) = k_2. \end{cases} \tag{3.33}$$

Remark 3.22 *The problem (3.33) is not a Cauchy problem (which would require the imposition of the values of y and y' in a fixed point $x_0 \in [a, b]$ (the same point for both values)). Instead, (3.33) requires that some suitable linear combinations of y and y', calculated on the extreme points (boundary) of the interval, be satisfied. Note, however, that such boundary conditions are just two, as the order of the equation.*

[55] Note that the other part of the solution is the general integral of the associated homogeneous equation, and hence it does not play any role in the construction of a particular solution of the non-homogeneous.

[56] Whose name is often referred as *Sturm-Liouville problems.*

Example 3.23 *(The vibrating chord)* Let us consider the problem

$$\begin{cases} y'' + \lambda y = 0 \text{ in }]0,1[\\ y(0) = y(1) = 0. \end{cases}$$

This problem corresponds to the choices $\Lambda = \mathbb{R}$, $[a,b] = [0,1]$, $p_0 = \lambda$, $p_1 = 0$, $p_2 = 1$, $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = k_1 = 0$, $\beta_0 = \beta_1 = 0$, $\beta_2 = 1$, $\beta_3 = k_2 = 0$.

If $\lambda < 0$, then the second order equation has the following general integral

$$y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x};$$

if $\lambda = 0$

$$y(x) = c_1 x + c_2,$$

and if $\lambda > 0$

$$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

It is obvious that, for any $\lambda \in \mathbb{R}$, the problem admits the null function $y \equiv 0$ as solution. The question is: for which values of λ, does the problem admit other solutions than the null one? Using the general integrals, we easily get the answer, just imposing the boundary conditions $y(0) = y(1) = 0$:
i)

$$\begin{aligned} &\lambda < 0 \implies c_1 + c_2 = e^{\sqrt{-\lambda}} c_1 + e^{-\sqrt{-\lambda}} c_2 = 0 \\ &\iff c_1 = c_2 = 0 \iff y \equiv 0; \end{aligned}$$

ii)

$$\lambda = 0 \iff c_2 = c_1 + c_2 = 0 \iff c_1 = c_2 = 0 \iff y \equiv 0;$$

iii)

$$\begin{aligned} &\lambda > 0 \iff c_1 = c_1 \cos(\sqrt{\lambda}) + c_2 \sin(\sqrt{\lambda}) = 0 \\ &\iff c_1 = c_2 = 0 \text{ or } c_1 = 0, c_2 \neq 0, \lambda = n^2\pi^2. \end{aligned}$$

Hence, the answer is: only for $\lambda = n^2\pi^2$ with $n \in \mathbb{N}$, and the solutions are of the form

$$y(x) = c \sin(k\pi x) \quad c \in \mathbb{R},\ k \in \mathbb{Z}.$$

Such solutions represent the modes of vibration of a chord hanged by its extremes[57]. The solutions corresponding to such values of λ are called *harmonics*.

[57] Think to the chord of a guitar.

For the general case (3.33), let[58]

$$y(x) = c_1\varphi_1(x,\lambda) + c_2\varphi_2(x,\lambda) + \psi(x,\lambda)$$

be the general solution of the equation, for every fixed $\lambda \in \Lambda$. Then for every λ, imposing the boundary data, we get the following 2×2 algebraic linear system in the unknown (c_1, c_2), (here for instance, φ_i' is the derivative with respect to $x \in [a,b]$, and if $x = a$ or $x = b$ it is the right derivative or, respectively, the left derivative):

$$\begin{cases} \Big(\alpha_0\varphi_1(a,\lambda) + \alpha_1\varphi_1'(a,\lambda) + \alpha_2\varphi_1(b,\lambda) + \alpha_3\varphi_1'(b,\lambda)\Big)c_1 \\ \quad +\Big(\alpha_0\varphi_2(a,\lambda) + \alpha_1\varphi_2'(a,\lambda) + \alpha_2\varphi_2(b,\lambda) + \alpha_3\varphi_2'(b,\lambda)\Big)c_2 \\ \quad\quad = k_1 - \alpha_0\psi(a,\lambda) - \alpha_1\psi'(a,\lambda) - \alpha_2\psi(b,\lambda) - \alpha_3\psi'(b,\lambda) \\ \\ \Big(\beta_0\varphi_1(a,\lambda) + \beta_1\varphi_1'(a,\lambda) + \beta_2\varphi_1(b,\lambda) + \beta_3\varphi_1'(b,\lambda)\Big)c_1 \\ \quad +\Big(\beta_0\varphi_2(a,\lambda) + \beta_1\varphi_2'(a,\lambda) + \beta_2\varphi_2(b,\lambda) + \beta_3\varphi_2'(b,\lambda)\Big)c_2 \\ \quad\quad = k_2 - \beta_0\psi(a,\lambda) - \beta_1\psi'(a,\lambda) - \beta_2\psi(b,\lambda) - \beta_3\psi'(b,\lambda) \end{cases}$$

Let $\Delta(\lambda)$ be the determinant of the system on the left-hand side. We easily conclude that, concerning the solutions of (3.33)[59],

i) $\Delta(\lambda) \neq 0 \iff$ exactly one solution,
(the null one in the homogeneous case $f \equiv 0 = k_1 = k_2$);
ii) $\Delta(\lambda) = 0 \iff$ infinitely many solutions for the homogeneous case,
either no solutions or infinitely many for the nonhomogeneous case.

In the homogeneous case, a value λ such that $\Delta(\lambda) = 0$ is said an <u>eigenvalue</u> of the problem and a corresponding non null solution is said an <u>eigenfunction</u>.

[58] General solution of the homogeneous plus a particular solution of the nonhomogeneous.
[59] Also recall the Rouché-Capelli theorem.

4 Integration of some nonlinear equations

In this section we give some methods for calculating the general integral (and hence, the solution of the Cauchy problem) for some special kinds of nonlinear scalar equations.

For convenience we first recall that, given the first order linear equation

$$z' = P(t)z(t) + Q(t),$$

its general integral is given by (recall (1.21))

$$z(t) = e^{\int P(t)dt}\left(c + \int Q(t)e^{-\int P(t)dt}dt\right). \tag{4.1}$$

4.1 Separation of variables

We consider equations of the following type

$$y' = f(t)g(y),$$

with $f : I \to \mathbb{R}, g : J \to \mathbb{R}$ continuous. If $y_0 \in J$ and $g(y_0) = 0$, then, it is obvious that the function $y(t) \equiv y_0$, $t \in I$, is a solution. Otherwise, if we are looking for solutions such that $g(y) \neq 0$, we can divide the equation by $g(y)$ and, formally, obtain

$$\frac{dy}{g(y)} = f(t)dt,$$

which, integrated, gives

$$G(y) = F(t) + k,$$

where G and F are primitives, respectively, of $1/g$ and f and $k \in \mathbb{R}$ is a constant of integration. If, at least locally, G is invertible, then we can get

$$y(t) = G^{-1}\left(F(t) + k\right),\ k \in \mathbb{R}$$

which gives the general integral.

Example 4.1 *(The catenary curve.)* We consider the equation in Example 1.5

$$y''(x) = \frac{g\mu}{c}\sqrt{1 + (y'(x))^2}.$$

We set $v = y'$, and hence we get

$$v'(x) = \frac{g\mu}{c}\sqrt{1 + v(x)^2},$$

which may be solved by separation of variables

$$\frac{dv}{\sqrt{1+v^2}} = \frac{g\mu}{c} \iff \mathrm{settsinh}(v) = \frac{g\mu}{c}x + k_1 \iff v(x) = \sinh\left(\frac{g\mu}{c}x + k_1\right),$$

from which

$$y(x) = \frac{c}{g\mu}\cosh\left(\frac{g\mu}{c}x + k_1\right) + k_2, \quad k_1, k_2 \in \mathbb{R}.$$

Hence, the shape attained by the chain is given by a piece of the graph of a hyperbolic cosine (suitably shifted by k_1 and k_2, depending on the data of the problem).

4.2 The Bernoulli equation

These are equations of the following type

$$y'(t) = P(t)y(t) + Q(t)y(t)^\alpha, \tag{4.2}$$

where P, Q are continuous, $\alpha \in \mathbb{R}$. The interesting cases are of course $\alpha \neq 0, 1$, otherwise the equation become linear, first order.

If $\alpha > 0$, then we certainly have the null solution $y \equiv 0$, whereas, if $\alpha < 0$, the null function is certainly not a solution. Also note that, when $0 < \alpha < 1$, the second member is not more Lipschitz in general, and so we may have non-uniqueness for the corresponding Cauchy problem, when $y_0 = 0$, as in the case of the example in Section 2.3.4. Then, let us suppose that y is a solution and that $y \neq 0$. We can divide the equation by y^α and obtain

$$\frac{y'}{y^\alpha} = P(t)y^{1-\alpha} + Q(t).$$

If we define $z = y^{1-\alpha}$, then z is a solution of the linear first order equation

$$z' = (1-\alpha)P(t)z + (1-\alpha)Q(t). \tag{4.3}$$

We easily find the general integral of (4.3) and then we get the general integral of (4.2), by imposing $y = z^{\frac{1}{1-\alpha}}$, and taking also account of $y \equiv 0$ if $\alpha > 0$[60].

Example 4.2 Find the general integral of

$$y'(t) = \frac{y(t)}{t} + y(t)^2.$$

Here we have $P(t) = \frac{1}{t}$, $Q(t) \equiv 1$, and $\alpha = 2$. Since $\alpha > 0$, we certainly have the null solution $y \equiv 0$. Moreover, all the solutions must be defined in a interval I not containing

[60]Pay attention to the fact that, if α is irrational then we can only search for solution $y \geq 0$ (otherwise y^α is meaningless), if $1-\alpha$ is a rational number with even numerator, then z must be nonnegative and we have the solutions $y = \pm z^{\frac{1}{1-\alpha}}$.

$t = 0$, otherwise P is not defined. Let us look for the general integral in $]0, +\infty[$. We have to look for the general integral of the linear equation (in $z = y^{-1}$)

$$z'(t) = -\frac{z(t)}{t} - 1,$$

which (for $t > 0$) is given by (see (4.1))

$$z(t) = \frac{2c - t^2}{2t}, \quad c \in \mathbb{R}$$

which gives the general integral of our equation (for $t > 0$)

$$\begin{array}{ll} y :]0, +\infty[\to \mathbb{R},\ t \mapsto \dfrac{2t}{2c - t^2} & c \leq 0, \\ y :]0, \sqrt{2c}[\to \mathbb{R},\ t \mapsto \dfrac{2t}{2c - t^2} & c > 0, \\ y :]\sqrt{2c}, +\infty[\to \mathbb{R},\ t \mapsto \dfrac{2t}{2c - t^2} & c > 0, \end{array}$$

to which we have to add the null function $y \equiv 0$.

The reader is invited to calculate the general integral for $t < 0$ and to draw the picture of all solutions in the (t, x) plane.

If, for instance, we now look for the solution of the Cauchy problem

$$\begin{cases} y' = \dfrac{y}{t} + y^2, \\ y(2) = 100, \end{cases}$$

we immediately get

$$y : \left]0, \sqrt{\frac{101}{25}}\right[\to \mathbb{R}, \quad t \mapsto \frac{2t}{\frac{101}{25} - t^2}.$$

4.3 Homogeneous equations

These are equations of the following type

$$y' = \varphi(t, y),$$

where φ is a homogeneous function of degree zero[61]:

$$\varphi(\lambda t, \lambda y) = \varphi(t, y) \ \ \forall (x, y),\ \forall \lambda \in \mathbb{R} \setminus \{0\}. \tag{4.4}$$

A typical example is

[61]Note that, here, the meaning of the word "homogeneous" is that the second member is a homogeneous function in the sense of (4.4), and not in the sense of "linear homogeneous". In particular, the equation is not linear!

$$\varphi(t,y) = f\left(\frac{y}{t}\right),$$

where f is a given function.

If y is a solution and if we set

$$z(t) = \frac{y(t)}{t},$$

then we get

$$z' = \frac{ty' - y}{t^2} = \frac{t\varphi(t,y) - y}{t^2} = \frac{t\varphi(1,z) - y}{t^2} = \frac{\varphi(1,z) - z}{t},$$

which is of the separation of variables type[62].

Example 4.3 Solve

$$y'(t) = \frac{y^3 + 3t^3}{ty^2}.$$

The equation, defined for $t \neq 0$, is homogeneous, with

$$\varphi(t,y) = \frac{y^3 + 3t^3}{ty^2},$$

which is the ratio of two homogeneous polynomial of degree three. Hence we have

$$z'(t) = \frac{3}{z^2 t},$$

which may be solved by separation of variables, getting

$$z(t) = (9\log|t| + k)^{\frac{1}{3}}, \quad t \neq 0, \ k \in \mathbb{R},$$

from which we get

$$y(t) = tz(t) = t\,(9\log|t| + k)^{\frac{1}{3}}, \quad t \neq 0, \ k \in \mathbb{R}. \tag{4.5}$$

However, the function y in (4.5) is not solution of our equation in whole $]-\infty, 0[\cup]0, +\infty[$. Indeed, it is not derivable in $t = \pm e^{-\frac{k}{9}}$. Hence, the general integral is given by

$$\begin{array}{lll}
y :]-\infty, -e^{-\frac{k}{9}}[\to \mathbb{R} & t \mapsto t\,(9\log(-t) + k)^{\frac{1}{3}} & k \in \mathbb{R}, \\
y :]-e^{-\frac{k}{9}}, 0[\to \mathbb{R} & t \mapsto t\,(9\log(-t) + k)^{\frac{1}{3}} & k \in \mathbb{R}, \\
y :]0, e^{-\frac{k}{9}}[\to \mathbb{R} & t \mapsto t\,(9\log t + k)^{\frac{1}{3}} & k \in \mathbb{R}, \\
y :]e^{-\frac{k}{9}}, +\infty[\to \mathbb{R} & t \mapsto t\,(9\log t + k)^{\frac{1}{3}} & k \in \mathbb{R}
\end{array}$$

[62] If the equation is $y' = f\left(\frac{y}{t}\right)$, then we get $z' = \frac{f(z)-z}{t}$.

The reader is invited to draw a picture of the general integral.

If for instance we are looking for the solution of the Cauchy problem

$$\begin{cases} y' = \dfrac{y^3 + 3t^3}{ty^2} \\ y(1) = 1, \end{cases}$$

we get

$$y :]e^{-1/9}, +\infty[\to \mathbb{R}, \quad x \mapsto t(9\log t + 1)^{1/3}.$$

Transformation of suitable equations into a homogeneous equation. By a suitable change of variables, the equations of the following kind (here $a, b, a_1, b_1 \in \mathbb{R}$)

$$y' = \frac{ax + by + c}{a_1 x + b_1 y + c_1} \tag{4.6}$$

may be transformed into an equivalent homogeneous equation. Indeed, let us suppose that

$$\det \begin{pmatrix} a & b \\ a_1 & b_1 \end{pmatrix} \neq 0, \tag{4.7}$$

and let us make the following change of variables

$$\begin{cases} u = ax + by + c \\ v = a_1 x + b_1 y + c_1. \end{cases} \tag{4.8}$$

Differentiating in (4.8), we get

$$\begin{cases} du = adx + bdy \\ dv = a_1 dx + b_1 dy, \end{cases}$$

which, by the hypothesis (4.7), may be solved (inverted) with respect to dx and dy, getting dx and dy as linear functions of du and dv:

$$dx = a'du + b'dv, \quad dy = a_1' du + b_1' dv,$$

which inserted in (4.6), together with (4.8), gives

$$\frac{du}{dv} = \frac{b'u - b_1' v}{a_1' v - a'u},$$

which is a homogeneous equation for the function $u(v)$. Once we know a solution $u(v)$ then we have an implicit representation for the solution $y(x)$ by

$$u(a_1 x + b_1 y + c_1) = ax + by + c,$$

which, if it is at least locally solving with respect to y, may give an explicit formula for $y(x)$.

On the other hand, if (4.7) does not hold, that is the matrix is not invertible, this means that the rows are proportional, and so there exists $k \in \mathbb{R}$ such that

$$a_1 = ka, \quad b_1 = kb.$$

Hence if we put $u = ax + by$, and so $du = adx + bdy$, we get[63]

$$dy = \frac{1}{b}(du - adx),$$

from which

$$\frac{du}{dx} = \left(a + b\frac{u + c}{ku + c_1}\right),$$

which is an equation for $u(x)$, which may be solved by separation of variables. Also in this case, once we know a solution $u(x)$, we may get a solution $y(x)$ just solving, with respect to y, the algebraic equation (if possible)

$$u(x) = ax + by$$

Example 4.4 Find the general integral of

$$y' = \frac{6x - 2y}{y - x + 3}. \tag{4.9}$$

The matrix is

$$\begin{pmatrix} 6 & -2 \\ -1 & 1 \end{pmatrix},$$

which has determinant $4 \neq 0$. Hence we have

$$\begin{cases} u = 6x - 2y \\ v = y - x + 3 \end{cases} \Longrightarrow \begin{cases} du = 6dx - 2dy \\ dv = dy - dx \end{cases} \Longrightarrow \begin{cases} dx = \dfrac{1}{4}du + \dfrac{1}{2}dv \\ dy = \dfrac{1}{4}du + \dfrac{3}{2}dv \end{cases} \tag{4.10}$$

We then get the homogeneous equation, for $u(v)$:

$$u' = \frac{2u - 6v}{v - u}$$

[63]Note that, in this particular case, we must have $b \neq 0$ (and hence also $b_1 \neq 0$), otherwise the equation (4.6) is not more a differential equation: there is not y on the right-hand side.

which is meaningful only if $u \neq v$. Hence, putting $z(v) = u/v$, and so $z \neq 1$, for $v \neq 0$[64], we get the equation of separation of variables

$$z' = \frac{z^2 + z - 6}{v(1 - z)}.$$

Imposing the condition $z^2 + z - 6 \neq 0$ we get

$$\frac{1 - z}{z^2 + z - 6} dz = \frac{dv}{v},$$

which, after some calculations, gives the general integral, in an implicit form[65]:

$$|v|^5 |z - 2||z + 3|^4 = k, \quad k \in \mathbb{R}, \; k > 0. \tag{4.11}$$

Recalling $z = u/v$ we get the implicit formula

$$|u - v||u + 3v|^4 = k, \quad k > 0,$$

from which, using (4.10), and letting drop the absolute values

$$(8x - 4y - 6)(3x + y + 9)^4 = m, \quad m \in \mathbb{R}, \; m \neq 0. \tag{4.12}$$

Now, we have to examine the case i) $u = v$, ii) $z^2 + z - 6 = 0$.

i) This case would correspond to the possible solution $y = (7x - 3)/3$. But such a function is not a solution of (4.9), as can be directly checked.

ii) This case gives the two possibilities $z = 2$ and $z = -3$ which correspond to $u = 2v$ and $u = -3v$, that is, referring to x and y, exactly to the cases given by (4.12) with $m = 0$.

Hence, we eventually get the general integral in an implicit form,

$$(8x - 4y - 6)(3x + y + 9)^4 = m, \quad m \in \mathbb{R}.$$

4.4 Exact differential equations and integrand factors

We consider the following type of equation (here, for similarity to other familiar notations, the independent variable of the unknown function is denoted by x instead of t):

$$y'(x) = -\frac{P(x, y(x))}{Q(x, y(x))}, \tag{4.13}$$

where $P, Q : A \to \mathbb{R}$ are two continuous functions, with $A \subseteq \mathbb{R}^2$ open, and $Q(x, y) \neq 0$ for all $(x, y) \in A$.

[64]Note that such condition is exactly the condition that the denominator of the right-hand side of (4.9) is not vanishing.

[65]This means that the values $z(v)$ assumed by the solution z are those such that the couple $(v, z(v))$ satisfies the equality (4.11).

As already done for the case of separation of variables, we can write (4.13) in the following way (at least formally):

$$P(x, y(x))dx + Q(x, y(x))dy = 0,$$

which is reminiscent of the differential 1-form

$$(x, y) \mapsto P(x, y)dx + Q(x, y)dy. \tag{4.14}$$

Definition 4.5 *The differential equation (4.13) is said to be an* <u>*exact differential equation*</u> *if the differential form (4.14) is exact in A.*

Remark 4.6 *Note that the separation of variable equations are particular cases of the exact differential equations.*

Theorem 4.7 *Let $A \subseteq \mathbb{R}^2$ be open and connected, and let the equation (4.13) be exact. Moreover, let $\varphi : A \to \mathbb{R}$ be a primitive of the differential form (4.14). Then, the general integral of (4.13) is given by*

$$\tilde{I} = \left\{ y : I \to \mathbb{R} \middle| y \in C^1(I), (x, y(x)) \in A \forall x \in I, \exists c \in \mathbb{R} \text{ such that } \varphi(x, y(x)) = c \forall x \in I \right\}. \tag{4.15}$$

Proof. Let us take $y : I \to \mathbb{R}$ such that $y \in C^1(I)$, and that $(x, y(x)) \in A$ for all $x \in I$. Then, we denote by ψ the function

$$\psi : I \to \mathbb{R}, \quad x \mapsto \varphi(x, y(x)).$$

If we prove that $\psi' \equiv 0$, if and only if y is a solution of (4.13) then we are done. This is indeed true, by the following chain of equalities (recalling also $Q \neq 0$):

$$\psi'(x) = \frac{\partial \varphi}{\partial x}(x, y(x)) + \frac{\partial \varphi}{\partial y}(x, y(x))y'(x) = P(x, y(x)) + Q(x, y(x))y'(x).$$

□

Remark 4.8 *Let φ be a primitive of (4.14), and, for every $c \in \mathbb{R}$, let us consider the level set c of φ*

$$E_c(\varphi) = \left\{ (x, y) \in A \middle| \varphi(x, y) = c \right\}.$$

If $y : I \to \mathbb{R}$ is a solution of the exact equation (4.13), then the couple $(x, y(x))$ belongs to the level set $E_c(\varphi)$, for some $c \in \mathbb{R}$, for all $x \in I$.

On the contrary, let $(x_0, y_0) \in A$ be such that $\varphi(x_0, y_0) = c \in \mathbb{R}$. Since $Q = \frac{\partial \varphi}{\partial y}$ is always not null by hypothesis, then, using the implicit function theorem, around (x_0, y_0)

we can represent, in a unique manner, E_c as a graph of a function $x \mapsto y(x)$. It is obvious that such a function is the unique local solution of the Cauchy problem

$$\begin{cases} y'(x) = \dfrac{P(x, y(x))}{Q(x, y(x))}, \\ y(x_0) = y_0. \end{cases}$$

Example 4.9 Write the general integral of the equation

$$y'(x) = \frac{2x - y(x)^3}{3xy(x)^2}.$$

Here we have

$$P(x, y) = y^3 - 2x, \quad Q(x, y) = 3xy^2,$$

and $Q \neq 0$ for $x \neq 0$ or $y \neq 0$, that is out of the coordinated axes. Let us look for solutions y such that the couple $(x, y(x))$ belongs to the first open quadrant A $(x > 0, y > 0)$. Let us check if the equation is exact:

$$\frac{\partial P}{\partial y}(x, y) = 3y^2 = \frac{\partial Q}{\partial x}(x, y),$$

and so, since A is simply connected, there exists a primitive φ of the associated differential form. That is the equation is exact. Let us look for a primitive φ.

$$\frac{\partial \varphi}{\partial x}(x, y) = P(x, y) = y^3 - 2x \iff \varphi(x, y) = xy^3 - x^2 + g(y),$$

where g is a C^1 continuous function, to be determined.

$$\frac{\partial \varphi}{\partial y}(x, y) = Q(x, y) \iff g'(y) \equiv 0,$$

which means g constant, and hence,

$$\varphi(x, y) = xy^3 - x^2$$

is a primitive of the differential form. From this, we immediately conclude that the general integral (with respect to the first quadrant) is

$$\begin{array}{lll} y :]0, +\infty[\to \mathbb{R} & y(x) = \left(\frac{x^2+c}{x}\right)^{\frac{1}{3}} & \text{if } c \geq 0 \\ y :]\sqrt{-c}, +\infty[\to \mathbb{R} & y(x) = \left(\frac{x^2+c}{x}\right)^{\frac{1}{3}} & \text{if } c < 0 \end{array}$$

If in particular, we search for the solution of the Cauchy problem

$$\begin{cases} y'(x) = \dfrac{2x - y(x)^3}{3xy(x)^2}, \\ y(1) = 2, \end{cases}$$

then we definitely get

$$y(x) = \left(\frac{x^2+7}{x}\right)^{\frac{1}{3}}, \ x > 0.$$

Sometimes[66] the equation (4.13) is not exact, that is the associated differential form (4.14) has no primitives. However, it may happen that, for some suitable never null function $h(x, y)$ the differential form

$$h(x, y)P(x, y)dx + h(x, y)Q(x, y)dy \tag{4.16}$$

becomes exact. We have then the following definition

Definition 4.10 *A never null function $h : A \to \mathbb{R}$ for which the differential form (4.16) is exact, is said to be an integrand factor for the differential form (4.14) (or for the equation (4.13)).*

Proposition 4.11 *If h is an integrand factor for (4.13), then the general integral of (4.13) is exactly the same as the general integral of the exact equation*

$$y'(x) = -\frac{h(x, y(x))P(x, y(x))}{h(x, y(x))Q(x, y(x))}.$$

Proof. Obvious. □

Here are two special cases where an integrand factor is easily found: if the function

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \quad \left(\text{respectively, } \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right)$$

is a function of the variable x only (respectively, y only), let us say $g(x)$ (respectively, $h(y)$), then

$$\mu(x) = e^{G(x)}, \quad (\text{respectively, } \mu(y) = e^H(y))$$

where G (respectively, H) is a primitive of g (respectively, of h), is an integrand factor.

[66]Better: almost always...

4.5 On the equations of the form $y = F(y')$

Here, we examine the first order scalar equation of the form

$$y = F(y')$$

where $F : A \to \mathbb{R}$ is of class C^1 on its domain $A \subseteq \mathbb{R}$. This is an equation in non normal form[67].

We consider the following change of variable

$$y' = p, \tag{4.17}$$

from which we get

$$\left.\begin{array}{l} y = F(p) \implies dy = F'(p)dp \\ y' = p \implies \dfrac{dy}{dx} = p \implies dx = \dfrac{dy}{p} \end{array}\right\} \implies dx = \frac{F'(p)}{p}dp.$$

Hence, the general integral of the equation is given, in a parametric form, by

$$\begin{cases} x = \displaystyle\int \frac{F'(p)}{p}dp, \\ y = F(p), \end{cases} \tag{4.18}$$

to which we have to add the constant function $y \equiv F(0)$, whenever $p = 0$ belongs to the domain of F.

The meaning of the parametric form (4.18) is the following. Let $\mathcal{F}$ be a primitive of $F'(p)/p$, hence (4.18) becomes

$$\begin{cases} x = \mathcal{F}(p) + C, \\ y = F(p), \end{cases} \tag{4.19}$$

where $C \in \mathbb{R}$ is any constant. If, for instance, we are looking for solutions of the Cauchy problem with initial datum $y(x_0) = y_0$, then we insert y_0 in the second line of (4.19), thus obtaining p_0 solving the equation (if possible). Then, inserting such a p_0, together with x_0, in the first line of (4.19), we may fix the constant $C = C_0$. Now, if $F'(p_0) \neq 0$, then around p_0 the function F is invertible, and we may consider $p = F^{-1}(y)$ getting x as function of y:

$$x(y) = \mathcal{F}\left(F^{-1}(y)\right) + C_0. \tag{4.20}$$

Since in this case $\mathcal{F} \circ F^{-1}$ has not null derivative in y_0[68], then we may locally invert (4.20) and obtain a formula for the solution $x \mapsto y(x)$.

[67] With similar techniques as the one exposed in this section, we can also study other equations in non normal form as $x = F(y')$ and $F(y, y', y'') = 0$.

[68] Just recall which is the derivative of $\mathcal{F}$ and what is the derivative of the inverse function F^{-1}.

On the other hand, if $F'(p_0) = 0$, and F is not invertible around p_0[69] we have two cases: p_0 is a relative minimum or p_0 is a relative maximum. In both cases, around (p_0, y_0), we have two different ways for choosing p such that $y = F(p)$, but we may proceed as before just inverting one of the two branches. However, in these cases, we possibly get a solution only defined on a right interval $[x_0, x_0 + \delta[$ or on a left interval $]x_0 - \delta, x_0]$.

Also note that if there are more than one way of choosing p_0 such that $F(p_0) = y_0$, then we may have multiplicity of the solution[70].

Finally , if there are not p_0 such that $y_0 = F(p_0)$, then the Cauchy problem has not solutions at all, whichever x_0 is.

Example 4.12 For any choice of the couple $(x_0, y_0) \in \mathbb{R}^2$, study existence and uniqueness for the following Cauchy problem:

$$\begin{cases} 2(y')^3 + (y')^2 - y = 0, \\ y(x_0) = y_0. \end{cases}$$

The equation is of the form $y = F(y')$ with

$$F(p) = 2p^3 + p^2.$$

We put $y' = p$ and, after some simple calculations, we get the parametric form

$$\begin{cases} x = 3p^2 + 2p + C, \\ y = 2p^3 + p^2, \end{cases}$$

to which we have to add the constant function $y \equiv F(0) = 0$. Studying the function $F(p) = 2p^3 + p^2$ we see that it tends to $\pm\infty$ as p goes to $\pm\infty$, it is negative for $p < -1/2$, and non negative for $p \geq -1/2$, it has a relative maximum in $p = -1/3$, that is $1/27$, and a relative minimum in $p = 0$, that is zero.

Hence we certainly have local existence and uniqueness for $y_0 < 0$ and for $y_0 > 1/27$ (independently from x_0). For $0 < y_0 < 1/27$ we may have three local solutions[71]. For $y_0 = 1/27$ and $y = 0$ we have three solutions.

As example, let us calculate the solutions for the initial datum $y(0) = 0$. In this case we have $x_0 = 0$, $y_0 = 0$ and we get the two values for p_0: $-1/2$ and 0. Let us first consider $p_0 = -1/2$. Then we get $C_0 = 1/4$. Hence, around (x_0, p_0) we can invert[72]

$$x = 3p^2 + 2p + \frac{1}{4},$$

[69] That is p_0 is not an inflection point, otherwise F is still invertible around p_0 and then we may similarly proceed as before.

[70] The multiplicity is not always guaranteed, since it may happens that with respect to different values of p_0, and so to different values the constant C_0, the function in (4.20) is anyway the same. However, multiplicity is very probable.

[71] And indeed we have three different solutions.

[72] In the general procedure above explained, we inverted $y = F(p)$, however, if, when the constant C_0 is fixed, also $x = \mathcal{F}(p) + c_0$ is invertible, we can then begin with that inversion.

getting

$$p(x) = \frac{-1 - \sqrt{\frac{1+12x}{4}}}{3}, \quad x \geq -\frac{1}{12},$$

and hence we get the solution

$$y_1(x) = 2\left(\frac{-1 - \sqrt{\frac{1+12x}{4}}}{3}\right)^3 + \left(\frac{-1 - \sqrt{\frac{1+12x}{4}}}{3}\right)^2, \quad x \geq -\frac{1}{12}.$$

If instead $p_0 = 0$, then we have $C_0 = 0$ and again, inverting $x = 3p^2 + 2p$ around (x_0, p_0), we get the solution

$$y_2(x) = 2\left(\frac{-1 + \sqrt{1+3x}}{3}\right)^3 + \left(\frac{-1 + \sqrt{1+3x}}{3}\right)^2, \quad x \geq -\frac{1}{3}.$$

Finally we also have the stationary solution $y_3(x) \equiv 0$.

Exercise. i) For the same problem as in Example 4.12, write the solutions for the initial datum $y(0) = 1/27$. Observe that some solutions may be defined only on a right (or left) interval around x_0.

ii) For the equation $yy' - y(y')^2 - 1 = 0$, study similar questions as in Example 4.12

5 Prolongation of solutions

In the local existence and uniqueness result for the Cauchy problem, Theorem 2.10, we have established the existence of $\delta > 0$ such that the solution exists unique in the interval $]t_0 - \delta, t_0 + \delta[$. Of course, such a constructed δ is not in general the optimal one[73], that is we can probably obtain existence and uniqueness also in some larger interval I. In other words, we can probably prolong the solution beyond $t_0 + \delta$ or beyond $t_0 - \delta$.

5.1 Maximal solutions and existence up to infinity

Let us consider the first order system of equations in normal form

$$y' = f(t, y), \tag{5.1}$$

where $f : A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^{n+1}$ open.

Definition 5.1 *A solution $\tilde{y} : \tilde{I} \to \mathbb{R}^n$ of (5.1), is said to be a maximal solution if there is not another solution $y : I \to \mathbb{R}^n$ such that $\tilde{I} \subseteq I$, $\tilde{I} \neq I$, and $y(t) = \tilde{y}(t)$ for all $t \in \tilde{I}$. In other words, the solution $\tilde{y}$ is maximal if it is not prolongable beyond its domain $\tilde{I}$.*

In a similar way we define a maximal solution for a Cauchy problem associated to (5.1)

Remark 5.2 *Let us note the difference of Definition 5.1 with the definition of the global solution, Definition 2.3, and with the definition of the unique global solution, Definition 2.12. In the first case we a priori fix an interval of existence (which we do not make in Definition 5.1), in the second case we require the uniqueness, whereas in Definition 5.1 we are not concerning with uniqueness: a maximal solution may exist without being unique as solution (even locally)[74]. However, as we are going to see, the two notions are strongly related.*

If the Cauchy problem associated to (5.1) has local existence and uniqueness for all initial data $(t_0, y_0) \in A$, then a maximal solution for the Cauchy problem exists and coincides with the unique global solution in Definition 2.12. This the statement of the next theorem, but first we need the following lemma.

Lemma 5.3 *Let us suppose that, for all $(t_0, x_0) \in A$, the Cauchy problem for (5.1) has a unique local solution. Then if two solutions of the equation $y' = f(t, y)$, let us say $\varphi : I \to \mathbb{R}^n$, I open, and $\psi : J \to \mathbb{R}^n$, J open, are equal in a point $\bar{t} \in I \cap J$, then they are equal in all their common interval of definition $I \cap J$.*

Proof. We are going to prove that the set

$$C = \left\{ t \in I \cap J \middle| \varphi(t) = \psi(t) \right\},$$

[73]we have only required that it was sufficiently small in order to be sure that our argumentation holds
[74]Recall, for instance, the example in Paragraph 2.3.4

is a nonempty open-closed subset of $I \cap J$ (for the induced topology)[75]. Hence, since $I \cap J$ is an interval and so connected, we must have $C = I \cap J$ which will conclude the proof.

First of all note that $C \neq \emptyset$ since $\bar{t} \in C$ by definition.

Let $t_n \in C$ converge to $t^* \in I \cap J$. Then, by definition of C, $\varphi(t_n) = \psi(t_n)$ for all n. Since both φ and ψ are continuous on $I \cap J$ (they are solutions), we get $\varphi(t^*) = \psi(t^*)$, and so $t^* \in C$, which turns out to be closed.

Let us now take any $t^* \in C$ and consider the Cauchy problem with datum (t^*, y^*) where $y^* = \varphi(t^*) = \psi(t^*)$. By hypothesis, such a problem has a unique local solution. Since both φ and ψ are solution of such a Cauchy problem, they must coincide in an interval $]t^* - \delta, t^* + \delta[$, which is then contained in C, which turns out to be open. $\square$

Theorem 5.4 *Let us suppose that, for all $(t_0, x_0) \in A$, the Cauchy problem for (5.1) has a unique local solution. Then for any initial datum, the Cauchy problem has a unique maximal solution $\tilde{y} : \tilde{I} \to \mathbb{R}^n$, which also turns out to be the unique global solution in the sense of Definition 2.12. The interval $\tilde{I}$ is said the maximal interval of existence.*

Proof. For a fixed initial datum, let us consider the general integral of the Cauchy problem, and define $\tilde{I}$ as the union of all intervals I such that there exists $y_I : I \to \mathbb{R}^n$ solution of the Cauchy problem.

First of all note that $\tilde{I}$ is an open interval containing t_0, since so are all the intervals I. Now we define

$$\tilde{y} : \tilde{I} \to \mathbb{R}^n, \quad y \mapsto y_I(t) \ \text{ if } t \in I. \tag{5.2}$$

Using Lemma 5.3 it is easy to see that (5.2) is a good definition[76], and also that $\tilde{y}$ is the unique maximal solution, as requested. $\square$

Remark 5.5 *If A is a strip: $]a, b[\times\mathbb{R}^n$, then the condition*

$$\exists\ c_1, c_2 \geq 0 \text{ such that } \|f(t, x)\| \leq c_1\|x\| + c_2 \ \forall (t, x) \in A, \tag{5.3}$$

together with the usual hypotheses of continuity and Lipschitz continuity (2.7), guarantees the existence of the maximal solution $\tilde{y}$ in the whole interval $]a, b[$[77]. Indeed, this is exactly what we have proven in the global existence and uniqueness Theorem 2.15, where

[75] Since $I \cap J$ is open, C is open in $I \cap J$ for the induced topology if an only if it is open in $\mathbb{R}$, and it is closed for the induced topology if and only if, whenever $t_n \to t^*$ as $n \to +\infty$, with $t_n \in C, t^* \in I \cap J$, then also $t^* \in C$.

[76] If $t \in I \cap J$ then $y_I(t) = y_J(t)$.

[77] Condition (5.3) says that if f, that is the derivative of $\tilde{y}$, has a linear behavior as $\|\tilde{y}\| \to +\infty$, then the solution $\tilde{y}$ exists for all times $]a, b[$. As example, think to the scalar case with $\tilde{y}, \tilde{y}' \geq 0$; hence we have $0 \leq \tilde{y}' \leq c_1\tilde{y} + c_2$, and it is believable that $\tilde{y}$ must stay under the solution of $y' = c_1 y' + c_2$ which is an exponential function. Since the exponential function exists for all times, that is it is finite for all times, then $\tilde{y}$ cannot go to $+\infty$ before the time b, and hence it must exist until b. In the next section we will formalize the comparison between solutions.

we used (2.12) which is nothing but (5.3), which, in that case, was given by the global uniform Lipschitz condition (2.11). Another simple condition which implies (5.3) is the boundedness of f.

Moreover if f,besides satisfying (5.3), is also defined and continuous in $[a, b[\times\mathbb{R}^n$ or $]a, b]\times\mathbb{R}^n$ (which implies a or b finite), then the limits $\lim_{t\to a^+}\tilde{y}(t)$ or $\lim_{t\to b^-}\tilde{y}(t)$ exist in $\mathbb{R}^n$, that is, the solution is prolongable till $[a, b[$ or $]a, b]$[78]*. This is easily seen just slightly modifying the proof of Theorem 2.15.*

Note that, if, for instance, $A =]a, +\infty[\times\mathbb{R}^n$, and f satisfies (2.7) and (5.3) then, by Remark 5.5, the maximal interval of existence is $]a, +\infty[$, that is the maximal solution exists for all times $t \to +\infty$. It similarly happens if $A =] - \infty, b[\times\mathbb{R}^n$.

In Remark 5.5, we gave some results about the behavior of the maximal solution on a strip when the extrema of the maximal interval are approached. Here we want to say something similar for the general case of A not a strip and in the case of strip but with a different condition than (5.3).

Proposition 5.6 *Let us suppose that $f : A \to \mathbb{R}^n$, with $A \subseteq \mathbb{R}^{n+1}$ open, satisfies the usual conditions for local existence and uniqueness of the Cauchy problem (see Theorem 2.10). Let $\tilde{y} : \tilde{I} \to \mathbb{R}^n$ be a maximal solution of the equation $y' = f(t, y)$, and also suppose that, defined $\beta = \sup\tilde{I}$, there exists $c \in \tilde{I}$ such that $\tilde{y}'$ is bounded in $[c, \beta[$. Then, we have the following alternative:*

i) $\beta = +\infty$

otherwise

ii) the limit $\lim_{t\to\beta^+}\tilde{y}(t) = x_\beta$ exists in $\mathbb{R}^n$, but $(\beta, x_\beta) \notin A$.

A similar conclusion holds for $\alpha = \inf\tilde{I}$.

Proof. By our hypothesis, $\tilde{y}$ is Lipschitz (and in particular uniformly continuous) in $[c, \beta[$, since its derivative is bounded. Hence, if $\beta < +\infty$, we can prolong $\tilde{y}$ up to the boundary β, that is the limit $x_\beta \in \mathbb{R}^n$ exists. However, if by absurd $(\beta, x_\beta) \in A$, which is open, then $\tilde{y}'_-(\beta) = f(\beta, x_\beta)$[79] and, again by hypothesis, we can extend the solution beyond β. This is a contradiction since $\tilde{y}$ is maximal. $\square$

Theorem 5.7 *Let f be as in Proposition 5.6, let $\tilde{y} : \tilde{I} \to \mathbb{R}^n$ be a maximal solution of $y' = f(t, y)$, and let $K \subset A$ be a compact set. Then there exists $[a, b] \subset\subset \tilde{I}$*[80] *such that*

$$(t, \tilde{y}(t)) \notin K \quad \forall\ t \in \tilde{I} \setminus [a, b].$$

We then say that the maximal solutions definitely exit from any compact set in the domain of f.

[78]This means, for instance, that $\lim_{t\to b^-}\tilde{y}(t) = x_b$ exists in $\mathbb{R}^n$ and that $\tilde{y}'_-(b) = f(b, x_b)$, where $\tilde{y}'_-$ is the left-derivative. The validity of the last formula is easily seen using the integral representation.

[79]See Footnote 78. Such an equality guarantees that the extension beyond β is really an extension of $\tilde{y}$, since it glues to it in $t = \beta$ in a C^1-manner.

[80]This means that $\inf\tilde{I} < a \le b < \sup\tilde{I}$.

Proof. Let us define $\beta = \sup \tilde{I}$, and by absurd, let us suppose that there exists a sequence $t_n \in \tilde{I}$ converging to β such that $(t_n, \tilde{y}(t_n)) \in K$ for all n. Hence, possibly extracting a subsequence, there exists $x_\beta \in \mathbb{R}^n$ such that

$$\lim_{n\to+\infty}(t_n, \tilde{y}(t_n)) = (\beta, x_\beta) \in K \subset A.$$

In particular, this means that $\beta < +\infty$. We now get the contradiction since, by virtue of the convergence, there exists a ball $B \subseteq A$ which contains (β, x_β) and $(t_n, \tilde{y}(t_n))$ for sufficiently large $n \geq \overline{n}$. If then we look back to the proof of the local existence and uniqueness Theorem 2.10, we see that there should exist a common $\delta > 0$ such that the unique local solution of any Cauchy problem

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_n) = \tilde{y}(t_n), \end{cases}$$

must exist at least until $t_n + \delta$. By uniqueness, such a local solution must be equal to $\tilde{y}$ itself for $t \leq \beta$. This means that, whenever $t_n > \beta - \delta$, we can prolong $\tilde{y}$ beyond β[81]. A contradiction to the maximality of $\tilde{y}$.

The proof for $\alpha = \inf \tilde{I}$ is similar. □

Remark 5.8 *In the particular case of an autonomous system $y' = f(y)$, with $f : A \to \mathbb{R}^n$ locally Lipschitz and $A \subseteq \mathbb{R}^n$ open, if $\tilde{y} : \tilde{I} \to \mathbb{R}^n$, $\tilde{I}$ open, is a maximal solution and $K \subseteq A$ is compact, then we have the following alternative*

i) $\tilde{y}$ definitely exits from K, that is there exist $a, b \in \tilde{I}$ such that

$$\tilde{y}(t) \notin K \ \forall \ t \in \tilde{I} \setminus [a, b],$$

otherwise

ii) $\sup \tilde{I} = +\infty$ *or* $\inf \tilde{I} = -\infty$.

Indeed, let us first note that we can think to the system as $y' = \tilde{f}(t, y)$, where $\tilde{f} : \tilde{A} \to \mathbb{R}^n$ with $\tilde{A} = \mathbb{R} \times A$, $\tilde{f}(t, x) = f(x)$. Hence, if i) is not verified, we have, for instance, $t_n \to \sup \tilde{I}^-$ with $\tilde{y}(t_n) \in K$. But then, if $\sup \tilde{I} \in \mathbb{R}$ we have, for a suitable compact interval J around it and for large n, $(t_n, \tilde{y}(t_n)) \in J \times K$, which is compact in $\tilde{A}$, and this is a contradiction to Theorem 5.7.

Remark 5.9 *Note that both Proposition 5.6 and Theorem 5.7 say that, if $\beta = \sup \tilde{I} < +\infty$ and A is bounded*[82]*, then*

$$\lim_{t\to\beta^-} dist\left((t, \tilde{y}(t), \partial A)\right) = 0, \tag{5.4}$$

that is the couple $(t, \tilde{y}(t))$ approximates the boundary ∂A when t approximates β. Moreover, Proposition 5.6 also says that, if $\tilde{y}'$ is bounded, then the limit $\lim_{t\to\beta^-} \tilde{y}(t)$ does exist in $\mathbb{R}^n$ (if $\tilde{y}'$ is not bounded, then the solution may oscillate).

[81] See Footnote 79.

[82] We are here supposing the boundedness of A in order to give an immediate meaning to (5.4), without ambiguities for $\|\tilde{y}\| \to +\infty$.

Now we give another criterium for the existence of the solution up to infinity.

Proposition 5.10 *Let f be again as in Proposition 5.6 and also suppose that A is the strip $]a, +\infty[\times\mathbb{R}^n$. Let $\tilde{y} : \tilde{I} \to \mathbb{R}^n$ be a maximal solution of $y' = f(t, y)$ such that, for some $\tau \in \tilde{I}$ and for some $c_1, c_2 \geq 0$, we have*[83]

$$\|\tilde{y}(t)\| \leq c_1 + c_2(t - \tau) \quad \forall\, t \geq \tau, \ t \in \tilde{I}. \tag{5.5}$$

Then, $\tilde{y}$ is prolongable up to infinity, that is $\beta = \sup \tilde{I} = +\infty$.

A similar result holds for the prolongation up to $-\infty$.

Proof. By absurd, let us suppose that $\beta < +\infty$. Hence, fixed any $B > c_1 + c_2(\beta - \tau)$, the compact cylinder

$$K = \left\{(t, x) \in \mathbb{R}^{n+1} \middle| t \in [\tau, \beta], \|x\| \leq B\right\},$$

is contained in A. Hence, by Theorem 5.7, the maximal solution $\tilde{y}$ must definitely exit from K when times approach β. By (5.5) and by our assumption on B we have, for any $t \geq \tau$, $t \in \tilde{I}$,

$$\|\tilde{y}(t)\| \leq c_1 + c_2(\beta - \tau) < B,$$

which implies that $\tilde{y}$ must exit from the cylinder K through the "wall" $\{(\beta, x) | \|x\| \leq B\}$. But this means, as before, that we can prolong $\tilde{y}$ beyond $\tilde{I}$, which is a contradiction since $\tilde{I}$ is the maximal interval. $\square$

5.2 Qualitative studies (I)

The results of the previous paragraph may be used to get information about the qualitative behavior of the solutions of scalar equations, even when an explicit formula for the solutions can not be found (or it is hard to be found). In this way it may be possible to draw a "qualitative" picture of the graphs of the solutions. Here is a list of points that are usually convenient to address.

1) Check the local existence and uniqueness, for instance verifying the hypothesis (2.7).

2) Check the prolongability of the solutions, for instance using (5.3).

3) Find the possible stationary solutions $y \equiv y_0$, which correspond to the values $y_0 \in \mathbb{R}$ such that $f(t, y_0) = 0$ for all t.

4) Find the region of the plane (t, x) where $f > 0$ and where $f < 0$ respectively, to get information about the monotonicity of the solutions. Also observe that, when passing from one region to another, the solution must have a relative extremum.

5) Observe that by uniqueness the graphs of the solutions cannot intersect each other.

[83] Also (5.5) is a condition of linearity up to infinity as (5.3), but here it is respect to time instead of space, and moreover it is requested to the solution instead of to the dynamics.

6) Study the limit $\lim_{t\to\pm\infty} y(t)$, when reasonable[84] and possible. Here, we can use some known facts as, for instance: if a C^1 function g satisfies $\lim_{t\to+\infty} g(y) = \ell \in \mathbb{R}$ then it cannot happen that $\lim_{t\to+\infty} g'(t) = \pm\infty$[85].

7) Sometimes it may be useful to check whether there are some symmetries in the dynamics f, since this fact may help in the study. For instance, if f has the following symmetry (oddness with respect to the vertical axis $t = 0$)

$$f(t, x) = -f(-t, x) \ \ \forall\ (t, x) \in \mathbb{R}^2,$$

then the behavior of the solutions on the second and third quadrants is specular with respect to the one in the first and fourth quadrants. Indeed, let $y :]a, b[\to \mathbb{R}$ be a solution with $0 \le a \le b$, then the function

$$\psi :]-b, -a[\to \mathbb{R}, \quad t \mapsto y(-t),$$

is also a solution. This can be easily checked. Let us fix $\tau \in]a, b[$ and consider $x_0 = y(\tau)$. Then we have

$$y(t) = x_0 + \int_\tau^t f(s, y(s))ds \ \ \forall\ t \in]a, b[,$$

and hence, for all $t \in]-b, -a[$, via the change of variable $\xi = -s$,

$$\begin{aligned}
&\psi(t) = y(-t) = x_0 + \int_\tau^{-t} f(s, y(s))ds = \\
&x_0 + \int_{-\tau}^{t} f(-\xi, y(-\xi))(-d\xi) = x_0 + \int_{-\tau}^{t} f(\xi, \psi(\xi))d\xi,
\end{aligned}$$

which means that ψ is solution[86].

8) Sometimes it may be useful to study the sign of the second derivative y''. This can be guessed just deriving the equation[87].

Example 5.11 Given the following scalar equation

$$y'(t) = (t^2 - y)\frac{\log(1 + y^2)}{1 + y^2}, \tag{5.6}$$

discuss existence, uniqueness, maximal interval of existence and draw a qualitative graph of the solutions.

We are going to analyze the above eight points.

[84] That is when the maximal interval is a left- and/or right- half line.

[85] The simple proof is left as exercise.

[86] Another simpler way to check that ψ is a solution is just to calculate its derivative.

[87] Concerning the existence of the second derivative, note that if $f \in C^m$ then any solution y belongs to C^{m+1}. Indeed, for instance, if $f \in C^1$, then, since the solutions are C^1 by definition, $y' = f(t, y) \in C^1$ and hence $y \in C^2$. We can then go on in this way.

1) Here we have

$$f(t,x) = (t^2 - x)\frac{\log(1+x^2)}{1+x^2},$$

which is of class C^1 in whole $\mathbb{R}^2$, which is a strip. Hence, there is local existence and uniqueness for the Cauchy problem with any initial datum $(t_0, x_0) \in \mathbb{R}^2$.

2) Since $\log(1+x^2) \leq 1+x^2$ for all $x \in \mathbb{R}$, then, for every $a > 0$ we have

$$|f(t,x)| \leq |x| + a^2 \;\; \forall\, (t,x) \in [-a,a] \times \mathbb{R},$$

which, by Remark 5.5, guarantees the existence of the maximal solutions in the whole interval $]-a,a[$. By the arbitrary of $a > 0$ we get the existence of the maximal solution for all times $t \in \mathbb{R}$.

3) The only stationary solution is $y(t) \equiv 0$.

4) The sign of f is given by

$$\begin{cases} f(t,x) > 0 & \text{if } x < t^2, \\ f(t,x) < 0 & \text{if } x > t^2. \end{cases}$$

This means that the solutions are decreasing if $y(t) > t^2$ and increasing if $y(t) < t^2$. Moreover, when the graph of a solution crosses the parabola $y = t^2$ at time $\bar{t}$, then, at that time, the solution has a relative extremum: a maximum if crosses in the second quadrant (where it can only pass from the increasing region to the decreasing region), and a minimum if crosses in the first quadrant[88].

5) Since the null function $y \equiv 0$ is a solution, then all the solutions which sometimes have a strictly positive value (respectively: a strictly negative value) are strictly positive (respectively: strictly negative) for all times $t \in \mathbb{R}$.

6) Let us consider a solution y negative. We must have $\lim_{t\to-\infty} y(t) = -\infty$. Indeed, since the negative solutions are all increasing, the alternative is $-\infty < \lim_{t\infty} y(t) = \ell < 0$. But this fact would imply that $\lim_{t\to-\infty} y'(t) \neq -\infty$, which is absurd since

$$\lim_{t\to-\infty} y'(t) = \lim_{t\to-\infty}(t^2 - y(t))\frac{\log(1+y(t)^2)}{1+y(t)^2} = \lim_{t\to-\infty} t^2 - \ell\frac{\log(1+\ell^2)}{1+\ell^2} = +\infty.$$

In a similar way, we have $\lim_{t\to+\infty} y(t) = 0$. Indeed, y is increasing and bounded above (by zero). Hence it must converge to a finite value ℓ. But, whenever $\ell \neq 0$ we get $\lim_{t\to+\infty} y'(t) = +\infty$ which is an absurd.

Now, let us consider a solution y positive. First of all, let us note that y must cross the parabola $y = t^2$ in the first quadrant, passing from the decreasing region to the increasing one, and remaining on the latter for the rest of the times. Hence we have $\lim_{t\to+\infty} y(t) = +\infty$. Indeed y is increasing and hence the alternative is the convergence

[88]Note that, for instance, a solution cannot cross the parabola in the second quadrant passing from the decreasing region to the increasing region, otherwise it should decrease a little bit in the increasing region too. Similarly it happens in the first quadrant.

to a finite value $\ell > 0$, but also in this case, looking to the behavior of the derivative, we would get an absurd. Concerning the behavior in the second quadrant, we note that every solution must cross the parabola in that quadrant too. Indeed, if not, we would have $\lim_{t\to-\infty} y(t) = +\infty$ with $y(t) \geq t^2$ for all $t \leq 0$ and y decreasing. But this implies

$$0 \geq \lim_{t\to-\infty} y'(t) = \lim_{t\to-\infty} (t^2 - y(t))\frac{\log(1 + y(t)^2)}{1 + y(t)^2} \geq \lim_{t\to-\infty} -y\frac{\log(1 + y(t)^2)}{1 + y(t)^2} = 0,$$

which is an absurd since, if it is true, we would not have $y(t) \geq t^2$ definitely for $t \to -\infty$. Hence, y must definitely belong to the increasing region and, as before, the only possibility is $\lim_{t\to-\infty} y(t) = 0$.

For this example, there are not evident symmetries and also we let drop point 8), since we already have sufficient information in order to draw a qualitative picture of the solutions. The drawing is left as an exercise.

6 Comparison and continuous dependence on data

In this section we first address the problem of comparing the solutions of two scalar equations, that is of being able to say if one is larger than the other just working on the equations without explicitly knowing the solutions (or more generally, to give some a-priori estimates on the solutions of a system of equations). A second problem we are going to address is the dependence from the initial data for the solution of a Cauchy problem.

Both problems rely on the following important lemma.

6.1 Gronwall Lemma

Theorem 6.1 *Let $y : [t_0, +\infty[\to [0, +\infty[$ be a continuous (nonnegative) function, where t_0 is a fixed real number. Let us also suppose that there exist a non decreasing continuous function $u : [t_0, +\infty[\to \mathbb{R}$ and a constant $L > 0$ such that*

$$0 \le y(t) \le u(t) + L \int_{t_0}^{t} y(s)ds \quad \forall t \ge t_0. \tag{6.1}$$

Then, we have

$$0 \le y(t) \le u(t)e^{L(t-t_0)} \quad \forall t \ge t_0.$$

Proof. For $\tau \ge t_0$, and recalling that $e^{-L\tau} > 0$, we have the following sequence of implications:

$$\begin{aligned}
&0 \le y(\tau) \le u(\tau) + L \int_{t_0}^{\tau} y(s)ds \implies \\
&0 \le e^{-L\tau}y(\tau) \le e^{-L\tau}u(\tau) + e^{-L\tau}L \int_{t_0}^{\tau} y(s)ds \implies \\
&e^{-L\tau}y(\tau) - e^{-L\tau}L \int_{t_0}^{\tau} y(s)ds \le e^{-L\tau}u(\tau) \implies \\
&\frac{d}{d\tau}\left(e^{-L\tau} \int_{t_0}^{\tau} y(s)ds\right) \le e^{-L\tau}u(\tau).
\end{aligned}$$

From this, for every $t \ge t_0$, using also the monotonicity of u and the positivity of the functions, we get

$$\begin{aligned}
&\int_{t_0}^{t} \frac{d}{d\tau}\left(e^{-L\tau} \int_{t_0}^{\tau} y(s)ds\right) d\tau \le \int_{t_0}^{t} e^{-L\tau}u(\tau)d\tau \le u(t) \int_{t_0}^{t} e^{-L\tau}d\tau \implies \\
&e^{-Lt} \int_{t_0}^{t} y(s)ds \le \frac{u(t)}{L}e^{-Lt_0} - \frac{u(t)}{L}e^{-Lt} \implies \\
&e^{-Lt}\left(u(t) + L \int_{t_0}^{t} y(s)ds\right) \le u(t)e^{-Lt_0} \implies \\
&u(t) + L \int_{t_0}^{t} y(s)ds \le u(t)e^{L(t-t_0)}.
\end{aligned} \tag{6.2}$$

Since, by hypothesis (6.1), the left-hand side of the last line of (6.2) is larger than or equal to $y(t)$, we get the conclusion. □

Remark 6.2 *In particular, if $u \equiv M$ is a constant, then we get the simpler formulation of the Gronwall Lemma*

$$0 \leq y(t) \leq M + L \int_{t_0}^{t} y(s)ds \implies y(t) \leq Me^{L(t-t_0)}.$$

Moreover, if $M = 0$, then we get

$$0 \leq y(t) \leq L \int_{t_0}^{t} y(s)ds \implies y(t) = 0,$$

that is: the only non negative continuous function which is smaller than its integral multiplied by any constant, is the null function.

6.2 Comparison of solutions and qualitative studies (II)

The first use of the Gronwall Lemma 6.1 is to obtain a-priori estimates on solutions.

Theorem 6.3 *Let $f : A \to \mathbb{R}^n$, with $A \subseteq \mathbb{R}^{n+1}$ open, satisfy the standard hypothesis for existence and uniqueness result of Theorem 2.10. Let us suppose that there exists a function*

$$\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (t, \xi) \mapsto \mu(t, \xi),$$

which is continuous and locally Lipschitz continuous with respect to ξ uniformly in t (the usual hypothesis (2.7)), and such that

$$\|f(t, x)\| \leq \mu(t, \|x\|) \ \ \forall\ (t, x) \in A. \tag{6.3}$$

Moreover, let us take $(t_0, x_0) \in A$, $\xi_0 \in \mathbb{R}$, with

$$\|x_0\| \leq \xi_0, \tag{6.4}$$

and consider the solutions y and v of the two following Cauchy problems, respectively

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = x_0, \end{cases} \qquad \begin{cases} v'(t) = \mu(t, v(t)) \\ v(t_0) = \xi_0. \end{cases}$$

Then, we have

$$\|y(t)\| \leq v(t), \quad \forall t \geq t_0,\ t \in I \tag{6.5}$$

where I is the common interval of existence of the solutions.

Proof. By absurd, let us suppose that there exists $\bar{t} \geq t_0$, $\bar{t} \in I$ such that

$$\|y(\bar{t})\| > v(\bar{t}). \tag{6.6}$$

Then, we define

$$t^* = \sup \left\{ t \in [t_0, \bar{t}] \middle| \|y(t)\| \leq v(t) \right\},$$

and note that by (6.4), by the continuity of the solutions, and by the absurd hypothesis (6.6), t^* exists and

$$t^* < \bar{t}, \quad \|y(t)\| > v(t) \quad \forall t \in]t^*, \bar{t}], \quad \|y(t^*)\| = v(t^*).$$

We consider the continuous non negative function, defined in $[t^*, \bar{t}]$,

$$w(t) = \|y(t)\| - v(t) > 0.$$

Since v and $\|y\|$ are continuous, then they are bounded in $[t^*, \bar{t}]$, and hence the couples $(t, \|y(t)\|), (t, v(t))$ do not exit from a compact set $K \subset \mathbb{R}^2$ for $t \in [t^*, \bar{t}]$. Let L_μ be a Lipschitz constant for μ in K. We get (also recall that $\|y(t^*)\| = v(t^*)$)

$$\begin{aligned}
&0 \leq w(t) = \left\| x_0 + \int_{t_0}^{t} f(s, y(s)) ds \right\| - \xi_0 - \int_{t_0}^{t} \mu(s, v(s)) ds = \\
&\left\| x_0 + \int_{t_0}^{t^*} f(s, y(s)) ds + \int_{t^*}^{t} f(s, y(s)) ds \right\| - \xi_0 - \int_{t_0}^{t^*} \mu(s, v(s)) ds - \int_{t^*}^{t} \mu(s, v(s)) ds \leq \\
&\|y(t^*)\| + \left\| \int_{t^*}^{t} f(s, y(s)) ds \right\| - v(t^*) - \int_{t^*}^{t} \mu(s, v(s)) ds = \\
&\left\| \int_{t^*}^{t} f(s, y(s)) ds \right\| - \int_{t^*}^{t} \mu(s, v(s)) ds \leq \int_{t^*}^{t} \Big(\|f(s, y(s))\| - \mu(s, v(s)) \Big) ds \leq \\
&\int_{t^*}^{t} \Big(\mu(s, \|y(s)\|) - \mu(s, v(s)) \Big) ds \leq L_\mu \int_{t^*}^{t} \Big(\|y(s)\| - v(s) \Big) ds = L_\mu \int_{t^*}^{t} w(s) ds.
\end{aligned}$$

By the Gronwall Lemma we get $w(t) = 0$ for all $t \in [t^*, \bar{t}]$, which is a contradiction. □

In the case of scalar equations, we obtain the following comparison result between solutions, without involving the absolute values. The proof is left as an exercise.

Proposition 6.4 *Let $f, g : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^2$ open, be continuous and Lipschitz continuous in x, such that*

$$f(t, x) \leq g(t, x) \quad \forall\ (t, x) \in A, \tag{6.7}$$

and moreover, let us consider $(t_0, x_0), (t_0, \xi_0)$ two admissible initial states with $x_0 \leq \xi_0$. Then, if $y, v : I \to \mathbb{R}$ are, respectively, the solutions of the Cauchy problems

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = x_0, \end{cases} \qquad \begin{cases} v'(t) = g(t, v(t)), \\ y(t_0) = \xi_0, \end{cases}$$

where I is the common interval of existence, we have

$$y(t) \leq v(t) \ \ \forall\, t \in I, \ t \geq t_0. \tag{6.8}$$

Remark 6.5 *If in Theorem 6.3 (respectively in Proposition 6.4)), μ is only continuous*[89] *(respectively g is only continuous), we still get (6.5) (respectively (6.8)), provided that (6.3) (respectively (6.7)) holds as strict inequality for all $(t,x) \in A$.*

One natural first use of the comparison result is to compare the (not analytically known) solution of a scalar Cauchy problem with the (analytically known) solution of another scalar Cauchy problem. This may permit to say something about the qualitative behavior of the unknown solution. Referring to the Example 5.11, this may be seen as the point 9) of a plan of procedure. It is explained in the following example.

Example 6.6 Study the qualitative behavior of the solutions of

$$y'(t) = ty\left(\max(1, y(t)) + \sin^2(y(t))\right).$$

We are going to analyze the eight points as in Example 5.11, plus the new point 9). First of all, we have to check the local existence and uniqueness. The dynamics is

$$f(t,x) = tx\left(\max(1,x) + \sin^2(x)\right),$$

which is defined on the whole plane $\mathbb{R}^2$, is continuous but not C^1, since the function $x \mapsto \max(1,x)$ is not derivable at $x = 1$. However, such a function of x is Lipschitz continuous and so f is locally Lipschitz continuous with respect to x uniformly in t[90].

Hence, there is existence and uniqueness for every initial datum $(t_0, w_0) \in \mathbb{R}^2$.

The only stationary solution is $y \equiv 0$ since other possible stationary solutions $y \equiv y_0 \neq 0$ would imply

$$\sin^2(y_0) = -\max(1, y_0) \leq -1$$

which is impossible.

The other non zero solutions are either always positive or always negative. They are increasing in the first and third quadrants, and they are decreasing in the second and fourth quadrants. This also means that the vertical line $t = 0$ is a line of minima for positive solutions and of maxima for negative solutions. Moreover the dynamics has the symmetry $f(t,x) = -f(-t,x)$ and hence we can study the behavior in the first and fourth quadrant only.

[89] And hence the corresponding Cauchy problem may not have uniqueness.

[90] Prove these facts as exercise. Hints: 1) prove that $|\max(1,x_1) - \max(1,x_2)| \leq |x_1 - x_2|$ for all $x_1, x_1 \in \mathbb{R}$; 2) observe that, just adding and subtracting a suitable term,

$$|f(t,x_1) - f(t,x_2)| \leq |t||x_1 - x_2||\max(1,x_1) - \sin^2 x_1| + |t||x_2||\max(1,x_1) - \max(1,x_2) + \sin^2 x_1 - \sin^2 x_2|,$$

from which the desired local Lipschitz property follows (also recall that $\sin^2$ is Lipschitz).

If $x < 0$ then, for all $t \in [-a, a]$, $a > 0$ arbitrary,

$$|f(t, x)| = \left|tx(1 + \sin^2(x))\right| \leq 2a|x|,$$

and hence the negative solutions are defined for all times $t \in]-\infty, +\infty[$. We easily get that, for negative solutions it is (the limit cannot be finite)

$$\lim_{t \to \pm\infty} y(t) = -\infty.$$

Now let us examine the positive solutions. Let $\tilde{y} : \tilde{I} \to \mathbb{R}$ be a maximal positive solution. Then $0 \in \tilde{I}$ since, otherwise, $\tilde{y}$ is prolongable to the left of $0 \leq \alpha = \inf \tilde{I}$[91]. Let us consider the case $\tilde{y}(0) = \tilde{y}_0 \geq 1$: for all positive t of its domain of definition we certainly have $\tilde{y}(t) \geq 1$ and hence we are not more able to get a linear estimate for f, since the best we can do for $x \geq 1$ is

$$|f(t, x)| \leq tx^2 + tx,$$

which is quadratic in x. But, if we take $c > 0$, $c \in \tilde{I}$, we then get

$$\tilde{y}'(t) = t\tilde{y}(t)(\tilde{y}(t) + \sin^2(\tilde{y}(t))) \geq c\tilde{y}^2(t) \quad \forall t \in \tilde{I}, t \geq c,$$

and hence, for $t \geq c$, $t \in \tilde{I}$, by the comparison results, $\tilde{y}$ stays above the solution y of the Cauchy problem

$$\begin{cases} y'(t) = cy^2(t), \\ y(c) = \tilde{y}(c), \end{cases}$$

which is easily computed as

$$y(t) = \frac{\tilde{y}(c)}{1 + c^2\tilde{y}(c) - c\tilde{y}(c)t}. \tag{6.9}$$

Since such a function y tends to $+\infty$ as

$$t \to \frac{1 + c^2\tilde{y}(c)}{c\tilde{y}(c)} > c,$$

we conclude that the maximal positive solutions such that $\tilde{y}(0) \geq 1$ do not live for all times but the maximal interval of existence is bounded, and they go to $+\infty$ when they approach the boundary of the maximal interval. On the other hand, if $0 < \tilde{y}(0) < 1$, then, after some lap of time, we must have $\tilde{y}(t) \geq 1$. Indeed, if it is not the case, by monotonicity, we should have $0 < \lim_{t \to +\infty} \tilde{y}(t) = \ell \leq 1$, which is impossible because, in that case, $\lim_{t \to +\infty} \tilde{y}'(t) = +\infty$. We then conclude that also the positive solutions starting at $t = 0$ below 1, after some lap of time stay above a function of the type (6.9), and hence they do not last for all times.

The qualitative picture of the graph is left as exercise.

[91] We certainly have $\lim_{t \to \alpha^+} \tilde{y}(t) \in \mathbb{R}$ because in the first quadrant $\tilde{y}$ is positive and increasing.

6.3 Continuous dependence on the initial datum and phase flow of solutions

Let us consider the equation $y' = f(t,y)$, with $f : A \to \mathbb{R}^n$, $A \subseteq \mathbb{R}^{n+1}$ open. In this section, for simplicity, we will always assume the following hypothesis, and use the following notation[92]

$$\begin{array}{l} A = \mathbb{R}^{n+1},\ f \text{ is continuous and satisfies the Lipschitz condition (2.7), moreover} \\ \text{all the maximal solutions are defined for all times } t \in \mathbb{R}, \text{ and for all } (t,x) \in \mathbb{R}^{n+1}; \\ \text{we denote by } \Phi_0(t,x), \text{ the state } y(t) \in \mathbb{R}^n \text{ of the solution } y \text{ which passes through } x \\ \text{at the time } t_0 = 0, \text{ that is } y(0) = x \end{array} \tag{6.10}$$

Definition 6.7 *The function $\Phi_0 : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $(t,x) \mapsto \Phi_0(t,x)$, is said the phase flow of the solutions.*

Let us note that by definition, for every fixed x the function $t \mapsto \Phi_0(t,x)$ is just the solution of the Cauchy problem

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(0) = x, \end{cases}$$

and hence it is continuous, and derivable.

Proposition 6.8 *Let (6.10) hold. Then, the flow Φ_0 is locally Lipschitz continuous.*

For proving Proposition 6.8, we first need the following result.

Lemma 6.9 *For every compact set $K \subset\subset \mathbb{R}^n$ and for every $T > 0$, there exists a compact set $\tilde{K} \subset\subset \mathbb{R}^n$ such that*

$$\Phi_0(t,x) \in \tilde{K} \quad \forall\ (t,x) \in [-T,T] \times K.$$

Proof of Lemma 6.9. Let us fix $T > 0$. We are going to prove that, for all $x \in \mathbb{R}^n$, there exist a neighborhood U of x and a compact set $K' \subset\subset \mathbb{R}^n$ such that

$$\Phi_0(t,z) \in K' \quad \forall\ (t,z) \in [-T,T] \times U.$$

From this fact the conclusion follows, since every compact set can be covered by a finite number of neighborhoods of a finite number of its points.

[92]Similar results as the ones stated in this section also hold, with suitable modifications, in the more general case when the prolongability to infinity does not hold and also in the case of $\phi(t,t_0,x)$: the state $y(t)$ at the time t of the solution such that $y(t_0) = x$ (our case $\Phi_0(t,x)$ corresponds to the particular case $t_0 = 0$).

Moreover, see Remark 6.11 for the easier case when f is also globally bounded and Lipschitz continuous.

Let us take K' as a tubular neighborhood of the trajectory

$$\Phi_{T,x} = \left\{ \Phi_0(t,x) \middle| t \in [-T,T] \right\} \subset \mathbb{R}^n,$$

that is we fix $\delta > 0$ and define

$$K' = \left\{ \xi \in \mathbb{R}^n \middle| dist(\xi, \Phi_{T,x}) \leq \delta \right\}.$$

By the continuity of the trajectory, we easily get the compactness of K'. Indeed it is (obviously) closed and it is also bounded since so is the trajectory $\Phi_{T,x}$.

Let $L > 0$ be the Lipschitz constant of f in $[-T,T] \times K'$. Let us take $z \in \mathbb{R}^n$ and suppose that there exists $\tau \in [-T,T]$ such that

$$\Phi_0(s,z) \in K' \ \forall \ |s| \leq \tau, \quad \Phi_0(\tau,z) \in \partial K'.$$

We have, for all $|s| \leq \tau$

$$\begin{aligned} &\|\Phi_0(s,z) - \Phi_0(s,x)\| \leq \|x - z\| + \left| \int_0^s \|f(s, \Phi_0(\eta,z)) - f(s, \Phi_0(\eta,x))\| d\eta \right| \leq \\ &\|x - z\| + L \left| \int_0^s \|\Phi_0(\eta,z) - \Phi_0(\eta,x)\| d\eta \right|. \end{aligned} \tag{6.11}$$

By the Gronwall Lemma, we then conclude

$$\|\Phi_0(s,z) - \Phi_0(s,x)\| \leq \|z - x\| e^{L|s|} \leq \|z - x\| e^{LT}. \tag{6.12}$$

Hence, if we define

$$U = \left\{ z \in \mathbb{R}^n \middle| \|z - x\| \leq \frac{\delta}{2} e^{-T} \right\},$$

we get the conclusion since the trajectory starting from z starts inside K' and cannot exits from it before the interval $[-T,T]$. $\square$

Remark 6.10 *Let us point out that Lemma 6.9 says that all the solutions starting at time $t = 0$ from any point of a fixed compact set do not exit from another compact set when time is bounded in a fixed interval $[-T,T]$.*

Proof of Proposition 6.8. Let us fix $T > 0$, $K \subset \mathbb{R}^n$ compact. Let $\tilde{K} \subset \mathbb{R}^{n+1}$ compact as in Lemma 6.9, and moreover let $M > 0$ and $L > 0$ be the bound and the Lipschitz constant of f in $[-T,T] \times \tilde{K}$. For every $(t,x), (\tau,z) \in [-T,T] \times \tilde{K}$, we have, also using (6.12),

$$\begin{aligned} &\|\Phi_0(t,x) - \Phi(\tau,z)\| \leq \|\Phi_0(t,x) - \Phi_0(t,z)\| + \|\Phi_0(t,z) - \Phi_0(\tau,z)\| \leq \\ &\|x - z\| e^{LT} + M|t - \tau|. \end{aligned} \tag{6.13}$$

$\square$

Remark 6.11 *We observe here that if, besides hypothesis (6.10), we also suppose that f is bounded in $\mathbb{R}^{n+1}$ and globally Lipschitz in x uniformly in t (i.e. not only locally Lispchitz, compare with (2.11) with $]a, b[=]-\infty, +\infty[$), then we do not need Lemma 6.9. Indeed such a Lemma is used in the proof of Proposition 6.8 just for being sure that we can use some constants L and M (since we are restricted inside the compact $[-T, T] \times \tilde{K}$), and so it is not more requested when we have global boundedness and Lipschitz property. In such hypothesis we then immediately get (6.11), (6.12) and (6.13), which hold for all $(s, x), (s, z), (t, z), (\tau, z) \in [0, T] \times \mathbb{R}^n$.*

Also the proof of the following Proposition 6.12 is obviously easier if we suppose global boundedeness and Lipschitz property.

From the local Lipschitz continuity of the phase flow, we get the following stability result.

Proposition 6.12 *Let (6.10) hold. If the initial data $z \in \mathbb{R}^n$ converge to $x \in \mathbb{R}^n$, then the corresponding trajectories $\phi_0(\cdot, z)$ converge to the trajectory $\Phi_0(\cdot, x)$, uniformly on the compact sets of time. That is, for every $T > 0$ fixed,*

$$\lim_{r\to 0^+} \sup_{\|z-x\|\leq r} \sup_{t\in[-T,T]} \|\Phi_0(t, z) - \Phi_0(t, x)\| = 0.$$

Proof. By (6.12), we get

$$0 \leq \sup_{\|z-x\|\leq r} \sup_{t\in[-T,T]} \|\Phi_0(t, z) - \Phi_0(t, x)\| \leq re^{LT},$$

where $L > 0$ is the Lipschitz constant of f in $[-T, T] \times \tilde{K}$, where $\tilde{K}$ is a compact set as in Lemma 6.9 with respect to $[-T, T] \times \overline{B}(x, r_0)$, where $r_0 \geq r > 0$. Then the conclusion immediately follows. $\square$

We now state, without proof, the following result on the differentiability of the flow.

Proposition 6.13 *If besides (6.10), f has also continuous partial derivatives with respect to x, that is: for every $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, for every $i \in \{1, \dots, n\}$*

$$\frac{\partial f}{\partial x_i}(t, x) \quad \text{exits and is continuous,}$$

then also all the partial derivatives

$$\frac{\partial \Phi_0}{\partial x_i}(t, x),$$

exist and are continuous for all (t, x). In particular this means that the flow is differentiable with respect to the state variable x. Moreover, also the mixed second derivatives

$$\frac{\partial^2 \Phi_0}{\partial t \partial x_i},$$

exist and are continuous.

Remark 6.14 *The point of view of the phase flow is a change in the way of looking to the Cauchy problems and to the solutions. The new way consists in interpreting the solutions as a map from* $\mathbb{R}^n$ *(the phase space) into itself: when we have fixed a time* t*, every point* $x \in \mathbb{R}^n$ *is mapped to another point which is the state at time* t *of the solution which passes through* x *at time* 0*. In some sense we think to the trajectory as a rule to transport points from one site to another one*[93]*. This is the point of view of a general theory whose name is* dynamical system theory*.*

By the uniqueness of the solution, we easily get the following (important) group property *of the phase flow*[94]*: for all* $x \in \mathbb{R}^n$*, and for all* $t, s \in \mathbb{R}$*, we have:*

$$\Phi_0(t+s, x) = \Phi_0(t, \Phi_0(s, x)),$$

that is: the state at the time $t+s$ *of the solution which passes through* x *at the time* 0 *is the state at the time* t *of the solution that passes through* $\Phi_0(s, x)$ *at time* 0*. Very roughly speaking we can say that "pieces of trajectories are trajectories suitably shifted in time".*

[93] As a parallelism: if we look to the stream inside a river we can fix a point and look to the water which passes through it, or we can put a paper-boat on that point and look to its movement: at every time t the boat will be in a new position, and its movement is along the integral curves (solutions) of the velocity field (the dynamics).

[94] Often, such a property holds only for positive times $t > 0$, and so it is referred as a semigroup property.

7 Autonomous systems

7.1 General results. First integrals

Let us consider an autonomous system

$$y'(t) = f(y(t)), \tag{7.1}$$

where $f : A \to \mathbb{R}^n$, with $A \subseteq \mathbb{R}^n$ open, is locally Lipschitz (and hence we have local existence and uniqueness for every initial state, and in particular existence and uniqueness of the maximal solution).

Definition 7.1 *If $y : I \to \mathbb{R}$ is a solution of (7.1), by trajectory or orbit we mean the image of y, that is the curve in $\mathbb{R}^n$ (the phase space) described by the function/parametrization y.*

The fact that the system is autonomous, that is the dynamics f does not explicitly depend on the time t, permits to study the solutions by studying their orbits in the phase space, and then to get several interesting results.

To simplify notations and proofs, we will often assume the following hypothesis:

$$A = \mathbb{R}^n, \quad \text{all the maximal solutions are defined for all time } t \in \mathbb{R}\ ^{95}. \tag{7.2}$$

Here are some first results that hold because of the autonomy.

Proposition 7.2 *i) If y is a solution of (7.1), and $c \in \mathbb{R}$, then the function*

$$\psi : t \mapsto y(t + c)$$

is still a solution.

ii) If y_1 and y_2 are two solutions such that, for some $t_1, t_2 \in \mathbb{R}$ it is

$$y_1(t_1) = y_2(t_2),$$

then we must have

$$y_1(t) = y_2(t + t_2 - t_1)\ \forall\ t \in \mathbb{R}.$$

Proof. i) Just deriving

$$\psi'(t) = y'(t + c) = f(y(t + c)) = f(\psi(t)).$$

ii) For the first point i), $\overline{y}_2(t) = y_2(t + t_2 - t_1)$ is a solution, and in particular $\overline{y}_2(t_1) = y_2(t_2) = y_1(t_1)$. Then, by uniqueness, $\overline{y}_2 = y_1$. □

[95] However, we will later see some quite easy controllable properties which guarantee the existence for all time.

Remark 7.3 *From Proposition 7.2, we have that any orbit corresponds to a one-parameter family of solutions* $y(t) = y(t + c)$*: if a solution describes an orbit, then every its translation in time describes the same orbit, moreover if two solutions describe the same orbits then they must be the same solution translated in time. Also, two orbits cannot intersect each other: the orbits give a partition of the phase space.*

Remark 7.4 *Let us note that every non autonomous system*

$$y'(t) = f(t, y(t))$$

can be written as an autonomous one just adding the fictitious state-variable t*. Indeed, writing* $\tilde{y} = (t, y)$ *and* $\tilde{f}(\tilde{y}) = (1, f(\tilde{y}))$*, we get*

$$\tilde{y}' = \tilde{f}(\tilde{y}).$$

Of course, in doing that, we have paid the fact that we passed to the larger dimension $n + 1$*.*

Definition 7.5 *A point* $x \in \mathbb{R}^n$ *is said an* <u>*equilibrium point*</u> *(or a* <u>*critic/singular point*</u>*) of the system if* $f(x) = 0$*. It is obvious that, if* x *is an equilibrium point, then the set* $\{x\} \subset \mathbb{R}^n$ *is an orbit, since the function* $y(t) \equiv x$ *is a solution. Such an orbit is sometimes called a* <u>*stationary orbit.*</u>

Proposition 7.6 *If* $x \in \mathbb{R}^n$ *is the limit of a solution when* $t \to \pm\infty$*, then* x *is an equilibrium point. Moreover, a non stationary solution* y *cannot pass through an equilibrium point.*

Proof. The second assertion is obvious, since an equilibrium point is an orbit. Let us prove the first one, for $t \to +\infty$. By absurd, let us suppose that $f(x) \neq 0$. Then there exists a unit versor $\nu \in \mathbb{R}^n$ such that $f(x) \cdot \nu > 0$. Let us take a small ball around x, B, such that, by the continuity of f, for a suitable fixed $\varepsilon > 0$,

$$y \in B \implies f(y) \cdot \nu > \varepsilon > 0.$$

Since by hypothesis of convergence $y(t) \in B$ definitely for $t \geq \bar{t}$ (because $y(t) \to x$ as $t \to +\infty$), for a suitable $\bar{t}$, we have, for all $t \geq \bar{t}$, and integrating in $[\bar{t}, t]$,

$$y'(t) \cdot \nu = f(y(t)) \cdot \nu > \varepsilon \implies y(t) \cdot \nu \geq y(\bar{t}) \cdot \nu + \varepsilon(t - \bar{t}) \to +\infty \text{ as } t \to +\infty,$$

which is an absurd since $y(t) \cdot \nu \to x \cdot \nu \in \mathbb{R}$. [96] □

[96] Roughly speaking: when $y(t) \in B$, by the absurd hypothesis, the trajectory has a scalar velocity which, with respect to the direction ν, is not less than $\varepsilon > 0$; $y(t) \in B$ for all time $t \geq \bar{t}$; these two facts imply that $y(t)$ must exit from the bounded set B. Contradiction.

Definition 7.7 *A solution y is said to be* <u>*periodic*</u> *if there exists a time $T > 0$ (the* <u>*period*</u>*) such that*

$$y(t+T) = y(t) \quad \forall t, \quad y(t+s) \neq y(t) \quad \forall s \in]0, T[.$$

Note that, by this definition, the constant trajectories (i.e. the equilibrium points) are not periodic.

Proposition 7.8 *For an autonomous system with uniqueness, there are only three types of orbits: singular equilibrium points, simple closed curves (i.e* <u>*cycles*</u> *without transversal/tangential self-intersections) which are periodic, and simple open curves (i.e open curves without transversal/tangential self-intersections).*

Proof. The fact that an orbit cannot have a self-intersection is obvious by uniqueness. It is also obvious that the orbit of a periodic solution is a cycle. Hence, we have only to prove that if a non constant solution y is such that $y(t_1) = y(t_2)$ for some $t_2 > t_1$, then it is periodic[97].

Let us define $\delta = t_2 - t_1$. We guess that $y(t + \delta) = y(t)$ for all t. Indeed, the function $\psi : t \mapsto y(t + \delta)$ is still a solution and it satisfies $\psi(t_1) = y(t_2) = y(t_1)$. Hence it coincides with y and the guess is proved. We obtain the periodicity of y, in the sense of Definition 7.7 if we prove that

$$T = \inf \left\{ \tau > 0 \middle| y(t + \tau) = y(t) \ \forall t \right\} > 0.$$

Let $\mathcal{P}$ be the set whose infimum we are going to consider. Note that $\mathcal{P}$ is not empty since $\delta \in \mathcal{P}$. Moreover, let us note that if $\tau \in \mathcal{P}$ then $m\tau \in \mathcal{P}$ for all positive integers m. Also, since y is continuous, $\mathcal{P}$ is closed in $]0, +\infty[$, that is if $\tau > 0$, $\tau_n \to \tau$ and $\{\tau_n\}_n \subset \mathcal{P}$, then $\tau \in \mathcal{P}$ (i.e. every strictly positive accumulation point of $\mathcal{P}$ belongs to $\mathcal{P}$ itself). By absurd, let us suppose that $T = 0$. Then, for every $\varepsilon > 0$ there exists $\tau \in \mathcal{P}$ such that $0 < \tau < \varepsilon$. Moreover, fixed such a τ, for any real number $c > 0$, for the archimedean property of $\mathbb{R}$, we find $m \in \mathbb{N} \setminus \{0\}$ such that

$$(m-1)\tau \leq c \leq m\tau \implies 0 \leq m\tau - c \leq \tau \leq \varepsilon,$$

which means that $c > 0$ is an accumulation point of $\mathcal{P}$ and so that $c \in \mathcal{P}$. For the arbitrariness of $c > 0$, we conclude that y is constant, which is a contradiction. $\square$

Definition 7.9 *A C^1 function $E : \mathbb{R}^n \to \mathbb{R}$ is said a* <u>*first integral*</u> *of the motion for the system if E is constant along any trajectory of the system. This in particular means that the function $t \mapsto E(y(t))$ is constant for any trajectory y, which is equivalent to say that its derivative is zero, that is*

[97] As stated in the beginning of the section, we are supposing that all the solutions are defined for all times. However, it can be easily proved that, if a non constant solution satisfies $y(t_1) = y(t_2)$ for some two different instants t_1, t_2 of its interval of definition, then it is prolongable for all times (and also periodic).

$$\nabla E(y(t)) \cdot y'(t) = 0 \implies \nabla E(y(t)) \cdot f(y(t)) = 0 \ \forall y(\cdot).$$

Since the trajectories are a partition of the phase space $\mathbb{R}^n$, we can equivalently say that E is a first integral if and only if

$$\nabla E(x) \cdot f(x) = 0 \ \ \forall \ x \in \mathbb{R}^n.$$

Remark 7.10 *If E is a first integral, then every orbit is entirely contained in a level set of E. Moreover, in the general case of a maximal orbit y defined in $\tilde{I}$, if y is contained in a bounded level set of E, then $\tilde{I} =]-\infty, +\infty[$. Indeed, in that case the derivatives are bounded. In particular, if E has all the level set bounded, then every solution is prolongable for all times.*

7.2 Bidimensional systems

The particular case of bidimensional system is quite favorable. Indeed the phase-space is the plane $\mathbb{R}^2$, where we can easier draw and analyze the orbits. Moreover, it can be easier to find a possible first integral and also, since the level sets of the first integrals are (generally) curves, the orbits coincide with (at least) pieces of such curves.

Proposition 7.11 *Let us consider the bidimensional system*

$$\begin{cases} x' = F(x, y) \\ y' = G(x, y). \end{cases} \tag{7.3}$$

i) The equilibrium points are the solution of the (nonlinear) algebraic system

$$\begin{cases} F(x, y) = 0 \\ G(x, y) = 0. \end{cases}$$

ii) If $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a potential of the differential 1-form

$$\omega(x, y) = G(x, y)dx - F(x, y)dy,$$

then φ is a first integral for the system (7.3).

iii) If a level set of a first integral E is a simple closed curve[98] *which does not contains equilibrium points for the systems, then it exactly coincides with a periodic orbit.*

Proof. The point i) is obvious by definition.

For the point ii), let $(x(\cdot), y(\cdot))$ be a solution, then we have

$$\frac{d}{dt}\varphi(x(t), y(t)) = G(x(t), y(t))x'(t) - F(x(t), y(t))y'(t) = \\ G(x(t), y(t))F(x(t), y(t)) - F(x(t), y(t))G(x(t), y(t)) = 0.$$

[98]Note that, in this setting, "closed curve" also implies that it is bounded and of finite length.

iii) Since the curve is bounded, then the solution is defined for all times. Since its scalar velocity $\|f(y(t))\|$ is uniformly greater then zero (there are no equilibrium points and the curve is compact), then the trajectory must pass two times through the same point. Hence, it is periodic. $\square$

Remark 7.12 *If the differential form $\omega = Gdx - Fdy$ is not exact, but it has an integrand factor $\lambda(x, y) > 0$, then a potential E of the form $\lambda\omega$ is still a first integral of the system.*

7.2.1 Qualitative studies (III)

Here we sketch a list of points which may be addressed when studying the qualitative behavior of the orbits for a bidimensional autonomous system.

i) Find the possible equilibrium points.

ii) Find a possible first integral (for instance searching for a potential of the associated differential form).

iii) If a first integral E exists, then study E: stationary points, relative maximum points, relative minimum points, saddle points, absolute maxima and minima... This may permit to understand the behavior of the level curves, which are the projections on $\mathbb{R}^2$ of the intersections in $\mathbb{R}^3$ between the graph of E and the horizontal planes. Another way may be directly study the level curves in $\mathbb{R}^2$ via their implicit formulations $E(x, y) = c$, at least when such equation is (easily) invertible with respect to x or to y. Finally, also some properties of E as convexity and coercivity[99] may be useful.

iv) Recall that: orbits may not intersect each other, the orbits form a partition of the phase space. Moreover if a level curve is a closed curve that does not contain equilibrium points, then it coincides with a cycle (a periodic orbit).

v) Check, if possible, whether the solutions are defined for all times or not. This can be done, for instance, looking to the boundedness of the level curves of E.

vi) Note that a closed level curve of E (and hence a cycle) must moves around a stationary point of E[100].

vii) Study some suitable level curves of E. For instance the ones passing through the equilibrium points, or the zero-level curves, which may be easier to be studied.

viii) Find the versus of moving along the orbits. This can be done by studying the sign of F and G respectively. Also note that, by the continuity of F and G, such a versus is "continuous", since it is the versus of the tangent vector (F, G). Hence, we cannot approximate an orbit with other orbits moving in opposite direction.

[99] "Coercivity" means that $\lim |E(x)| = +\infty$ when $\|x\| \to +\infty$, or more generally, when x approximates the boundary of the domain of E.

[100] Again, by "closed curve" we also mean that it is bounded, and hence it is compact (since, being a cycle, it is certainly "topologically closed"). Since it is a level curve of E, that is E is constant on it, then, by a simple generalization of the one-dimensional Rolle theorem, there must be a stationary point of E in the region inside the curve. Also note that the stationary points of E are strictly related to the equilibrium points (they almost always coincide).

Example 7.13 Let us recall the Lotka-Volterra system

$$\begin{cases} x' = (\alpha - \beta y)x \\ y' = (-\gamma + \delta x)y, \end{cases}$$

where $\alpha, \beta, \gamma, \delta > 0$ are fixed, and we are looking for solutions $(x(t), y(t))$ in the first quadrant only, that is $x(t), y(t) > 0$.

The only equilibrium point is $(x_0, y_0) = (\gamma/\delta, \alpha/\beta)$, and a first integral is[101]

$$E(x, y) = -\gamma \log x + \delta x - \alpha \log y + \beta y.$$

The study of E gives that: (x_0, y_0) is the only stationary point of E and it is the absolute minimum, E is strictly convex[102], E is coercive, that is it tends to $+\infty$ when x or y tends to 0 (i.e. when the point (x, y) tends to the axes.), and also when $x, y \to +\infty$. Hence, its level curves are closed curves around (x_0, y_0).

We easily conclude that the orbits are periodic (cycles), they are defined for all times and that they counterclockwise move around the equilibrium point.

7.2.2 Some exercises

1) Let us consider the second order autonomous scalar equation

$$y'' = f(y).$$

As usual we can transform it in a first order autonomous bidimensional system

$$\begin{cases} y_1' = y_2 \\ y_2' = f(y_1). \end{cases}$$

Prove that, if F is a primitive of f, then

$$E(y_1, y_2) = \frac{y_2^2}{2} - F(y_1)$$

is a first integral for the system[103].

2) For the following systems/second order equations, plot a qualitative picture of the orbits, and check, if possible, whether the solutions are defined for all times, and whether the equilibrium points are stable, asymptotically stable or unstable[104].

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[101] It can be found using the integrand factor $1/(xy)$.

[102] Its Hessian matrix is everywhere positively definite.

[103] In a mechanical point of view, y_1 is the position and y_2 is the velocity. Hence E is the "total energy" of the system: kinetic energy plus potential energy. Since the trajectories move along the level curves of E, that is E is constant along the trajectories, then the system is conservative: the total energy is kept constant.

[104] For the concept of stability see next paragraph.

2i)

$$\begin{cases} x' = (3-x)(x+2y-6) \\ y' = (y-3)(2x+y-6). \end{cases}$$

2ii)

$$\begin{cases} x' = 3y^2 \\ y' = 3x^2. \end{cases}$$

2iii)

$$y'' = y^2 + 2y.$$

2iv) (pendulum without friction)

$$y'' = -k \sin y, \quad k > 0.$$

Some words. The equation is the model for the oscillations of a point of mass m hanged to an extremum of a rigid rod which has negligible mass, length equal to ℓ and is free to rotate in a vertical plane around its other (fixed) extremum, only subject to the gravity force. Denoting by φ the radiant angle of the rod with respect to the downward position, the Newton equation of the motion is

$$m\ell\varphi''(t) = -mg \sin(\varphi(t)),$$

which corresponds to our equation with $y = \varphi$ and $k = g/\ell$.

Following the first exercise of this section, we write our equation as the autonomous bidimensional system ($z_1 = y$ angle, $z_2 = y'$ angular velocity)

$$\begin{cases} z_1' = z_2 \\ z_2' = -k \sin z_1, \end{cases}$$

which, by periodicity, may be studied only for $z_1 \in [-\pi, \pi]$, and hence the equilibrium points are $(-\pi, 0), (0, 0), (\pi, 0)$. A first integral is

$$E(z_1, z_2) = \frac{1}{2} z_2^2 + (k - k \cos z_1) \geq 0,$$

and hence the level curves are the curves of equations

$$z_2 = \pm\sqrt{2(c - k + k \cos z_1)}, \quad c \geq 0.$$

Analyzing all the case for $c \geq 0$ we get that there exist: 1) cycles around the equilibrium point $(0, 0)$, 2) heteroclinic[105] orbits connecting the equilibrium points $(-\pi, 0), (\pi, 0)$, 3) open orbits (not connecting any equilibrium points: they are open in the strip $[-\pi, \pi] \times \mathbb{R}$, but they indeed reply by periodicity in the whole $\mathbb{R}^2$).

2v)

$$y'' = -ye^y.$$

[105] In general, orbits connecting two different equilibrium points are called "heteroclinic", whereas orbits connecting the same equilibrium point are called "homoclinic".

7.3 Stability

To study the stability of the system means to understand how much the trajectories are sensible in (small) change of the initial value: if we little change the initial value, what happens to the trajectory? Does it remain "near" to the initial one or not?

We have already met a "stability result": Proposition 6.12. It says that, on the compact sets of time, if we little change the initial point x, then the trajectories do not "change to much": they uniformly converge to the trajectory starting from x. But that proposition says nothing about what happens when $t \to +\infty$: a stability result for compact set of time does not imply stability for all time, the distance between the trajectories may diverge when $t \to +\infty$.

In this section we assume that all the solutions are defined for $t \in [0, +\infty[$ and we address their behavior when $t \to +\infty$ (a similar analysis may be done when $t \to -\infty$). We are going to use the (flow-) notation $\Phi_0(\cdot, x)$ for the solution starting from x at $t = 0$. For all this section we assume that the autonomous system $y' = f(y)$ satisfies the usual hypothesis for existence and uniqueness.

This section is rather sketched.

Definition 7.14 *Let $x \in \mathbb{R}^n$ be fixed. The solution $\Phi_0(\cdot, x)$ is said:*
i) <u>stable</u> if: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$z \in B(x, \delta) \implies \|\Phi_0(t, x) - \Phi_0(t, z)\| \leq \varepsilon \ \forall\ t \geq 0;$$

ii) <u>asymptotically stable</u> if: it is stable and moreover

$$\lim_{t \to +\infty} \|\Phi_0(t, x) - \Phi_0(t, z)\| = 0 \ \forall\ z \in B(x, \delta);$$

iii) <u>unstable</u> in all the other cases.

Remark 7.15 *Point i) of the previous definition says that the trajectory $\Phi_0(\cdot, x)$ is stable if for every "tubular" neighborhood of it, there exists a ball around x such that, starting from any point of such a ball, we remain inside the tube for all $t \geq 0$. Point ii) does not only require the remaining inside the tube, but also requires that we better and better approximate the trajectory $\Phi_0(\cdot, x)$.*

It is interesting, both from a theoretical and applicative point of view, to study the stability of the equilibrium points. In that case, the trajectory is just the point, and hence it is stable if we can remain as a close to it as we want. It is asymptotically stable if we also converge to the equilibrium point.

In the case of asymptotically stable equilibrium point x, we define its <u>basin of attraction</u> as

$$\Omega = \left\{ z \in \mathbb{R}^n \middle| \lim_{t \to +\infty} \Phi_0(t, z) = x \right\}.$$

We say that x is globally asymptotically stable if $\Omega = \mathbb{R}^n$, that is, whichever the initial point is, we converge to x.

If x is an equilibrium point, that is $f(x) = 0$, then, by the change of variable and dynamics: $z = y - x$ and $g(z) = f(z + x)$, we get the equivalent system $z' = g(z)$ with $\zeta = 0$ as equilibrium point. Hence we may restrict the study to the case where the equilibrium point is the origin.

7.3.1 Stability for linear systems

For the linear (homogeneous) systems $y' = Ay$, the study of the stability of the equilibrium point $x = 0$ is rather easy[106].

Proposition 7.16 *Given the linear homogeneous system $y' = Ay$, the origin is:*

i) globally asymptotically stable if and only if $Re(\lambda) < 0$ for all $\lambda \in \mathbb{C}$ eigenvalues of the matrix A;

ii) stable (but not asymptotically) if and only if $Re(\lambda) \leq 0$, there exists a pure imaginary eigenvalue (i.e. $Re(\lambda) = 0$), and all the pure imaginary eigenvalues has algebraic multiplicity 1;

iii) unstable in all the other cases.

Proof (very sketched). Just arguing as for the linear n-order scalar equation, it can be easily seen that the solutions of the linear system are linear combination of addenda of the following type

$$ht^m e^{Re(\lambda)t}(\cos(Im(\lambda)t) \pm \sin(Im(\lambda)t)),$$

where $h \in \mathbb{R}^n$ is a suitable non null vector. Hence, point i) is almost immediate. For point ii) just observe that we are requiring that, if $Re(\lambda) = 0$, then the multiplicity is 1. This implies that $m = 0$ and hence the addendum is $h(\cos(Im(\lambda)t) \pm \sin(Im(\lambda)t))$ which does not converge to the origin but stays there around. $\square$

7.3.2 On the Liapunov function

For the general case of a nonlinear system $y' = f(y)$, with $f(0) = 0$, the stability of the origin can be studied by the help of a suitable function.

Definition 7.17 *Let $A \subseteq \mathbb{R}^n$ be a neighborhood of the origin. A C^1 function $V : A \to \mathbb{R}$ is said a Liapunov function for the system if*

i) (positively definite) $V(x) \geq 0$ for all $x \in A$ and $V(x) = 0 \iff x = 0$

ii) (decreasing along trajectories) $\nabla V(x) \cdot f(x) \leq 0$ for all $x \in A$.

[106] Note that $x = 0$ is always an equilibrium point for a linear homogeneous system, since $A0 = 0$.

Note that the condition ii) says that, given a trajectory $y(\cdot)$ in A, then the function of time $t \mapsto V(y(t))$ is not increasing.

We do not prove the following theorem.

Theorem 7.18 *If there exists a Liapunov function, then the origin is stable. Moreover, if the Liapunov function is strictly decreasing along the trajectories (i.e. $\nabla V(x) \cdot f(x) < 0$), then the origin is asymptotically stable.*

Remark 7.19 *The Liapunov function is something like a first integral (even if in general is harder to be found). The difference is the following: for the case of the first integral, the trajectories live for all times inside the level set, instead for the Liapunov function, the trajectories tend to leave the level set and to point towards the origin (the lowest value of V). For instance, in the case of strict decreasing along the trajectories, let S_c be the level set of value $c > 0$ for V. Then ∇V is orthogonal to S_c and points towards the higher value of V. Hence the condition $\nabla V \cdot f < 0$ means that the field f is strictly pointing towards the lower value of V and so the trajectory $y(t)$ enters in the region $V < c$. Since this happens at all times, in the limit the trajectory tends to the origin (the lowest value).*

7.3.3 On the linearization method

Another way to study the stability in the nonlinear case is, as it usually happens for nonlinear problems, to make a linearization and try to apply the already known results for the linear case.

Proposition 7.20 *Let the origin be an equilibrium point for $y' = f(y)$, with f of class C^1. We may expand f with the first-order Taylor formula around the origin*[107]

$$f(x) = Df(0)x + o(\|x\|) \quad x \to 0,$$

where $Df(0)$ is the Jacobian matrix of f in 0. Let us consider the linearized system $y' = Df(0)y$, and suppose that $Df(0)$ is not singular. If the origin is asymptotically stable for the linearized system, then it is also asymptotically stable for the nonlinear system. In general, the converse is not true.

7.3.4 On the limit cycles

We know that if a trajectory converges to a point x for $t \to +\infty$, then x is an equilibrium point. However, the trajectories may in general have other behaviors as $t \to +\infty$. For instance they may approximate (or tend to) a cycle given by a periodic orbit. In that case, the behavior of the trajectory is something like a spiral which moves around the cycle and tends to it, without reaching it, of course. In that case we say that such a cycle is a <u>limit cycle</u>.

To check the existence of a limit cycle is of course harder than checking the existence of an equilibrium point. We state the following theorem without proof. It holds for bidimensional systems.

[107]Since the origin is an equilibrium, then $f(0) = 0$.

Theorem 7.21 *(Poincaré-Bendixson). Let us consider a bidimensional autonomous system $y' = f(y)$, with $f \in C^1$. If there exists a bounded open set $\Omega \subseteq \mathbb{R}^2$ such that: every trajectory which enters in it will stay inside Ω for all the other times, and such that it does not contains equilibrium points, then there exists a limit cycle inside Ω.*

8 Appendix

8.1 Uniform continuity

Definition 8.1 *Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a set and a function. We say that f is* <u>*uniformly continuous on A*</u> *if*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \ x, y \in A, \ \|x - y\|_{\mathbb{R}^n} \leq \delta \implies \|f(x) - f(y)\|_{\mathbb{R}^m} \leq \varepsilon.$$

Note that such a definition, with respect to the usual $\varepsilon - \delta$ definition of continuity, says that, when $\varepsilon > 0$ is fixed, the amplitude $\delta > 0$ can be taken uniformly all around A, that is independently from the points x and y.

Definition 8.2 *A function $\omega : [0, +\infty[\to [0, +\infty[, \ r \mapsto \omega(r)$ is a* <u>*modulus of continuity*</u> *if $\omega(0) = 0$ and if it is increasing and continuous at $r = 0$.*

The following proposition is a useful characterization of the uniform continuity.

Proposition 8.3 *Referring to the same notations as in Definition 8.1, f is uniformly continuous if and only if there exists a modulus of continuity ω such that*

$$\|f(x) - f(y)\|_{\mathbb{R}^m} \leq \omega(\|x - y\|_{\mathbb{R}^n}) \ \ \forall \ x, y \in A.$$

Proof. The sufficiency is easy. For the necessity, just define

$$\omega(r) := \sup_{x,y \in A \ \|x-y\| \leq r} \|f(x) - f(y)\|$$

and prove that it is a modulus of continuity satisfying the request. $\square$

The following theorem is a very popular result. The proof may be found on every text book.

Theorem 8.4 *If A is compact and f is continuous in A, then f is also uniformly continuous on A.*

Also the following proposition is often useful.

Proposition 8.5 *Let $A \subseteq \mathbb{R}^n$ be a nonempty set, and $f : A \to \mathbb{R}^m$ be a uniform continuous function. Then, there exists a unique uniformly continuous function $\tilde{f} : \overline{A} \to \mathbb{R}^n$, where $\overline{A}$ is the closure of A, which extends f, that is $\tilde{f}(x) = f(x)$ for every $x \in A$.*

Proof. Let us fix $\overline{x} \in \overline{A}$. Then there exists a sequence x_k of points of A such that $x_k \to \overline{x}$ when $k \to +\infty$. Hence, being convergent, x_k is a Cauchy sequence in $\mathbb{R}^n$, that is, for every $\varepsilon > 0$ there exists $\overline{k} \in \mathbb{N}$ such that $\|x_i - x_j\|_{\mathbb{R}^n} \leq \varepsilon$ for every $i, j \geq \overline{k}$. We consider the sequence $f(x_k)$ in $\mathbb{R}^m$. We are going to prove that it is a Cauchy sequence and hence, being $\mathbb{R}^m$ complete, convergent[108]. Let us fix $\varepsilon > 0$, then, by the uniform continuity of f, there exists $\delta > 0$ such that, for every $x, y \in A$, with $\|x - y\| \leq \delta$, we have $\|f(x) - f(y)\| \leq \varepsilon$. Now, let us take $\overline{k}$ such that, for $i, j \geq \overline{k}$, $\|x_i - x_j\| \leq \delta$. Hence we also have $\|f(x_i) - f(x_j)\| \leq \varepsilon$.

Now, let $y \in \mathbb{R}^m$ be the limit of the sequence $f(x_k)$, we prove that y is independent from the particular sequence in A, convergent to $\overline{x}$. Indeed, let x'_k be another sequence in A convergent to $\overline{x}$. By the same argument as above, the sequence $f(x'_k)$ converges to a point y'. But, for every $\varepsilon > 0$, we find $\overline{k}$ such that, for $k \geq \overline{k}$, $\|x_k - x'_k\| \leq \varepsilon$ (since they are converging to the same point $\overline{x}$). Hence, by the uniform continuity of f, we have $\|f(x_k) - f(x'_k)\| \leq \omega_f(\|x_k - x'_k\|) \leq \omega_f(\varepsilon)$, where ω_f is a modulus of continuity for f on A. We then get that $f(x_k)$ and $f(x'_k)$ must converge to the same value $y = y'$.

The following definition is then well-defined, $\tilde{f} : \overline{A} \to \mathbb{R}^m$

$$\tilde{f}(x) = y \quad \text{where } y \text{ is defined as above.}$$

Note that, if $x \in A$, then applying the construction of $\tilde{f}(x) = y$ as above, we immediately get $y = f(x)$: just take the constant sequence $x_k \equiv x$. Hence $\tilde{f}$ is an extension of f.

The reader is invited to prove that $\tilde{f}$ is uniformly continuous in $\overline{A}$ and that it is the unique possible uniform extension of f (take another $g : \overline{A} \to \mathbb{R}^m$ which extends f and is uniformly continuous and prove that $\tilde{f} = g$). □

8.2 Uniform convergence of functions

Definition 8.6 *Let $f_n : A \subseteq \mathbb{R}^p \to \mathbb{R}^m$ be a sequence of functions, with $A \subseteq \mathbb{R}^p$ a subset. Moreover, let $f : A \to \mathbb{R}^m$ be a function. We say that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ uniformly converges to f, on A if*

$$\forall \varepsilon > 0 \ \exists \overline{n} \in \mathbb{N} \textit{ such that } n \geq \overline{n} \implies \|f_n(x) - f(x)\|_{\mathbb{R}^m} \leq \varepsilon \ \forall \ x \in A,$$

which is equivalent to say that

$$\lim_{n\to+\infty} \sup_{a\in A} \|f_n(x) - f(x)\|_{\mathbb{R}^m} = \lim_{n\to+\infty} \|f_n - f\|_{C^0(A;\mathbb{R}^m)} = 0$$

We immediately note the difference of such a definition with the other definition of pointwise convergence to f on A:

$$\lim_{n\to+\infty} f_n(x) = f(x) \quad \forall \ x \in A.$$

[108] The completeness means that every Cauchy sequence is convergent. And it is well-known that $\mathbb{R}^m$ has such a property, for every m.

Indeed, the pointwise convergence says that, for every fixed $x \in A$, the sequence of vectors $(f_n(x))_{n\in\mathbb{N}}$ converges in $\mathbb{R}^m$ to the vector $f(x)$, but it does not say anything about the "velocity of such a convergence": for any point x the convergence velocity may be completely independent from the velocity of the convergence in other points. In particular, once $\varepsilon > 0$ is fixed, it may be not possible to find n such that the values $f_n(x)$ is far from the limit value $f(x)$ not more than ε, and such that this happens for all $x \in A$. Instead, this is exactly what the uniform convergence says. From a geometrical point of view, the uniform convergence says that, for every width (ε) of a tubular neighborhood (a tube) around the graph of f over A, we can find $\overline{n}$ such that, from $\overline{n}$ on, the graphs of all the functions f_n are entrapped inside such a neighborhood.

A first interesting result about the uniform convergence is the following, whose proof may be found in any text book.

Proposition 8.7 *If $f_n : A \to \mathbb{R}^m$ is a sequence of continuous functions uniformly converging on A to a function $f : A \to \mathbb{R}^m$, then f is continuous too.*

8.2.1 Change between the limit sign and the sign of integral

One of the major point of interest of the uniform convergence is the following Proposition about the passage of the limit sign under the sign of integral. The proof may be found on any text book.

Proposition 8.8 *Let $f_n : A \to \mathbb{R}^m$ be a sequence of integrable functions uniformly converging to a function $f : A \to \mathbb{R}^m$ on A*[109]*. Then f is also integrable on A and*

$$\int_A f(x)dx = \left(= \int_A \left(\lim_{n\to+\infty} f_n(x) \right) dx \right) = \lim_{n\to+\infty} \int_A f_n(x)dx. \tag{8.1}$$

Let us note that the uniform convergence is only a sufficient condition for passing with the limit inside the integral. It is not necessary: we may require a little less than the uniform convergence, still preserving the sufficiency for (8.1). However, the pointwise convergence is not more sufficient: it is too few. Take for instance the sequence of functions in $[0,1]$

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x \le \frac{1}{n}, \\ -n^2 x + 2n & \text{if } \frac{1}{n} \le x \le \frac{2}{n}, \\ 0 & \text{if } \frac{2}{n} \le x \le 1. \end{cases}$$

The sequence f_n pointwise converges to the null function $f \equiv 0$, but the integrals do not converge

$$\int_0^1 f_n(x)dx = 1 \ \forall n > 0, \ \int_0^1 f(x)dx = 0.$$

[109] The integral of a vector-valued function is defined as the vector whose components are the integral of the (scalar-valued) components of the function.

Proposition 8.9 *Let us consider $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$, $f : A \to \mathbb{R}^p$ uniformly continuous, and $z_k : B \to A$ uniformly convergent to $z : B \to A$ on B. Then, the sequence of functions $f \circ z_k : B \to \mathbb{R}^p$ uniformly converges to the function $f \circ z$ on B.*

Proof. The thesis immediately follows from our hypotheses and from the following inequality, which holds for every $x, y \in B$, and for all $k \in \mathbb{N}$,

$$\|f(z_k(x)) - f(z(x))\|_{\mathbb{R}^p} \leq \omega_f(\|z_k(x) - z(x)\|_{\mathbb{R}^n}),$$

where ω_f is a modulus of continuity for f on A. □

8.2.2 The Ascoli-Arzelà Theorem

The following theorem is of fundamental importance for proving existence results in the theory of ordinary differential equations[110].

Theorem 8.10 *Let $K \subset \mathbb{R}^m$ be a compact set and $f_n : K \to \mathbb{R}^p$ be a sequence of functions. Let us suppose that the following hypotheses are satisfied*

i) equiboundedness: there exists a real number $M > 0$ such that

$$\|f_n(x)\|_{\mathbb{R}^p} \leq M \ \forall\ x \in K \ \forall\ n;$$

ii) equicontinuity[111]*: for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$x, y \in K,\ \|x - y\|_{\mathbb{R}^m} \leq \delta \implies \|f_n(x) - f_n(y)\|_{\mathbb{R}^p} \leq \varepsilon \ \forall\ n.$$

Then there exist a continuous function $f : K \to \mathbb{R}^p$ and a subsequence f_{n_i} such that f_{n_i} uniformly converges to f on K, as $i \to +\infty$.

Proof. Let $A \subseteq K$ be a numerable set which is dense in K[112]. Let us enumerate A

$$A = \{x_1, x_2, x_3, \ldots, x_h, \ldots\}.$$

We first consider the sequence in $\mathbb{R}^p$: $\{f_n(x_1)\}_n$[113]. By the equiboundedness, such a sequence is bounded in $\mathbb{R}^p$, and hence it has a convergent subsequence. We denote the limit by y_1 and the subsequence by

$$f_{n^1_j(x_1)}, \quad \lim_{j\to+\infty} f_{n^1_j}(x_1) = y_1 \in \mathbb{R}^p.$$

[110] Of course, it is of fundamental importance in many other contexts.

[111] Actually, this is an equi-uniform continuity

[112] For any $n \in \mathbb{N} \setminus \{0\}$ take a covering of K given by a finite number of balls of radius $1/n$. For any of such balls take a point belonging to K.

[113] Note that it is a sequence of vectorial elements (in $\mathbb{R}^p$). Here convergence is obviously equivalent to the convergence of the sequences of the components.

Hence, we have a sequence of functions, $f_{n_j^1}$, which converges in x_1, Let us evaluate such a sequence in x_2, obtaining, as before, the bounded sequence $f_{n_j^1}(x_2)$ in $\mathbb{R}^p$. Again, such a sequence has a convergent subsequence. We denote the limit by y_2 and the subsequence by

$$f_{n_j^{1,2}}(x_2), \quad \lim_{j\to+\infty} f_{n_j^{1,2}}(x_2) = y_2 \in \mathbb{R}^p.$$

Now, as before, we evaluate such a sequence of functions in x_3 and we extract another subsequence converging in x_3. We then have

$$\lim_{j\to+\infty} f_{n_j^{1,2,3}}(x_3) = y_3 \in \mathbb{R}^p.$$

Proceeding in this way, at the step h we have a subsequence of functions, denoted by

$$f_{n_j^{1,2,\cdots,h-1,h}}$$

which converges in x_i, for every $i = 1, 2, \cdots, h-1, h$, respectively to points $y_i \in \mathbb{R}^p$.

Now, for every $i \in \mathbb{N}$ we define the index (*diagonal procedure*)

$$n_i := n_i^{1,\cdots,i-1,i}.$$

Let us first note that $\lim_{i\to+\infty} n_i = +\infty$ (that is n_i is really a subsequence of the sequence of indices n). Indeed, by our construction we have

$$n_i = n_i^{1,\cdots,i} \le n_i^{1,\cdots,i,i+1} < n_{i+1}^{1,\cdots,i,i+1} = n_{i+1}, \tag{8.2}$$

where the first inequality comes from the fact that the sequence of indices $\{n_j^{1,\cdots,i+1}\}_j$ is extracted from the sequence $\{n_j^{1,\cdots,i}\}_j$.

We leave the proof of the following statement to the reader.

The subsequence of functions f_{n_i} pointwise converges to the function f on A, where $f(x_h) = y_h$ for every $h \in \mathbb{N}$. That is

$$\lim_{i\to+\infty} f_{n_i}(x) = f(x) \quad \forall x \in A.$$

Moreover, f is uniformly continuous in A. (Hint for the pointwise convergence: take $x_h \in A$ and $\varepsilon > 0$ and show that there exists $j_h \ge h$ such that $\|f_{n_{j_h}}^{1,\dots,h}(x_h) - y_h\| \le \varepsilon$ and conclude by (8.2). Hint for the uniform continuity: take $\varepsilon > 0$ and then $\delta > 0$ given by the equicontinuity of f_h; take any $x, y \in A$ with $\|x - y\| \le \delta$ and then i such that $\|f_{n_i}(x) - f(x)\|, \|f_{n_i}(y) - f(y)\| \le \varepsilon$.)

Since f is uniformly continuous on A, it uniquely extends to a uniformly continuous function, still denoted by f, defined on the whole set K, which is the closure of A.

We are done if we prove that f_{n_i} uniformly converges to f on K. By absurd, let us suppose that it is false. Hence, there exist $\overline{\varepsilon} > 0$, a subsequence $i_s \to +\infty$ as $s \to +\infty$, and a sequence of points $x_{i_s} \in K$ such that

$$\|f_{n_{i_s}}(x_{i_s}) - f(x_{i_s})\| > \overline{\varepsilon} \ \ \forall s.$$

By the compactness of K the sequence of points has a subsequence which converges to a point $\overline{x} \in K$. We still denote such a subsequence by x_{i_s}, that is

$$x_{i_s} \to \overline{x} \ \text{ as } s \to +\infty.$$

By the equicontinuity of the functions $f_{n_{i_s}}$ and the uniform continuity of f, there exists $\delta > 0$ such that

$$\xi, \eta \in K, \ \|\xi - \eta\|_{\mathbb{R}^m} \leq \delta \implies \|f_{n_{i_s}}(\xi) - f_{n_{i_s}}(\eta)\|_{\mathbb{R}^p} \leq \frac{\overline{\varepsilon}}{5} \ \forall s, \text{ and } \|f(\xi) - f(\eta)\|_{\mathbb{R}^p} \leq \frac{\overline{\varepsilon}}{5}.$$

Now, we take $\overline{s} \in \mathbb{N}$ such that $\|x_{i_s} - \overline{x}\| \leq \delta$ for $s \geq \overline{s}$, we take $\tilde{x} \in A$ such that $\|\tilde{x} - \overline{x}\| \leq \delta$ (remember that A is dense in K), and finally we possibly modify $\overline{s}$ in such a way that $\|f_{n_{i_s}}(\tilde{x}) - f(\tilde{x})\| \leq \overline{\varepsilon}/5$ for all $s \geq \overline{s}$. Hence, for $s \geq \overline{s}$ we have the following inequality which contradicts the absurd hypothesis:

$$\begin{aligned}
&\|f_{n_{i_s}}(x_{i_s}) - f(x_{i_s})\| \\
&\leq \|f_{n_{i_s}}(x_{i_s}) - f_{n_{i_s}}(\overline{x})\| + \|f_{n_{i_s}}(\overline{x}) - f_{n_{i_s}}(\tilde{x})\| + \|f_{n_{i_s}}(\tilde{x}) - f(\tilde{x})\| \\
&+ \|f(\tilde{x}) - f(\overline{x})\| + \|f(\overline{x}) - f(x_{i_s})\| \\
&\leq \frac{\overline{\varepsilon}}{5} + \frac{\overline{\varepsilon}}{5} + \frac{\overline{\varepsilon}}{5} + \frac{\overline{\varepsilon}}{5} + \frac{\overline{\varepsilon}}{5} + \overline{\varepsilon}.
\end{aligned}$$

$\square$

Now we exhibit three examples where we separately let drop only one of the previous hypotheses (compactness of K, equiboundedness, equicontinuity) and where the thesis of the theorem does not hold anymore.

Example 8.11 (Lacking of compactness.) Let us consider the sequence $f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq n \text{ or } x \geq n+1, \\ x - n & \text{if } n \leq x \leq n + \frac{1}{2}, \\ -x + n + 1 & \text{if } n + \frac{1}{2} \leq x \leq n + 1. \end{cases}$$

Such a sequence is equicontinuous and equibounded on $\mathbb{R}$, but no subsequence uniformly converges on $\mathbb{R}$. Indeed, the sequence pointwise converges to the null function $f \equiv 0$ and then, if a subsequence is uniformly convergent it must converge to $f \equiv 0$[114], but

$$\sup_{x \in \mathbb{R}} |f_n(x)| = \frac{1}{2} > 0 \ \forall \ n.$$

[114] The uniform convergence implies the pointwise convergence.

Example 8.12 (Lacking of equiboundedness). Let us consider the sequence $f_n : [0,1] \to \mathbb{R}$, with $f_n(x) \equiv n$ for all n. Such a sequence is equicontinuous on the compact set $[0,1]$ but it is not equibounded. And, of course, no subsequence is uniformly convergent.

Example 8.13 (Lacking of equicontinuity.) Let us consider the sequence $f_n : [-1,1] \to \mathbb{R}$

$$f_n(x) = \begin{cases} -1 & \text{if } -1 \le x \le -\frac{1}{n}, \\ nx & \text{if } -\frac{1}{n} \le x \le \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \le x \le 1. \end{cases}$$

Such a sequence is equibounded on the compact set $[-1,1]$ but it is not equicontinuous (even if every f_n is continuous). No subsequence may uniformly converge. Indeed, the sequence is pointwise converging to the function

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \end{cases}$$

which is discontinuous. If a subsequence uniformly converges, then it must converge to f, but all the f_n are continuous and the limit f is discontinuous. This is impossible. Hence there are not uniformly convergent subsequence.

8.3 Metric spaces

Definition 8.14 *A non empty set X is said a <u>metric space</u>[115] if there exists a nonnegative function*

$$d : X \times X \to [0, +\infty[$$

satisfying:
i) (positive definition) $d(x,y) = 0$ if and only if $x = y$;
ii) (symmetry) $d(x,y) = d(y,x)$ for all $x, y \in X$;
iii) (triangular inequality) $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in X$.
The function d is also said a <u>distance on X</u> (or a <u>metrics</u>).

It is easy to see that a metric space is also a topological space where the convergence of a sequence x_n to a point $\overline{x}$ is equivalent to the convergence to 0 (in $\mathbb{R}$) of the distances[116], i.e.:

$$\lim_{n \to +\infty} x_n = \overline{x} \iff \lim_{n \to +\infty} d(x_n, \overline{x}) = 0.$$

[115] Actually, we should say that a couple (X, d) is a metric space if X is a non empty set and d is a function from $X \times X$ to $[0, +\infty[$ satisfying....

[116] A more precise characterization is that the balls $B(x, \delta) = \{y \in X \big| d(x,y) < \delta\}$, when we vary $x \in X$ and $\delta > 0$, form a basis for the topology (the set of all open sets of X)

8.3.1 Completeness and fixed point theorems

Definition 8.15 *A metric space X, with distance d, is said a complete metric space if every Cauchy sequence in it is convergent. A Cauchy sequence in X is a sequence of points x_n of X such that*

$$\forall\ \varepsilon > 0\ \exists\ \overline{n} \in \mathbb{N} \text{ such that } n, m \geq \overline{n} \implies d(x_n, x_m) \leq \varepsilon.$$

Definition 8.16 *Let us consider a metric space X and a function $F : X \to X$. We say that F is a contraction (on X) if there exists a constant L, with $0 \leq L < 1$ such that*

$$d\left(F(x), F(y)\right) \leq Ld(x, y), \quad \forall\ x, y \in X.$$

We say that a point $x \in X$ is a fixed point for (of) F if

$$F(x) = x.$$

The following theorem is one of the most powerful tool in mathematical analysis.

Theorem 8.17 *(Contraction Lemma). If X is a complete metric space and $F : X \to X$ is a contraction, then there exists a unique fixed point for F, that is*

$$\exists!\ x \in X \text{ such that } F(x) = x.$$

Note that this is an existence as well as a uniqueness result: it says that a fixed point exists, and it also says that it is unique.

Proof of Theorem 8.17. i) (Uniqueness.) We first prove that there exists at most one fixed point. Indeed, let $x, y \in X$ be two fixed points for F, and suppose that they are distinct (i.e. $d(x, y) > 0$). We immediately get the following contradiction:

$$d(x, y) = d\left(F(x), F(y)\right) \leq Ld(x, y) < d(x, y).$$

ii) (Existence.) Now we prove that a fixed point really exists. Let us take a point $x \in X$ and consider the following sequence in X:

$$x_0 := x,\ x_1 := F(x_0),\ x_2 := F(x_1),\ x_3 := F(x_2),\ \ldots,\ x_n := F(x_{n-1}),\ \ldots$$

We now prove that such a sequence is a Cauchy sequence and that its limit (which exists since X is complete) is a fixed point for F. We first observe that

$$x_n = F(x_{n-1}) = F(F(x_{n-2})) = F(F(F(x_{n-3}))) = \cdots = F^n(x),$$

where, with obvious notation, F^n stays for the composition $F \circ F \circ \cdots \circ F$ of F n times with itself (with F^0 the identity). Using such a notation we have, for every $n \geq 1$,

$$d(x_{n+1}, x_n) = d\left(F^{n+1}(x), F^n(x)\right) = d\left(F\left(F^n(x)\right), F\left(F^{n-1}(x)\right)\right) \leq Ld\left(F^n(x), F^{n-1}(x)\right).$$

Repeating such an estimate n times and setting $C = d(x_1, x) \geq 0$, we get, for every n,

$$d(x_{n+1}, x_n) \leq L^n d(x_1, x) := L^n C \geq 0,$$

and hence, for all $n \geq m \geq \overline{n}$ (recall $L \geq 0$)

$$0 \leq d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \leq C \sum_{i=m}^{n-1} L^i < C \sum_{i=\overline{n}}^{+\infty} L^i,$$

from which we get the Cauchy property of the sequence since $\sum_{i=\overline{n}}^{+\infty} L^i$ is the rest of the converging series of powers of L (remember that $0 \leq L < 1$), and so it is infinitesimal as $\overline{n} \to +\infty$.

Now, let $\overline{x} \in X$ be the limit of the sequence. Hence we have

$$\begin{aligned} &0 \leq d(F(\overline{x}), \overline{x}) \leq d\left(F(\overline{x}), F^{n+1}(x)\right) + d\left(F^{n+1}(x), \overline{x}\right) \\ &\leq Ld(\overline{x}, F^n(x)) + d\left(F^{n+1}(x), \overline{x}\right) \ \to \ 0, \text{ as } n \to +\infty. \end{aligned}$$

We then get $d(F(\overline{x}), \overline{x}) = 0$, from which the conclusion $F(\overline{x}) = \overline{x}$. □

8.4 Matrix algebra

8.4.1 The exponential matrix and its properties.

Theorem 8.18 *Let A be a $n \times n$ matrix. Then the matrix series*[117]

$$\sum_{k=0}^{+\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

is convergent[118] *to a $n \times n$ matrix which we call the exponential matrix of A and we denote by e^A.*

Proof. We recall that, given the $n \times n$ matrix

$$A = (a_{ij})_{i,j=1,\dots,n},$$

its norm is defined as

$$\|A\| = \sum_{i,j=1}^{n} |a_{ij}|,$$

[117] Where I is the identity matrix and A^k means $AA \cdots A$ k times.

[118] You can regard the convergence of a series of $n \times n$ matrices as convergence of a sequence of vectors in $\mathbb{R}^{n^2}$.

which, from a topological point of view, is equivalent to the Euclidean norm of $\mathbb{R}^{n^2}$. Since it is a norm it satisfies the sub-additivity property. Moreover, also the Cauchy-Schwarz inequality holds: for every A, B $n \times n$ matrices

$$\|AB\| \le \|A\|\|B\|.$$

Now, we prove that our matrices series is a Cauchy series, from which the conclusion follows. Indeed, for every $p, q \in \mathbb{N}$ we have

$$\left\|\sum_{k=0}^{p+q} \frac{A^k}{k!} - \sum_{k=0}^{p} \frac{A^k}{k!}\right\| = \left\|\sum_{k=p+1}^{p+q} \frac{A^k}{k!}\right\| \le \sum_{k=p+1}^{p+q} \frac{\|A^k\|}{k!} \le \sum_{k=p+1}^{p+q} \frac{\|A\|^k}{k!},$$

and we conclude, since the last term is the rest of the (converging) real exponential series of $\|A\| \in \mathbb{R}$. $\square$

We do not prove the following Proposition, whose proof may be found on every text book.

Proposition 8.19 *For every $n \times n$ matrix, the exponential matrix e^A is nonsingular (i.e. determinant different from zero), indeed*

$$\det\left(e^A\right) = e^{tr(A)},$$

where $tr(A)$ is the trace of A, $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

Proposition 8.20 *Let us denote by $\mathcal{M}_n$ the space of the $n \times n$ real matrices, and fix $A \in \mathcal{M}_n$. Then, the function*

$$\mathbb{R} \ni t \mapsto e^{tA} \in \mathcal{M}_n$$

is derivable and its derivative is

$$\frac{d}{dt} e^{tA} = A e^{tA}. \tag{8.3}$$

Proof. We just prove (8.3). In the same way as we proved the convergence of the series, we can also prove the uniform convergence, on every compact set, of the time-dependent series

$$\sum_{k=0}^{+\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!},$$

and of the series of its derivatives

$$\sum_{k=1}^{+\infty} \frac{k t^{k-1} A^k}{k!}.$$

Hence, we can derive term by term, and we get

$$\begin{aligned}\frac{d}{dt}e^{tA} &= \frac{d}{dt}\sum_{k=0}^{+\infty}\frac{t^kA^k}{k!} = \sum_{k=0}^{+\infty}\frac{A^k}{k!}\frac{d}{dt}t^k \\ &= \sum_{k=1}^{+\infty}\frac{A^k}{(k-1)!}t^{k-1} = A\sum_{k=0}^{+\infty}\frac{t^kA^k}{k!} = Ae^{tA}.\end{aligned}$$

□

Remark 8.21 *By its very definition we have $e^O = I$ where O is the null matrix. Moreover, for every $t, s \in \mathbb{R}$, A and the exponential matrices e^{tA}, e^{sA} commute and we have*

$$Ae^{tA} = e^{tA}A, \quad e^{tA}e^{sA} = e^{(t+s)A}.$$

8.4.2 On the calculus of the exponential matrix

In general, to explicitly calculate the exponential matrix e^A is hard[119]. Here, we illustrate some particular favorable cases for the computation of e^A.

(*Diagonal matrix.*) If A is diagonal, then the exponential matrix e^A is the diagonal matrix whose principal diagonal coefficients c_{jj} are given by $e^{\lambda_{jj}}$, where λ_{jj} are the corresponding coefficients of the principal diagonal of A. To check this fact, just use the definition.

(*Diagonalizable matrix.*) If A is diagonalizable (on $\mathbb{R}$), i.e. there exist two (real) matrices B, invertible, and D, diagonal, such that

$$D = B^{-1}AB,$$

then, using just the definition, an easy calculation shows that, for every $t \in \mathbb{R}$,

$$e^{tA} = Be^{tD}B^{-1},$$

where e^{tD} is diagonal as in the previous case.

(*Three special cases for* 3×3 *matrices.*[120])

i) The matrix A has only one eigenvalue $\lambda \in \mathbb{R}$. Then, for every $t \in \mathbb{R}$:

$$e^{tA} = e^{\lambda t}\left(I + t\,(A - \lambda I) + \frac{1}{2}t^2\,(A - \lambda I)^2\right).$$

ii) The matrix A has three distinct eigenvalues $\lambda, \mu, \nu \in \mathbb{R}$. Then, for every $t \in \mathbb{R}$:

[119] In particular, e^A is not the matrix whose coefficients are the exponential of the coefficients of A.

[120] For the proof, see, for instance, T.M. Apostol: "Calcolo, Volume terzo, Analisi 2", Bollati-Boringhieri, 1992.

$$e^{tA} = e^{\lambda t}\frac{(A-\mu I)\,(A-\nu I)}{(\lambda-\mu)(\lambda-\nu)} + e^{\mu t}\frac{(A-\lambda I)\,(A-\nu I)}{(\mu-\lambda)(\mu-\nu)} + e^{\nu t}\frac{(A-\lambda I)\,(A-\mu I)}{(\nu-\lambda)(\nu-\mu)}.$$

iii) The matrix A has only two eigenvalues $\lambda, \mu \in \mathbb{R}$, where λ has (algebraic) multiplicity 2 and μ has (algebraic) multiplicity 1. Then, for every $t \in \mathbb{R}$:

$$e^{tA} = e^{\lambda t}\left(I + t(A-\lambda I)\right) + \frac{e^{\mu t}-e^{\lambda t}}{(\mu-\lambda)^2}\,(A-\lambda I)^2 - \frac{te^{\lambda t}}{\mu-\lambda}\,(A-\lambda I)^2\,.$$

8.5 Remarks on the space $C([a,b];\mathbb{R}^n)$

Let $[a,b]$ be a compact interval of the real line $\mathbb{R}$. Then we define the set of all continuous functions (up to the boundary) from $[a,b]$ to $\mathbb{R}^n$ as[121]

$$C([a,b];\mathbb{R}^n) = \left\{ f : [a,b] \to \mathbb{R}^n \middle| f \text{ is continuous in } [a,b] \right\}.$$

It is well-known that it is a vectorial space on $\mathbb{R}$. We endow it with the following metrics[122]

$$d(f,g) = \sup_{x\in[a,b]} \|f(x)-g(x)\|_{\mathbb{R}^n} =: \|f-g\|_\infty. \tag{8.4}$$

Remark 8.22 *Note that, by definition, a sequence $f_k \in C([a,b];\mathbb{R}^n)$ converges to a function $f \in C([a,b];\mathbb{R}^n)$ with respect to the metrics in (8.4) if and only if the sequence uniformly converges to f in $[a,b]$.*

Proposition 8.23 *Endowed with the metrics (8.4), $C([a,b];\mathbb{R}^n)$ is a complete metric space.*

Proof. Let us consider a Cauchy sequence, that is a sequence of functions $f_k \in C([a,b];\mathbb{R}^n)$ such that, for every $\varepsilon > 0$, there exists $\overline{k} \in \mathbb{N}$ such that

$$i,j \geq \overline{k} \implies \sup_{x\in[a,b]} |f_i(x) - f_j(x)| \leq \varepsilon. \tag{8.5}$$

But then, for every $x \in [a,b]$ fixed, the sequence of vectors $f_k(x)$ is also a Cauchy sequence in $\mathbb{R}^n$. Hence, for the completeness of $\mathbb{R}^n$, it converges. Let us denote by $f(x)$ the limit of such a sequence. We are going to prove that $f \in C([a,b];\mathbb{R}^n)$ and that

$$\lim_{k\to+\infty} \|f_k - f\|_\infty = 0,$$

[121]Sometimes it is also denoted by $C^0([a,b];\mathbb{R}^n)$.
[122]As exercise, prove that it is a metrics.

which will conclude the proof. Let us fix $\varepsilon > 0$ and take $\overline{k}$ as in (8.5). Hence, for $i, j \geq \overline{k}$, we have

$$\begin{aligned}
&\|f_i - f_j\|_\infty \leq \varepsilon \Longrightarrow \|f_i(x) - f_j(x)\| \leq \varepsilon \ \forall x \in [a,b] \\
&\Longrightarrow \lim_{j\to+\infty} \|f_i(x) - f_j(x)\| \leq \varepsilon \ \forall x \in [a,b] \Longrightarrow \|f_i(x) - f(x)\| \leq \varepsilon \ \forall x \in [a,b] \\
&\Longrightarrow \|f_i - f\|_\infty \leq \varepsilon.
\end{aligned}$$

Since the convergence in metrics is the uniform convergence, then the limit is also continuous: $f \in C([a,b];\mathbb{R}^n)$, and the proof is finished. $\square$

Remark 8.24 *If $B \subseteq \mathbb{R}^n$ is a set, we can also consider the space*

$$C([a,b];B) = \left\{ f : [a,b] \to B \middle| f \text{ is continuous in } [a,b] \right\},$$

which is a subset of $C([a,b];\mathbb{R}^n)$, and we can think to it as endowed with the same metrics (8.4). If B is closed, then $C([a,b];B)$ is also closed in $C([a,b];\mathbb{R}^n)$ and hence, as metric space, it is a complete metric space too.

8.6 Remarks on differential 1-forms in two variables

Definition 8.25 *A differential 1-form in two variables is a function between an open set A of the plane $\mathbb{R}^2$ and the set $\mathcal{L}$ of the linear functions from $\mathbb{R}^2$ to $\mathbb{R}$:*

$$\omega : A \to \mathcal{L},\ (x,y) \mapsto \omega(x,y) : \mathbb{R}^2 \to \mathbb{R} \ (\textit{linear}).$$

If ω is such a 1-form, then it is not hard to see that there exist two functions $P, Q : A \to \mathbb{R}$ such that $\omega(x,y) = P(x,y)dx + Q(x,y)dy$, where dx and dy are the projection to the x and the to y axis, respectively[123]*. We say that ω is a continuous 1-form (respectively: a C^1 1-form) if P and Q are continuous (respectively: of class C^1).*

If $\omega = Pdx + Qdy$ is a continuous 1-form on $A \subseteq \mathbb{R}^2$, we say that a C^1 function $\varphi : A \to \mathbb{R}$ is a primitive (or a potential) of the differential form if

$$\frac{\partial \varphi}{\partial x}(x,y) = P(x,y), \quad \frac{\partial \varphi}{\partial y}(x,y) = Q(x,y) \ \ \forall\ (x,y) \in A,$$

and, in such a case, we say that w is exact.

The following theorem is well-known.

Theorem 8.26 *Let $\omega : A \to \mathcal{L}$ be a C^1 1-form on $A \subseteq \mathbb{R}^2$, open. A necessary condition for $\omega = Pdx + Qdy$ be exact is*

$$\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y) \ \ \forall (x,y) \in A. \tag{8.6}$$

The condition (8.6) becomes also sufficient if the domain A is simply connected[124]*.*

[123] They are a basis of the linear space $\mathcal{L}$.

[124] A is simply connected if it is connected and it has no "holes", in the sense that every closed curve in it is homotopic to a point, with homotopy which makes the curve not exit from A.

Once we know that a continuous differential form $\omega = Pdx + Qdy$ is exact, how can we calculate a primitive? Here is one of the possible methods. If φ is a primitive, then

$$\frac{\partial \varphi}{\partial x}(x, y) = P(x, y),$$

and so, using indefinite integrals, we also have

$$\int \frac{\partial \varphi}{\partial x}(x, y)dx = \int P(x, y)dx \implies \varphi(x, y) = \mathcal{P}(x, y) + g(y),$$

where $\mathcal{P}$ is a primitive of P with respect to x only, and $g(y)$ is the constant of integration which of course depends on y[125]. Then we get

$$\frac{\partial \varphi}{\partial y}(x, y) = Q(x, y) \implies \frac{\partial \mathcal{P}}{\partial y}(x, y) + g'(y) = Q(x, y),$$

from which

$$g'(y) = Q(x, y) - \frac{\partial \mathcal{P}}{\partial y}(x, y),$$

and, if ω is exact, the second member is really a function of the only variable y. Hence

$$g(y) = \tilde{\mathcal{Q}}(x, y) + k$$

where $k \in \mathbb{R}$ and $\tilde{\mathcal{Q}}$ is a primitive of the function of y: $Q(x, y) - \frac{\partial \mathcal{P}}{\partial y}(x, y)$. We conclude that the functions

$$\varphi(x, y) = \mathcal{P}(x, y) + \tilde{\mathcal{Q}}(x, y) + k,$$

are all the primitives of ω[126].

[125]It is a constant with respect to the integration variable x only.
[126]Actually, they are "all" the primitives if A is connected.